Eta expressions for (some) modular forms

LuCaNT 2025 - Lightning talks

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Eta quotients in the LMFDB

eta.yaml

Newform orbit 32.2.a.a

Newspace parameters

Level:	N	=	$32 = 2^5$
Weight:	k	=	2
Character orbit:	$[\chi]$	=	32.a (trivial)

Expression as an eta quotient

$$f(z) = \eta(4z)^2\eta(8z)^2 = q\prod_{n=1}^{\infty}(1-q^{4n})^2(1-q^{8n})^2$$

- The LMFDB only stores quotients
- There are many more expressions scattered throughout the literature (and beyond!)

Finding eta expressions

Example

Let f be the cusp form with label `35.2.a.a`. Then

$$\begin{aligned} f(z) &= \eta(z)^2\eta(35z)^2 + \eta(5z)^2\eta(7z)^2 \\ &= \eta_{35}[2,0,0,2] + \eta_{35}[0,2,2,0]. \end{aligned}$$

(the divisors of 35 are $\{1,5,7,35\}$)

The eta quotients in $M_2(35)$ are

$\eta_{35}[3,-1,-1,3], \eta_{35}[2,0,0,2], \eta_{35}[1,1,1,1], \eta_{35}[0,2,2,0], \eta_{35}[-1,3,3,-1].$

Linear algebra in $M_2(35) \rightsquigarrow$ the expression above.

But optmising the expression is usually harder than this!

Technical challenges

Problem 1: Not enough quotients. There are no eta quotients in $M_2(37)$!
Solution 1: Raise the level. Both `37.2.a.a` and `37.2.a.b` are in the span of eta quotients in $M_2(37 \cdot 8)$.

Problem 2: Too many quotients. Ideally we would like an expression with the fewest terms and small coefficients. `37.2.a.a` requires between 7 and 21 quotients in $M_2(37 \cdot 8)$, and `37.2.a.b` requires between 7 and 16.

- Writing down the eta quotients becomes expensive as the weight and (the number of divisors of) the level grow.
- Finding a "nice" expression becomes *very* expensive as the number of quotients grows. Our current "best" expression for `89.2.a.b` has 160 terms!

What has been, and what one day may be

In principle, $f \in M_k(N)$ can be written as a rational function in eta quotients of level at most $4N$ (Kilford 2007). Eta expressions can be found sporadically all over the place (including physics papers!)

Someday, the LMFDB might have an expression for *every* cusp form that is proven to be "maximally nice". Or at least every form in some range.

The method we used in `arXiv:2407.05748`, joint with Elisabeth (Yin Ting) Chan, misses 4 forms in the range weight $k = 2$, level $N = 1, \ldots, 100$, and uses level raises greater than $4N$. Even this modest range seems unfeasible.

Some theoretical advances are needed!

Reduction types of curves in p-adic families

Jakab Schrettner

July 10, 2025

Motivation: Tate's algorithm

A particular corollary of Tate's algorithm is the following continuity result:

Theorem (Corollary of Tate's algorithm)

For an elliptic curve $E/\mathbb{Q}_p$ in Weierstrass form:

$$E : y^2 = x^3 + ax + b,$$

there exists an integer $N \geq 1$ (depending on a, b) such that the reduction type of E depends only on the values of a, b mod p^N.

Alternatively: in the family of elliptic curves obtained by varying the values of a and b, the reduction type is *locally constant*.

Reduction types in families

Goal: in a family of (smooth projective) curves obtained by varying the coefficients in the defining equations, the reduction type is locally constant.

Why do we care about this? Possible applications:

- Local-to-global arguments: given a curve $C/\mathbb{Q}_p$, we may perturb its coefficients slightly to get a curve $C'/\mathbb{Q}$ with the same reduction type.
- Computation: to compute the reduction type of C, it suffices to know coefficients up to some finite precision.

Reduction types in families

We can show the following:

Theorem

Let $C/\mathbb{Q}_p$ be a curve given as a complete intersection (n equations in $n+1$ variables). Let C' be a curve obtained by perturbing the coefficients of the equations defining C.
If the perturbation is sufficiently small, then C and C' have regular models with the same special fibre.

Example

Example: bihyperelliptic curves

$$C : \begin{cases} y^2 = f(x) \\ z^2 = g(x) \end{cases} \qquad \text{and} \qquad C' : \begin{cases} y^2 = \widetilde{f}(x) \\ z^2 = \widetilde{g}(x) \end{cases}$$

for some $f, \widetilde{f}, g, \widetilde{g} \in \mathbb{Z}_p[x]$.

Previous theorem implies that if $f \equiv \widetilde{f} \mod p^N$ and $g \equiv \widetilde{g} \mod p^N$ for some sufficiently large N, then C and C' have the same reduction type.

An algorithm for isolated j-invariants arising from $X_0(n)$

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July 10, 2025

Isolated points

Let C be a smooth, projective, and geometrically integral curve defined over a number field k.

Definition

A closed point $x \in C$ of degree d is **isolated** if it is not part of an infinite family of degree d points parametrized by a geometric object.

Let $\Phi_d : C^{(d)} \to \mathrm{Jac}(C)$.

- $\mathbb{P}^1$-parametrized: A rational degree d map $f : C \to \mathbb{P}^1$ where $x \in f^{-1}(\mathbb{P}^1(\mathbb{Q}))$.
- AV-parametrized: A positive rank abelian subvariety A, such that $\Phi_d(x) + A \subseteq \mathrm{im}(\Phi_d)$.

If $x = (E, \langle P \rangle) \in X_0(n)$ is isolated, then we call the j-invariant $j(E)$ an **isolated j-invariant**.

Motivations

For $X_1(n)$:

- Merel proved a uniform boundedness theorem which states that for a fixed d, there are no non-cuspidal points on $X_1(n)$ with degree d, for n sufficiently large.
- A classification of degree d points is known for $d \leq 4$.

Analogue of Merel's theorem for $X_0(n)$ is false — quadratic non-cuspidal CM points exist for infinitely many n. What can we say for non-CM?

- The set of n for which there are infinitely many quadratic points on $X_0(n)$: known.
- The set of n for which there is a nonempty, finite set of quadratic non-CM non-cuspidal points on $X_0(n)$: **unknown**.

  Quadratic points **are isolated** when there are finitely many of them.

- Recent conjecture of Balakrishnan and Mazur (2024): there are finitely many such n.

Overview of non-CM j-invariants algorithm

Input: A non-CM, rational j-invariant.
Output: A finite list

$$(a_1, d_1), ..., (a_k, d_k),$$

such that j is isolated if and only if there exists an isolated point $x \in X_0(a_i)$ of degree d_i with $j(x) = j$ for some pair (a_i, d_i) in the list.

(1) Construct an elliptic curve $E/\mathbb{Q}$ with $j(E) = j$.
(2) Compute adelic image of $E/\mathbb{Q}$, using Zywina's open image algorithm.
(3) Find m_0, the level of the m-adic Galois representation associated to E, where m is the product of $2, 3$, and all non-surjective primes.
(4) Obtain finite list $\mathcal{P}(E)$ as (level, degree) pairs.
(5) Riemann-Roch filter: Remove (level, degree) pairs such that degree $> \mathrm{genus}(X_0(\text{level}))$.
(6) Genus 0 filter: Remove (level, degree) pairs such that, if $a_i =$ level, $\mathrm{genus}(\text{the mod } a_i \text{ Galois representation of } E/\mathbb{Q}) = 0$.

Magma implementation

j-invariant	n
-121	11
-24729001	11
46969655/32768	15
-121945/32	15
-349938025/8	15
-25/2	15
-882216989/131072	17
-297756989/2	17
3375/2	21
-189613868625/128	21
-1159088625/2097152	21
-140625/8	21
-9317	37
-162677523113838677	37

Table: Known rational non-CM isolated j-invariants for $X_0(n)$

GitHub repository:

Identifying the rational elliptic curves that provably admit infinitely many twists satisfying full BSD

Barinder S. Banwait and Xiaoyu (Coco) Huang

Boston University/MIT and Temple University

~~Rump session~~ Lightning talk
LuCaNT 2025 @ ICERM
Thursday, 10^{th} July 2025

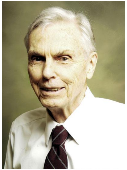
Possibly the hardest way to become a millionaire ...

(A) *If g is the number of generators of $\mathcal{A}$ of infinite order, then $f(P) \sim C(\log P)^g$ as $P \to \infty$, and $\zeta_\Gamma(s) \sim C'(s-1)^{g-1}$ as $s \to 1$, for some constants C, C' depending on Γ.*

The second conjecture of Birch and Swinnerton-Dyer, which refines the first, is

(B) $$L^*(s) \sim \frac{[Ш]\,|\det \langle a_i', a_j \rangle|}{[A(K)_{tors}][A'(K)_{tors}]}(s-1)^r, \quad \text{as} \quad s \to 1,$$

where the quantities on the right are now to be explained.



Theorem 1 *A majority of elliptic curves over $\mathbb{Q}$, when ordered by height, satisfy the Birch and Swinnerton-Dyer rank conjecture.*

In fact, we will prove that the words "A majority" in Theorems 1 and 2 can be replaced with "At least 66.48%".

What about full BSD?

Theorem (Tian, 2014, CM case)

The elliptic modular curve $X_0(32)$ admits infinitely many twists that satisfy full BSD.

Theorem (Burungale–Skinner–Tian–Wan, 2024, non-CM case)

The elliptic curve `46.a2` admits infinitely many twists that satisfy full BSD.

Theorem 10.12. *Let E be a semistable elliptic curve defined over $\mathbb{Q}$ with conductor N, and $M > 1$ a square-free integer with $(M, N) = 1$. Let $E^{(M)}$ denote the quadratic twist of E by the character associated to the quadratic extension $\mathbb{Q}(\sqrt{M})/\mathbb{Q}$. Suppose that the following conditions hold.*

(i) *We have*

$$L(1, E^{(M)}) \neq 0,$$

(ii) *The 2-part of the BSD formula holds for $E^{(M)}$,*

(iii) $a_3(E) = 0$,

(iv) *For all odd primes p, $E[p]$ is an absolutely irreducible $G_{\mathbb{Q}}$-representation,*

(v) *For any prime $p|N$, there exists a multiplicative prime $q \neq p$ at which $E[p]$ is ramified,*

(vi) *E has ordinary reduction at prime divisors of M.*

Then the Birch and Swinnerton-Dyer conjecture is true for $E^{(M)}$, that is, $E^{(M)}(\mathbb{Q})$ and $\text{Ш}(E^{(M)})$ are finite, and

$$\frac{L(1, E^{(M)})}{\Omega_{E^{(M)}}} = \frac{\#\text{Ш}(E^{(M)}) \cdot \prod_{\ell \nmid \infty} c_\ell(E^{(M)})}{\#E^{(M)}(\mathbb{Q})_{\text{tor}}^2}.$$

Question

1. *Which elliptic curves in the LMFDB admit infinitely many twists satisfying full BSD?*
2. *For each such curve E, given a squarefree M, do we know that full BSD holds for $E^{(M)}$?*

```python
    if cond_lower_bound is not None:
        ecq_query['conductor'] = {'$gte' : cond_lower_bound}
print(f"ecq_query={ecq_query}")
ecq_payload = ecq.search(ecq_query, projection=ECQ_COLS)
df = pd.DataFrame(list(ecq_payload))
assert df['lmfdb_iso'].nunique() == len(df), "Values in the 'lmfdb_iso' column are not uniqu

# CONDITION 1b: a3 in {-2, -1, 0, 1, 2}
classdata_df = get_a3_a5_a7(df)
lmfdb_iso_labels_a3_cond = classdata_df[classdata_df['a3'].isin({-2, -1, 0, 1, 2})]
df = pd.merge(df, lmfdb_iso_labels_a3_cond, on='lmfdb_iso', how='inner')

# CONDITION 1c: 2 is the only isogeny prime
# CONDITION 1d: ramification CONDITION at bad primes
df = filter_CONDITIONs_c_d(df)

#----------------------
# check the residual criteria from [CLZ20, Theorem 1.5]
#----------------------
# CONDITION 1g: E(Q)[2] = Z/2Z
df_CLZ20 = filter_CONDITIONs_g(df)

# CONDITION 1h: 2-ord of special_value/(real_period*regulator) = -1
df_CLZ20 = merge_mwbsd(df_CLZ20)
df_CLZ20['L_alg'] = (df_CLZ20['special_value']/(df_CLZ20['real_period'] * df_CLZ20['regulato
df_CLZ20['L_alg'] = df_CLZ20['L_alg'].apply(lambda x: QQ(RR(x)).valuation(2))    # now L_alg
df_CLZ20 = df_CLZ20[df_CLZ20['L_alg'] == -1]

# condtion 1i: sha(E') = 1
df_CLZ20 = filter_CONDITIONs_i(df_CLZ20)

# give the source of the data
df_CLZ20['source'] = 'CLZ20'
#----------------------
# check the residual criteria from [Zha16, Theorem 1.1 - 1.4]
#----------------------
# CONDITION 1g:
df_Zha16 = df
df_Zha16 = merge_mwbsd(df_Zha16)
df_Zha16['real_components'] = df_Zha16['ainvs'].apply(lambda x: EllipticCurve(x).real_compon
df_Zha16['L_alg_ord_2'] = (df_Zha16['special_value'] * df_Zha16['real_components']/(df_Zha16
```

data > lmfdb_labels_new.txt

```
Generated at 18:49 (eastern) on Monday 07 July 2025
Params: cond_upper_bound=None, cond_lower_bound=None
Run took: 108 minutes 0 seconds

46.a2, CLZ20
69.a2, CLZ20
70.a4, CLZ20
77.c2, CLZ20
78.a4, CLZ20
82.a2, CLZ20
85.a1, CLZ20
94.a2, CLZ20
102.c5, CLZ20
114.a1, CLZ20
130.b4, CLZ20
138.c4, CLZ20
141.b2, CLZ20
142.b2, CLZ20
154.c4, CLZ20
155.b2, CLZ20
158.e2, CLZ20
174.c4, CLZ20
178.a2, CLZ20
182.c4, CLZ20
194.a2, CLZ20
203.c2, CLZ20
205.b1, CLZ20
205.c2, CLZ20
206.a2, CLZ20
210.a3, CLZ20
213.a2, CLZ20
219.c2, Zha16
221.a2, CLZ20
221.b1, CLZ20
222.b4, CLZ20
226.a2, CLZ20
238.a2, Zha16
238.d3, CLZ20
246.f3, CLZ20
254.c2, CLZ20
258.a1, CLZ20
258.g1, CLZ20
259.a2, CLZ20
274.b1, CLZ20
```

BSD formula

$$0.962477969 \approx L(E,1) = \frac{\#\text{Ш}(E/\mathbb{Q}) \cdot \Omega_E \cdot \mathrm{Reg}(E/\mathbb{Q}) \cdot \prod_p c_p}{\#E(\mathbb{Q})_{\mathrm{tor}}^2} \approx \frac{1 \cdot 3.849912 \cdot 1.000000 \cdot 4}{4^2} \approx 0.962477969$$

This elliptic curve admits infinitely many quadratic twists known to unconditionally satisfy full BSD.

Enter squarefree M to check if BSD is known at this twist: $M =$ 311 Compute This twist satisfies full BSD.

Why do we care?

1. Part of a larger project on studying how $|Ш|$ varies in a quadratic twist family, for which there is a conjecture due to Radziwill–Soundararajan.

straightforward, and combining this with the Keating-Snaith conjecture for $L\left(\frac{1}{2}, E_d\right)$, we are led to formulate the following conjecture.

Conjecture 1 *Let E be given by the model $y^2 = f(x)$ for a monic cubic polynomial f with integer coefficients. Let K denote the splitting field of f over $\mathbb{Q}$, and let $G = Gal(K/\mathbb{Q})$ denote the associated Galois group, which we view as a subgroup of S_3. Given $g \in G$ let $c(g)$ denote one plus the number of fixed points of g (viewed as an element of S_3), and define*

$$\mu(E) = -\frac{1}{2} - \frac{1}{|G|}\sum_{g\in G} \log c(g) \text{ and } \sigma(E) = 1 + \frac{1}{|G|}\sum_{g\in G} (\log c(g))^2 .$$

As d ranges over $\mathcal{E}$, the distribution of $\log\left(|Ш(E_d)|/\sqrt{|d|}\right)$ is approximately Gaussian with mean $\mu(E)\log\log|d|$ and variance $\sigma(E)^2\log\log|d|$. More precisely, for any fixed $V \in \mathbb{R}$ and as $X \to \infty$,

$$\left|\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log\left(|Ш(E_d)|/\sqrt{|d|}\right) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}} \ge V\right\}\right|$$

is

$$\sim |\{d \in \mathcal{E}, |d| \le X\}| \left(\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-\frac{x^2}{2}}dx\right).$$

2. Could also be used as a training set for an Artificial Superintelligence when it proves BSD! 🎉 ✨

Fin

Quaternionic analogues of Zagier's sums of binary quadratic forms

LuCaNT ⚡ - Thursday July 10, 2025

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Background - Zagier

- For $k > 0$ odd and Δ a positive integer, define $F_{k,\Delta} : \mathbb{R} \to \mathbb{R}$ by

$$F_{k,\Delta}(x) \overset{\text{def.}}{=} \sum_{\substack{a,b,c\in\mathbb{Z},a<0 \\ b^2-4ac=\Delta}} \max(0, ax^2 + bx + c)^k.$$

Theorem 1: Zagier[1].

- Fix $k > 0$ odd. The vector space

$$\operatorname{span}_{\mathbb{R}}\{F_{k,\Delta} \mid \Delta > 0 \text{ nonsquare discriminant}\}$$

is finite dimensional, with dimension is equal to the dimension of the vector space of modular forms of weight $2k + 2$ for $\mathrm{SL}(2, \mathbb{Z})$.

- **Example:** When $k = 1$ or 3, there are no cusp forms of weight $2k + 2$.
  - If Δ is the discriminant of a real quadratic field K, then Zagier shows

$$F_{1,\Delta}(x) = 60\zeta_K(-1) \quad \text{and} \quad F_{3,\Delta}(x) = 120\zeta_K(-3).$$

---

[1] Zagier D (1999) From quadratic functions to modular functions. In: Győry K, Iwaniec H, Urbanowicz J (eds) Number Theory in Progress. De Gruyter, Berlin, Boston, pp 1147–1178

Background - Karabulut

- Let $\mathcal{O}$ be the ring of integers in an Euclidean imaginary quadratic field.
- For $k > 0$ odd and Δ a positive integer, define $H_{k,\Delta} : \mathbb{C} \to \mathbb{R}$ by

$$H_{k,\Delta}(z) \stackrel{\text{def.}}{=} \sum_{\substack{a,c\in\mathbb{Z}, a<0, b\in\mathcal{O} \\ \bar{b}b-ac=\Delta}} \max\left(0, a\bar{z}z + bz + \overline{bz} + c\right)^k.$$

Theorem 2: Karabulut[2].

- Fix $k > 0$ odd. The vector space

$$\operatorname{span}_{\mathbb{R}}\left\{H_{k,\Delta} \mid \Delta > 0 \text{ not a norm of any } b \in \mathcal{O}\right\}$$

is finite dimensional, with dimension $\approx$ to dimension of the vector space of Bianchi modular forms of parallel weight $(k+2, k+2)$ for $\mathrm{SL}(2, \mathcal{O})$.

- **Example:** Specialize to the case when $\mathcal{O} = \mathbb{Z}\left[\left(1+\sqrt{-3}\right)/2\right]$ and $k = 3$.
  - If $\Delta \neq \overline{\beta}\beta$ for any $\beta \in \mathcal{O}$, then Karabulut shows

$$H_{3,\Delta}(z) \sim Z(-\Delta, 4).$$

[2] Karabulut C (2022) From binary Hermitian forms to parabolic cycles of Euclidean Bianchi groups. J Number Theory 236:71–115

Main Result

- Let $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, where $\mathbb{H}$ is Hamilton's quaternions.
- Let $\mathcal{O} \subset K$ be a discrete norm-Euclidean subring. **Example:** Hurwitz order

$$\Lambda(2) = \left\{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2}\right\}.$$

- For $k > 0$ odd and Δ a positive integer, define $H_{k,\Delta,\mathcal{O}} : K \to \mathbb{R}$ by

$$H_{k,\Delta,\mathcal{O}}(z) \overset{\text{def.}}{=} \sum_{\substack{a,c\in\mathbb{Z},a<0,b\in\mathcal{O} \\ \overline{b}b-ac=\Delta}} \max\left(0, a\overline{z}z + \overline{z}b + \overline{b}z + c\right)^k$$
$$- \frac{1}{2k+2} \sum_{b\in\mathcal{O},\overline{b}b=\Delta} \overline{\mathbb{B}}_{k+1}\left(zb + \overline{b}\overline{z}\right).$$

Theorem 3: Chinta - Y.

Fix $k > 1$ odd. The vector space

$$\operatorname{span}_{\mathbb{R}}\left\{H_{k,\Delta,\mathcal{O}} \mid \Delta > 0\right\}$$

is finite dimensional, with dimension $\approx$ to dimension of $H^1_{\text{par}}(\mathrm{SL}(2, \mathcal{O}), V_k)$.

Table of Dimensions

dim span $H_{k,\Delta,\mathcal{O}}$	$k=3$	$k=5$	$k=7$	$k=9$	$k=11$
$\mathcal{O}=\mathbb{Z}$	**1**	2	2	2	3
$\mathcal{O}=\mathbb{Z}[i]$	2	2	3	3	4
$\mathcal{O}=\mathbb{Z}\left[\sqrt{-2}\right]$	3	4	5	6	7
$\mathcal{O}=\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$	**1**	2	2	2	3
$\mathcal{O}=\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$	2	3	4	4	5
$\mathcal{O}=\mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right]$	3	4	5	6	7
$\mathcal{O}=\Lambda(2)$	**1**	2	2	2	3
$\mathcal{O}=\Lambda(3)$	2	2	3	3	3
$\mathcal{O}=\Lambda(5)$	2	4	4	5	6

$$H_{3,\Delta,\Lambda(2)}(z)=\frac{1}{10}\sigma_{\mathrm{odd}}(\Delta)+\sigma_3(\Delta)+24\sum_{N=1}^{\Delta-1}\sigma_3(N)\sigma_{\mathrm{odd}}(\Delta-N).$$

Reduction types of genus 2 curves

LuCaNT 2025

Edwina Aylward (UCL)

July 10th, 2025

Joint work with Vladimir Dokchitser and Elvira Lupoian.

Art in mathematics

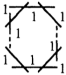
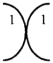
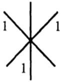
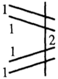
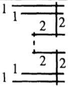
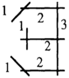
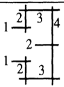
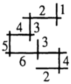
Kodaira symbol	I_0	I_n $(n \geq 1)$	II	III	IV	I_0^*	I_n^* $(n \geq 1)$	IV^*	III^*	II^*
Special fiber $\tilde{\mathcal{C}}$ (The numbers indicate multiplicities)										
$m =$ number of irred. components	1	n	1	2	3	5	$5+n$	7	8	9
$E(K)/E_0(K) \cong \tilde{\mathcal{E}}(k)/\tilde{\mathcal{E}}^0(k)$	(0)	$\frac{\mathbb{Z}}{n\mathbb{Z}}$	(0)	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ n even; $\frac{\mathbb{Z}}{4\mathbb{Z}}$ n odd	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	(0)
$\tilde{\mathcal{E}}^0(k)$	$\tilde{E}(k)$	k^*	k^+	k^+	k^+	k^+	k^+	k^+	k^+	k^+
Entries below this line only valid for $\mathrm{char}(k) = p$ as indicated										
$\mathrm{char}(k) = p$			$p \neq 2,3$	$p \neq 2$	$p \neq 3$	$p \neq 2$	$p \neq 2$	$p \neq 3$	$p \neq 2$	$p \neq 2,3$
$v(\mathcal{D}_{E/K})$ (discriminant)	0	n	2	3	4	6	$6+n$	8	9	10
$f(E/K)$ (conductor)	0	1	2	2	2	2	2	2	2	2
behavior of j	$v(j) \geq 0$	$v(j) = -n$	$\tilde{j} = 0$	$\tilde{j} = 1728$	$\tilde{j} = 0$	$v(j) \geq 0$	$v(j) = -n$	$\tilde{j} = 0$	$\tilde{j} = 1728$	$\tilde{j} = 0$

Table 4.1: A Table of Reduction Types

Genus 2

Hyperelliptic: $y^2 = f(x)$, $\deg f = 6$.

100+ families of reduction types (Namikawa-Ueno labels). Split into 7 semistable types:

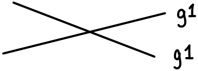
Good (18)

One node (12)

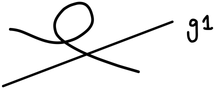
Two nodes (5)

Two $\mathbb{P}^1$s $\cap$ at 3 pts (6)

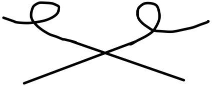
Good EC$\times$Good EC (42)



Mult EC$\times$Good EC (16)



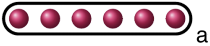
Mult EC$\times$Mult EC (5)



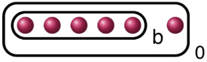
Potentially good reduction

Consider $C: y^2 = c \cdot \prod_{i=1}^{6}(x - r_i), \quad r_i \in \overline{\mathcal{K}},\ c \in \mathcal{K}^\times$ (res. char $\mathcal{K} > 5$).

If $C/\mathcal{K}$ has potentially good reduction, one can obtain an equation for C with one of the following associated cluster pictures:



$a \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$



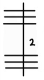
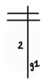
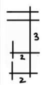
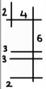
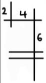
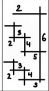
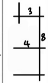
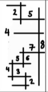
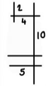
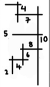
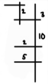
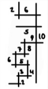
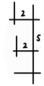
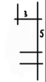
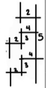
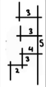
$b \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$

(Faraggi-Nowell)

The reduction type is determined by the pair:

$$(\mathrm{a},\ v(c) \bmod 2) \quad \text{or} \quad (\mathrm{b},\ v(c) \bmod 2)$$

For your consideration

Namikawa-Ueno label	I_0	I_0^*	II	III	IV	V	V^*	VI	VII	VII^*	VIII-1	VIII-2	VIII-3	VIII-4	IX-1	IX-2	IX-3	IX-4
Special fibre																		
Number of components	1	7	3	7	6	5	12	7	5	12	5	9	6	13	6	5	11	9
$\Phi(\bar{k})$	(0)	$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^4$	(0)	$\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2$	(0)	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	(0)	(0)	(0)	(0)	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$
$\mathrm{char}(\bar{k}) \neq 2, 3, 5$																		
 a	0	0	1/2, 1/2	1/3, 2/3	1/3, 2/3	1/6, 5/6	1/6, 5/6				4/5	2/5	3/5	1/5	3/5	1/5	4/5	2/5
 b	0	0						1/2, 1/2	1/4, 3/4	1/4, 3/4	1/5	3/5	2/5	4/5	2/5	4/5	1/5	3/5
f (conductor)	0	4	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
$v(\Delta_{\min})$	0	10	15	10	20	5	15	10	5	15	4	12	18	16	8	6	14	12

A Table of Potentially Good Reduction Types

■ : $v(c) \equiv 0 \mod 2$,
■ : $v(c) \equiv 1 \mod 2$.

Marcella Manivel: Lightning Talk

LuCaNT 2025
7/10/25

$$-\Delta \circlearrowleft L^2\left(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})\right)$$

$$\Delta = y^2\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)$$

$$C : End_{\mathbb{C}}(\mathfrak{g}) \to \mathfrak{g}\otimes\mathfrak{g}^* \to \mathfrak{g}\otimes\mathfrak{g} \to A\mathfrak{g} \to U\mathfrak{g}$$

$$D : End_{\mathbb{R}}(\mathfrak{p}) \to \mathfrak{p}\otimes\mathfrak{p}^* \to \mathfrak{p}\otimes\mathfrak{p} \to Cl(\mathfrak{p})\otimes U\mathfrak{g}$$

Hamiltonians and Energy of a System

eigenvalues $\Longleftrightarrow$ energy levels
eigenvectors $\Longleftrightarrow$ energy states
spectrum $\Longrightarrow$ spectral decomposition

$\mathbb{R}$

$$-\frac{d^2}{dx^2} + x^2$$

$\Gamma\backslash\mathfrak{H}$

$$-y^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) + (\mathbb{D}E_1^*)^2$$

$\Gamma\backslash G/K$

$$-\Delta + (\mathbb{D}E_1^*)^2$$

Sands' Hamiltonian

$$S = -y^2\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)+(\mathbb{D}E_1^*)^2$$

Dirac Operator:
$\mathbb{D}^2 = -\Delta$

Kronecker's Limit formula:
$$2\pi\left(\gamma - \log 2 - \log(\sqrt{y}\,|\,\eta(z)\,|^2)\right)$$

$\sigma(S) = \frac{6}{\pi}, ?$

eigenfunction: $e^{-E_1^*}$, ?

My Research

- Find the rest of $\sigma(S)$
- Show $-\Delta + \left(\mathbb{D}E_1^*\right)^2$ has discrete spectrum
- Growth condition: potential $\gg \Sigma_{\alpha\in\Phi^+} |\alpha(m)|^{\epsilon}$
- Characterization of automorphic Schwartz space and comparison to existing characterizations

Ask me about:

- How this relates to 0's of L-functions
- How to use this Hamiltonian to characterize automorphic Schwartz space
- Epstein Zeta Functions

Tell me about:

- Epstein Zeta Functions
- Automorphic Hermite Polynomials
- Physics (Type II string theory)
- Suggestions for numerically computing spectra of operators

Classify all possible images of $\rho_E\colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\widehat{\mathbb{Z}})$ that occur for E a non-CM elliptic curve over $\mathbb{Q}$.

Curves E whose adelic image lies in $G \Leftrightarrow$ rational points on the modular curve $X_G/\mathbb{Q}$ that classifies G-level structures (up to twisting issues).

Work of Zywina, Jones, Rakvi, etc. (maybe folklore) has shown it is better to consider the *agreeable closure*[1] $\mathcal{G}$ of the image $\mathrm{im}(\rho_E)$.

Zywina has produced a collection of 841 agreeable subgroups $\mathcal{G}$ of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ such that

> *Determination of $X_{\mathcal{G}}(\mathbb{Q})$ for such $\mathcal{G}$, plus a few ℓ-adic holdouts, plus the resolution of Serre's uniformity conjecture for normalizers of nonsplit Cartans at prime level, will complete*[2] *Mazur's Program B.*

Work of Mayle-Rouse handles all modular curves that contain a rank 0 elliptic curve as a simple isogeny factor. But if this is not the case, you will want something more automated than considering hundreds of curves on a case-by-case level.

This is where model-free Chabauty comes in.

---

[1] Then $\pm\mathrm{im}(\rho_E)$ will be precisely an infinite *family* of groups parametrized by $H^1(\mathrm{Gal}_{\mathbb{Q}}, \mathcal{G}^{ab}/\widehat{\mathbb{Z}}^\times)$. See also: Karabiyik's software demo.

[2] This does not cover Atkin-Lehner quotients: see Hashimoto-Keller-Le Fourn for recent progress on this.

Let $X/\mathbb{Q}$ be a curve, genus at least 2, and let J be its Jacobian.

Pairing $J(\mathbb{Q}) \times H_0(X_{\mathbb{Q}_p}, \Omega^1_{X/\mathbb{Q}_p}) \to \mathbb{Q}_p$ given by "Coleman integration": $(D, \omega) \mapsto \int_D \omega$. These integrals do exactly as expected, except integrating between residue disks requires following a "Frobenius-invariant path".

$J(\mathbb{Q})$ rank r, X genus $g \Rightarrow$ at least $g - r$ dim's worth of ω that kills $J(\mathbb{Q})$. Call these $\mathcal{A}$, the *annihilating differentials*.[3]

Suppose $b \in X(\mathbb{Q})$. For $P \in X(\overline{\mathbb{Q}})$ to be $\mathbb{Q}$-rational, necessary to have $\int_b^P \omega = 0$ for all $\omega \in \mathcal{A}$. So why not find that zero set?

In practice: work residue disk by residue disk. For each $\mathbb{F}_p$-disk $]U[$, choose basepoint[4] $b_U \in X(\mathbb{Q}_p)$ and uniformizer t_U, and then solve the system of $g - r$ equations $\int_b^{b_U} \omega + \int_{t_U(b_U)}^{t_U} \omega = 0$ as ω varies in $\mathcal{A}$.

By Newton polygon nonsense, this solution set is finite and contains $X(\mathbb{Q})$.

Now either use LLL to recognize rationals, or Mordell-Weil sieve[5] to shrink the search space for $X(\mathbb{Q})$ until you provably found everything.

---

[3] If $r > g$, then replace J with a "Selmer variety" $\mathrm{Sel}_n(X)$ for a large enough n. For $n > 2$ beyond the Tate motive case, see work of Corwin, Kantor, Corwin-Zehavi.

[4] Usually, these will be Teichmuller lifts, but for modular curves, easy to use CM lifts.

[5] Mordell-Weil sieving: the art of divining numerical coincidences on $\#J(\mathbb{F}_\ell)$ for varying ℓ.

Modular curves: weight 2 newforms $\Leftrightarrow$ simple isogeny factors of J.
For each factor, if its dim $= d$, then rank is one of $\{0, d, 2d, \dots\}$. So you only need to check analytic rank 0 to see[6] if in $\mathcal{A}$.

Computing cusp forms: Go between $\Gamma(N)$ and $\Gamma_0(N^2) \cap \Gamma_1(N)$ via $q \mapsto q^N$. Try to find a basis of weight 2 cusp forms[7] as *eta quotients* (see e.g. Chan-Combes).

Compute action of $\mathrm{SL}_2(N)$ on cusp forms. Hardest part is numerically recognizing the Fricke pseudoeigenvalues as algebraic numbers[8]. Matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right]$ acts as Galois action $\zeta_N \mapsto \zeta_N^d$. Thus: get action of $\mathrm{GL}_2(N)$. Take G-invariants.

Computing b_U and enumerating the $]U[$. Choose lifts of CM points, and then use group theory to determine which level structures yield $\mathbb{F}_p$-pts.

Large integrals $\Rightarrow$ tiny integrals: Use the formula
$\left[\int_P^Q \omega_i\right]_i = \left[\sum_{j=0}^p \int_{Q_j}^Q \omega_i - \int_{P_j}^P \omega_i\right](p+1-T_p)^{-1}$. These are tiny integrals by Eichler-Shimura and by agreeability (namely, G contains all scalar matrices).

For the formula above, compute the Hecke action on differentials using action of $\mathrm{SL}_2(N)$ and matrices $\left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} 1 & p-1 \\ 0 & p \end{smallmatrix}\right], \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]$.[9]

[6] Logachev c. 1990s: for modular ab. vars, analytic rank $r \Rightarrow$ algebraic rank r for $r \in \{0, d\}$.
[7] $f \mapsto f\, dq/q$ turns a weight 2 cusp form into a holomorphic differential.
[8] You can make this fast using complementary results of Zywina and Brunault-Neururer.
[9] Crucially, they are as considered in $\mathrm{GL}_2^+(\mathbb{R})$, as opposed to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right] \in \mathrm{GL}_2(N)$.

We are left with the what turns out to be the most challenging part of our computation. **Big issue:** Tate p-adic uniformization only takes you so far from the cusps. No choice but to resort to complex numbers.

Choose uniformizer t_U to be $(j - j(b_U))/C$ generically, $j^{1/3}/C$ if $j(b_U) = 0$, $(j - 1728)^{1/2}/C$ if $j(b_U) = 1728$, and if you're at a cusp then just slash f appropriately.

The mystery constant C is a number that comes from having to shrink your radius of convergence so as to avoid all of the ramification that comes with the map that the uniformizer defines. In particular it is composed of primes dividing $Nj(b_U)(j(b_U) - 1728)$.
You can compute C by playing around with elliptic curves.[10]

Compute expansions of $f(q_N)\,dq_N/q_N$ in terms of t_U via Cauchy integral (read: FFT) techniques[11]. The point is that if you started off with f being defined over $\mathbb{Z}$, then with the choice of C above, you will get *algebraically integral* coefficients.

Recognizing these coefficients: LLL is far too weak to handle the large degree field extensions that occur for the chosen b_U. Instead, compute expansions at conjugates of b_U and then perform another FFT(!)[12] to win.

[10] Preprint available at `https://chrisxudoesmath.com`!

[11] Chekhov's gun: eta quotients speed up the sampling thanks to sparsity of coefficients.

[12] Chekhov's gun: we are abelian over an imaginary quadratic field!

Results for 10.60.2.f.1 and unanswered questions

This is X_G for $G := \langle \left[\begin{smallmatrix} 4 & 3 \\ 7 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 6 & 9 \\ 3 & 4 \end{smallmatrix}\right] \rangle \leq \mathrm{GL}_2(10)$. I have written in `MAGMA` a truly marvellous computation, but for which the slide margins are too small to contain the outputs of. Have some handouts instead.

Everything I said generalizes to arbitrary modular curves as long as a rank 0 quotient exists.

Model-free quadratic Chabauty? Upcoming work of Rendell (student of Dogra) will produce an example of this. For the general case, you need: p-adic Gross-Zagier formula[13] for the global height, and a complete understanding of semistable models[14] for the heights at bad reduction places.

What can you do if you can't find an eta expression for a newform? First try to find one for the twist-minimal newform in its twist orbit space, and barring that, you can try to compute products of 4-tuples of weight 1/2 theta series.[15]

Ask me afterwards about the details. (How did I overcome the possible failure of coarse base change? What sort of precision analysis did I do?) I'm on the job market this fall! Thanks for listening!

---

[13] Hashimoto gets something for $X_0(N)$, and I suspect you can use a result of Disegni to get something for arbitrary G.

[14] Weinstein's paper does this, except that he does not cover the $p = 2$ case!

[15] A theorem of Serre-Stark says that these span $M_{1/2}(4N, \chi)$ for all N and (even) χ. It is my suspicion that these guys come very close to generating the canonical ring of $X_1(N)$, if not outright.