Quadratic Chabauty Experiments on Genus 2 Bielliptic Modular Curves in the LMFDB

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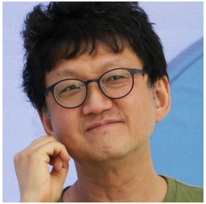
Rational Points on Curves

- Let X be a projective, smooth, and absolutely integral curve over $\mathbb{Q}$ of genus 2 or greater. What is $X(\mathbb{Q})$?
- Faltings, 1983: The set of rational points on a curve $X/\mathbb{Q}$ of genus 2 or more is always finite.
- Kim's nonabelian Chabauty: construct a sequence of subsets of p-adic points using Selmer varieties and their iterated p-adic integrals:
  $X(\mathbb{Q}_p) \supseteq X(\mathbb{Q}_p)_1 \supseteq X(\mathbb{Q}_p)_2 \supseteq \ldots \supseteq X(\mathbb{Q})$.



Kim's Conjectures

- For sufficiently large n, the set $X(\mathbb{Q}_p)_n$ is finite.
- For sufficiently large n, we have $X(\mathbb{Q}_p)_n = X(\mathbb{Q})$.

Past Work

- Balakrishnan and Dogra established the finiteness of $X(\mathbb{Q}_p)_2$ subject to $r < g - 1 + rank(NS(J))$, where J is the Jacobian of X.
- They define a finite set $X(\mathbb{Q}_p)_Z \subset X(\mathbb{Q}_p)$, dependent on a choice of correspondence $Z \subset X \times X$, that contains $X(\mathbb{Q}_p)_2$.

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- We will explicitly describe and compute a finite subset of $\mathbb{Q}_p$, Ω, and a function $\widetilde{\rho}$ with $\widetilde{\rho} : X(\mathbb{Q}_p) \to \mathbb{Q}_p$ such that the set of rational points is contained in this finite set:

$$A = \{z \in X(\mathbb{Q}_p) : \widetilde{\rho}(z) \in \Omega\}.$$

- Bianchi and Padurariu used this method to compute the set of rational points on bielliptic curves of genus and rank 2 in the LMFDB.

Our Goal

- LMFDB (Beta) now has a large database of modular curves, including many of genus 2 and ranks ≤ 2 that are also amenable to these methods.
- Expand the previous methods to perform quadratic Chabauty over number fields K for modular curves X such that J has rank 1 over $\mathbb{Q}$ but rank 2 over K.

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- LMFDB (Beta) now has a large database of modular curves, including many of genus 2 and ranks ≤ 2 that are also amenable to these methods.
- Expand the previous methods to perform quadratic Chabauty over number fields K for modular curves X such that J has rank 1 over $\mathbb{Q}$ but rank 2 over K.
- Compute the quadratic Chabauty loci for these genus 2 modular curves of rank 1 and 2 and study what they look like.
- Recall: $A := \{z \in X(\mathbb{Q}_p) : \widetilde{\rho}(z) \in \Omega\}$.

Question

Are there algebraic points among the mock rational points, $A \setminus X(\mathbb{Q})$?

The Bielliptic Setup

Let X be a non-singular genus 2 bielliptic curve

$$X : y^2 = a_6x^6 + a_4x^4 + a_2x^2 + a_0, a_i \in \mathbb{Z},$$

and consider the two elliptic curves

$$E_1 : y^2 = x^3 + a_4x^2 + a_2a_6x + a_0a_6^2,$$

$$E_2 : y^2 = x^3 + a_2x^2 + a_4a_0x + a_6a_0^2.$$

We have degree 2 maps $\phi_i : X \to E_i$:

$$\phi_1(x, y) = (a_6x^2, a_6y) \qquad \phi_2(x, y) = (a_0x^{-2}, a_0yx^{-3})$$

Any genus 2 curve with a degree 2 map to an elliptic curve has a model of this form.

Notation

- Let log denote the p-adic logarithm extended to $\mathbb{Q}_p^\times$ by setting $\log(p) = 0$.
- For $P \in E_i(\mathbb{Q}_p)$, we let $Log(P) = \int_\infty^P \frac{dx}{2y}$.
- Given a prime q, we let ord_q denote the q-adic valuation on $\mathbb{Q}_q$, normalized to be surjective onto the integers and let $|\cdot|_q$ denote the standard absolute value on $\mathbb{Q}_q$.

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Proposition (Bianchi–Padurariu, 2022)

Let $q \neq p$ and let E be a minimal model of an elliptic curve. If $P \in E(\mathbb{Q}_q)$ reduces to a nonsingular point modulo q, then the local height $\lambda_q(P) = \log(max\{1, |x(P)|_q\})$.

Let W_q^E be the set of values attained by λ_q on points in $(x, y) \in E(\mathbb{Q}_q)$ with $x, y \in \mathbb{Z}_q$. Then W_q^E is finite, explicitly computable, and equal to $\{0\}$ for all but finitely many q.

In particular, $W_q^E \subseteq \{0\}$ at all primes of good reduction for the given model of E.

Defining $\widetilde{\rho}$ and Ω

Theorem (Bianchi–Padurariu, 2022)

Suppose that each of E_1 and E_2 are minimal of rank 1 over $\mathbb{Q}$. Let $P_i \in E_i(\mathbb{Q})$ be a point of infinite order. Define $\alpha_i = \frac{h_p(P_i)}{Log^2(P_i)}$. Then α_i is independent of choice of P_i and the function $\rho : X(\mathbb{Q}_p) \setminus \{P : x(P) \in \{0, \infty\}\} \to \mathbb{Q}_p$ given by

$$\rho(z) = \lambda_p(\phi_1(z)) - \lambda_p(\phi_2(z)) - 2\log(x(z)) - \alpha_1 Log^2(\phi_1(z)) + \alpha_2 Log^2(\phi_2(z))$$

can be continued to a locally analytic function $\widetilde{\rho} : X(\mathbb{Q}_p) \to \mathbb{Q}_p$.

Also, for a prime $q \neq p$, let

$$\Omega_q = (-W_q^{E_1} + W_q^{E_2} + \{-n \log q : -ord_q(a_6) \leq n \leq ord_q(a_0), n \equiv 0 \bmod 2\}) \\ \cup(\log|a_0|_q - W_q^{E_1}) \cup (-\log|a_6|_q + W_q^{E_2})$$

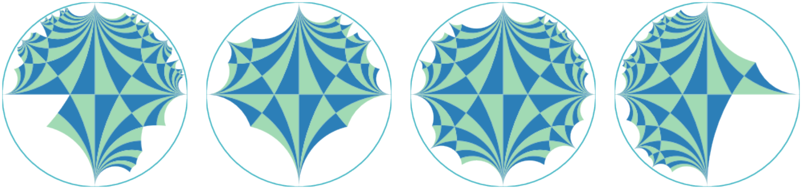
and set $\Omega = \{\sum_{q\ bad} w_q : w_q \in \Omega_q\}$. Then Ω is finite and contains $\widetilde{\rho}(X(\mathbb{Q}))$.

The data

The LMFDB (Beta) contains modular curves X_H with $\det(H) = \widehat{\mathbb{Z}}^\times$ such that at least one of the following hold:

1. H has level ≤ 70 or prime-power level ≤ 335.
2. H contains $-I$ and has genus ≤ 24 and level ≤ 335.
3. H does not contain $-I$ and has genus ≤ 8 and level ≤ 335.

Total: 209 rank 2 genus 2 curves and 1237 rank 1 genus 2 curves of non-simple Jacobian with an available model to work with.



On Choice of Model

A given curve does not have a unique model $y^2 = a_6x^6 + a_4x^4 + a_2x^2 + a_0$. A model with a larger discriminant will produce a larger set of p-adic points during quadratic Chabauty.

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We can represent transformations that change the discriminant by a factor of $e^{20}(ad - bc)^{-30}$ with matrices of the form

$$M(a, b, c, d, e) := \begin{pmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{pmatrix}.$$

Proposition (F., 2025)

Let $X : y^2 = a_6x^6 + a_4x^4 + a_2x^2 + a_0$ with a_i all integers. For $M(a, b, c, d, e)$ to yield an isomorphic curve of the same form with integer coefficients, there are seven explicit conditions involving $a, b, c, d, e, a_0, a_2, a_4,$ and a_6 that are necessary and sufficient.

Analysis - Rank 2

- Consider the 30 unique bielliptic curves up to isomorphism with a known rational point.
- For each rank 2 bielliptic curve X with $J(X) \sim E_1 \times E_2$ and at least one known point in $X(\mathbb{Q})$, perform quadratic Chabauty using every eligible prime p of good, ordinary reduction for both E_i such that $3 < p < 100$.
- Recover the list of mock rational points for each prime.
- For each mock rational point, check to see if either of the coordinates satisfy low-degree relations.
- Perform the Mordell-Weil sieve following the methodology of Bianchi and Padurariu.

Analysis - Rank 1

- Consider the 121 unique bielliptic curves up to isomorphism with a known rational point.
- For each rank 1 bielliptic curve X with $J(X) \sim E_1 \times E_2$ and a known $P \in X(\mathbb{Q})$, obtain the list of eligible primes as in the rank 2 case, with $3 < p < 100$.
- For each prime, find the number fields $K = \mathbb{Q}(\sqrt{-D})$ such that: the rank of each E_i over K is 1 and p splits in K, and $1 < D < 20$ or $D \in \{43, 67, 163\}$.
- Recover the list of mock rational points using quadratic Chabauty over K for pairs (p, D) five times or until no further pairs exist.
- As before, for each mock rational point, check to see if either of the coordinates satisfy low-degree relations.
- Perform the Mordell-Weil sieve using built-in MAGMA functionality.

Generalized Theorem

We need to account for height contributions of primes that divide the discriminant of the number field K.

More precisely, we can modify the previous theorem which defined $\widetilde{\rho}$ and Ω.

Theorem (F., 2025)

Suppose that each E_i has rank 1 over an imaginary quadratic $K/\mathbb{Q}$. Let $P_i \in E_i(\mathbb{Q})$ be of infinite order. Define $\alpha_i = \frac{h_p(P_i)}{[K:\mathbb{Q}]Log^2(P_i)}$. Then α_i is independent of choice of P_i and:

1. *The function $\rho : X(\mathbb{Q}_p) \setminus \{P : x(P) \in \{0, \infty\}\} \to \mathbb{Q}_p$, definition unchanged, can be continued to a locally analytic function $\widetilde{\rho} : X(\mathbb{Q}_p) \to \mathbb{Q}_p$.*
2. *For a prime $q \neq p$, let Ω_q be as before and set $\Omega = \{\sum_{q \in S} w_q : w_q \in \Omega_q\}$, where S consists of the bad primes for each E_i, the prime factors of a_0 and a_6 for X, and the prime divisors of the discriminant of K. Then Ω is finite and contains $\widetilde{\rho}(X(\mathbb{Q}))$.*

Proposition (F., 2025)

Among the genus 2 Jacobian rank 2 modular curves satisfying conditions for inclusion in the LMFDB, we provide a complete list of curves with points over the listed number fields through quadratic Chabauty for some prime $3 < p < 100$ of good, ordinary reduction, along with the smallest prime for which they appear, up to precision $O(p^{25})$.

Table 4: Rank 2 Non-Weierstrass Points

Label	Model	Prime	Number Field	Points
14.48.2.g.1	$y^2 = -x^6 + 83x^4 - 19x^2 + 1$	37	$\mathbb{Q}(\sqrt{-3}, \sqrt{21})$	$(\pm\frac{\sqrt{-3}}{3}, \pm\frac{8\sqrt{21}}{9})$
15.60.2.d.1	$y^2 = -45x^6 + 75x^4 - 15x^2 + 1$	19	$\mathbb{Q}(\sqrt{5})$	$(\frac{\sqrt{5}}{5}, \pm\frac{4}{5})$
16.48.2.a.1	$y^2 = x^6 - 5x^4 - 5x^2 + 1$	11	$\mathbb{Q}(\sqrt{-2})$	$(\pm 1, \pm 2\sqrt{2})$
18.72.2.f.1	$y^2 = 9x^6 - 99x^4 + 27x^2 - 1$	7	$\mathbb{Q}(\sqrt{-3})$	$(\pm\frac{\sqrt{-3}}{3}, \pm\frac{8\sqrt{-3}}{3})$
28.48.2.j.1	$y^2 = x^6 - 83x^4 + 19x^2 - 1$	19	$\mathbb{Q}(\sqrt{-3}, \sqrt{7})$	$(\pm\frac{\sqrt{-3}}{3}, \pm\frac{8\sqrt{-21}}{9})$
36.72.2.d.1	$y^2 = -9x^6 + 99x^4 - 27x^2 + 1$	13	$\mathbb{Q}(\sqrt{-3}, \sqrt{3})$	$(\pm\frac{\sqrt{-3}}{3}, \pm\frac{8\sqrt{3}}{3})$
48.48.2.dk.1	$y^2 = 12x^6 + 54x^4 + 48x^2 + 12$	73	$\mathbb{Q}(\sqrt{3})$	$(0, \pm 2\sqrt{3})$
48.48.2.dm.1	$y^2 = -24x^6 + 54x^4 - 24x^2 + 3$	73	$\mathbb{Q}(\sqrt{3})$	$(0, \pm\sqrt{3})$
48.48.2.dw.1	$y^2 = 27x^6 - 72x^4 + 54x^2 - 8$	73	$\mathbb{Q}(\sqrt{6})$	$(\pm\frac{\sqrt{6}}{9}, \pm 2)$

Statement of Rank 1 Results

Proposition (F., 2025)

Among the genus 2 Jacobian rank 1 modular curves satisfying satisfying conditions for inclusion in the LMFDB, the listed curves in the same paper have points over the listed number fields through quadratic Chabauty for some prime $3 < p < 100$ of good, ordinary reduction up to precision $O(p^{25})$.

Table 5: Rank 1 Non-Weierstrass Points

Label	Model	(D, p)	Number Field	Points
11.66.2.a.1	$y^2 = 11x^6 + 11x^4 - 7x^2 + 1$	[7,23]	$\mathbb{Q}(\sqrt{-7})$	$(\pm\frac{\sqrt{-7}}{3}, \frac{76}{27})$
16.24.2.f.1	$y^2 = 4x^6 + 6x^4 + 4x^2 + 1$	[6,5]	$\mathbb{Q}(i)$	$(\pm i, \pm i)$
16.24.2.f.1	$y^2 = 4x^6 + 6x^4 + 4x^2 + 1$	[7,37]	$\mathbb{Q}(\sqrt{-7})$	$(\pm\frac{2\sqrt{-7}}{7}, \pm\frac{5\sqrt{-7}}{49})$
16.48.2.bm.1	$y^2 = 8x^6 + 16x^4 + 9x^2 + 1$	[5,41]	$\mathbb{Q}(\sqrt{-5})$	$(\pm\frac{3\sqrt{-5}}{7}, \pm\frac{62}{343})$
16.48.2.bx.1	$y^2 = -x^6 + 7x^4 - 7x^2 + 1$	[1,5]	$\mathbb{Q}(i)$	$(\pm i, \pm 4)$
16.48.2.bx.1	$y^2 = -x^6 + 7x^4 - 7x^2 + 1$	[17,7]	$\mathbb{Q}(\sqrt{-17})$	$(\pm\sqrt{-17}, \pm 84)$
16.48.2.c.1	$y^2 = -x^6 - 5x^4 + 5x^2 + 1$	[7,11]	$\mathbb{Q}(\sqrt{-7})$	$(\pm\sqrt{-7}, \pm 8)$
16.48.2.c.1	$y^2 = -x^6 - 5x^4 + 5x^2 + 1$	[1,17]	$\mathbb{Q}(i, \sqrt{2})$	$(\pm i, \pm 2\sqrt{-2})$
16.48.2.c.2	$y^2 = -2x^6 - 10x^4 + 10x^2 + 2$	[19,11]	$\mathbb{Q}(\sqrt{-19})$	$(\pm\frac{9\sqrt{-19}}{19}, \pm\frac{680\sqrt{-19}}{361})$

Conjectures

Conjecture

Let X be a bielliptic modular curve of genus 2 and level N. If $Jac(X)$ is of rank 2 over $\mathbb{Q}$ and X has a mock rational point over $\mathbb{Q}(\sqrt{D})$ then D must be a divisor of N.

Conjecture

Let X be a bielliptic modular curve of genus 2 and level N. If $Jac(X)$ is of rank 1 and X has a mock rational point over $\mathbb{Q}(\sqrt{D})$ arise through quadratic Chabauty when computing over $\mathbb{Q}(\sqrt{D'})$, then at least one of the following must hold:

1. D divides N.
2. D divides D'.
3. $D = 10$.

A Rank 2 Example

We highlight points found on the rank 2 modular curve $X_{ns}^{+}(15)$.

The model we use is $y^2 = -45x^6 + 75x^4 - 15x^2 + 1$.

prime	# rational points	# mock rational p-adic points
19	14	284
31	14	404
61	14	780

Among these mock rational points, we find four quadratic points: $\{(\pm\frac{\sqrt{5}}{5}, \pm\frac{4}{5})\}$.

A Rank 1 Example

We now highlight points over different number fields found on the rank 1 modular curve with LMFDB label 16.24.2.f.1.

The model we use is $y^2 = 4x^6 + 6x^4 + 4x^2 + 1$.

prime	number field	# mock rational p-adic points	Algebraic points
5	$\mathbb{Q}(\sqrt{-6})$	66	$(\pm i, \pm i)$
37	$\mathbb{Q}(\sqrt{-7})$	566	$(\pm i, \pm i), (\pm \frac{2\sqrt{-7}}{7}, \pm \frac{5\sqrt{-7}}{49})$

Each analysis also recovered the two known rational points for 16.24.2.f.1.

Future work

- All points found were over quadratic or biquadratic fields. Does this generalize to modular curves of different genus or ranks of the Jacobian, or to other hyperelliptic curves in general?
- It is also the case that upon observing one algebraic point, we always find the other points in its Galois orbit. Once again, how does this generalize?
- Could some of the algebraic points found be linearly equivalent to those in the Mordell-Weil group of the Jacobian over the base field? If they have the same heights away from p, could this explain their appearance in the analysis?
- If we find algebraic points over number fields for a given curve, how does the Mordell-Weil group change when computed over these fields or their compositum versus the rationals?

Thank you!