Hilbert modular forms from orthogonal modular forms on binary lattices

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joint work with John Voight (University of Sydney)

July 8, LuCaNT 2025

Orthogonal modular forms

Let $Q\colon V \to F$ be a nondegenerate quadratic form over a totally real number field F with $R = \mathbb{Z}_F$ and Λ be a R-lattice in V.

$$\begin{aligned}
\mathrm{O}(V) &\coloneqq \{f \in \mathrm{GL}(V)\colon Q(f(x)) = Q(x)\ \forall x \in V\} \\
\mathrm{GSO}(V) &\coloneqq \{(f,u) \in \mathrm{SL}(V) \times F^{\times}\colon Q(f(x)) = uQ(x)\ \forall x \in V\} \\
\mathrm{GSO}(\Lambda) &\coloneqq \text{ the subgroup of } \mathrm{GSO}(V) \text{ that preserves } \Lambda \\
\mathrm{Gen}_{\mathrm{GSO}}(\Lambda) &\coloneqq \{\Lambda' \subset V\colon \Lambda_{\mathfrak{p}} \simeq_{\mathrm{GSO}} \Lambda'_{\mathfrak{p}} \text{ for all prime } \mathfrak{p} \text{ of } R\} \\
\mathrm{Cls}_{\mathrm{GSO}}(\Lambda) &\coloneqq \mathrm{Gen}_{\mathrm{GSO}}(\Lambda)/{\simeq_{\mathrm{GSO}}}
\end{aligned}$$

The space of **orthogonal modular forms** for Λ (with trivial weight) is

$$M(\Lambda) \coloneqq \mathrm{Map}(\mathrm{Cls}_{\mathrm{GSO}}(\Lambda), \mathbb{C}).$$

Theorem

Suppose that $\dim_F V = 2$. Then $\mathrm{Cls}_{\mathrm{GSO}}(\Lambda) \simeq \mathrm{Cl}(S)$ where $S = \mathrm{Clf}^0(Q_\Lambda)$ is the even Clifford algebra of Λ.

Hecke operators

For $\mathfrak{p} \nmid \operatorname{sgndisc}(\Lambda)$ with $\mathfrak{p} = \mathfrak{P}\mathfrak{P}'$, define the **Hecke operator**

$$\begin{aligned} T_{\mathfrak{P}} \colon M(\mathrm{Cl}(S)) &\to M(\mathrm{Cl}(S)) \\ f &\mapsto \Big([\mathfrak{A}] \mapsto f([\mathfrak{A}\mathfrak{P}])\Big). \end{aligned}$$

The eigenforms are multiplicative characters on $\mathrm{Cl}(S) \Rightarrow$ algebraic Grössencharacters Ψ.

Given a nonzero ideal class $[\mathfrak{B}] \in \mathrm{Cl}(S)$, define the **theta series** with respect to $[\mathfrak{B}]$ to be

$$\theta_{\mathfrak{B}}(z) := \sum_{\nu \in (\mathfrak{B}^{-1})_{>0}} e^{2\pi i \operatorname{Tr}(\mathrm{Nm}_{K/F}(\nu) z)}.$$

Hilbert modular forms

Theorem (Eichler, Walling, Voight–Wu)

Suppose that $\# \mathrm{Cl}^{+}(F) = 1$ and Λ has squarefree discriminant $\mathfrak{N}$. Then

$$M(\Lambda) \hookrightarrow M_1(\Gamma_0(\mathfrak{N}), \chi)$$

where $\chi(\gamma) = \left(\frac{\mathrm{sgndisc}(Q)}{d}\right)$ for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N})$.

Example

Let $F = \mathbb{Q}(\sqrt{5})$ and $Q \colon F^2 \to F$ be defined by $Q(x, y) = x^2 + xy + 6y^2$. Let $\Lambda = \mathbb{Z}^2$. Then $\# \operatorname{Cls}_{\mathrm{O}}(\Lambda) = 2$. We find eigenvectors $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $f = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \in M(\Lambda)$. We have $T_2(f) = -f$.

Let $K = F(\sqrt{-23})$ and Ψ be a Hecke character on $\mathbb{Z}_K$. Now $2 = \mathfrak{P}_1\mathfrak{P}_2$ in K. We compute that $\Psi(\mathfrak{P}_1) + \Psi(\mathfrak{P}_2) = -1$.

Computing genus 2 curves over $\mathbb{Q}$ whose Jacobians have good reduction away from 2

LMFDB, Computation, and Number Theory (LuCaNT) 2025, ICERM

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8 July 2025

Genus 2 curves

Problem (Poonen 1996)

List all genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2.

Smart (1997) computed all 366 genus 2 curves with good reduction outside 2. But there are more! Some examples of other curves $C/\mathbb{Q}$ where $\mathrm{Jac}(C)$ good outside 2:

- $C : y^2 = x^5 - 14x^3 + 81x$ has bad reduction at $\{2, 3\}$.
- $C : y^2 = 2x^5 - 9x^4 - 24x^3 + 22x^2 + 78x - 41$ has bad reduction at $\{2, 5\}$.
- $C : y^2 = 2x^5 + x^4 - 16x^3 - 72x^2 + 240x + 136$ has bad reduction at $\{2, 7\}$.
- $C : y^2 = x^5 + 478x^3 + 57122x$ has bad reduction at $\{2, 13\}$.
- $C : y^2 = x^5 + 28x^4 - 868x^3 - 6160x^2 + 43076x - 149072$ has bad reduction at $\{2, 3, 11\}$.

So far, we've found 512 examples of genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2.

Genus 2 curves good outside S

Easier Problem

Fix a small set of primes S. List all genus 2 curves $C/\mathbb{Q}$ with good reduction outside S and whose Jacobians have good reduction away from 2.

Smart's S-unit strategy:

- Let $C/\mathbb{Q} : y^2 = c(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)(x-\alpha_6)$ be such a curve, where $\alpha_i \in \mathbb{Q}(J[2])$.
- Use Siegel's identity: $\dfrac{(\alpha_i-\alpha_j)(\alpha_k-\alpha_\ell)}{(\alpha_i-\alpha_k)(\alpha_j-\alpha_\ell)} + \dfrac{(\alpha_i-\alpha_\ell)(\alpha_j-\alpha_k)}{(\alpha_i-\alpha_k)(\alpha_j-\alpha_\ell)} = 1$
- Compute all possible 2-torsion fields $\mathbb{Q}(J[2])$.
- Solve the T-unit equations $x+y=1$ for $x, y \in O_T^\times$ over $\mathbb{Q}(J[2])$, where T is the primes in $\mathbb{Q}(J[2])$ lying above S.

Solving a linear system

Let $\psi_1, \psi_2, \ldots, \psi_t$ be a set of T-unit generators over $\mathbb{Q}(J[2])$. Let $a_{k,i,j} \in \mathbb{Z}$ be given by

$$\alpha_i - \alpha_j = \psi_1^{a_{1,i,j}} \psi_2^{a_{2,i,j}} \cdots \psi_t^{a_{t,i,j}}.$$

We can compute the integers $a_{k,i,j}$ as solutions to a big linear system of equations!

Examples of linear constraints on $a_{k,i,j}$:

- Galois constraints: For all $\sigma \in \mathrm{Gal}(\mathbb{Q}(J[2])/\mathbb{Q})$, $a_{f_\sigma(k), g_\sigma(i), g_\sigma(j)} = a_{k,i,j}$.
- Cluster pictures (using that J has good reduction at all odd primes).
- Solving simple T-unit equations (i.e. $\tau + \sigma(\tau) = 1$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(J[2])/\mathbb{Q})$).

How to solve the linear system:

- Brute force.
- Closest vector problem (CVP).
- Integer linear programming (ILP).

Summary - bit.ly/genus2

Theorem (V. 2025)

There are at least 512 $\mathbb{Q}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2. This list includes all such curves $C/\mathbb{Q}$ which have good reduction away from S and where $\mathbb{Q}(J[2]) \subseteq M$, where (S, M) can be any of the following 18 pairs:

$$\big(\{2,3\},\mathbb{Q}(\zeta_{16})\big),\ \big(\{2,3\},\mathbb{Q}(\zeta_8,\sqrt[4]{2})\big),\ \big(\{2,3\},\mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),\ \big(\{2,5\},\mathbb{Q}(\zeta_{16})\big),$$
$$\big(\{2,5\},\mathbb{Q}(\zeta_8,\sqrt[4]{2})\big),\ \big(\{2,5\},\mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),\big(\{2,7\},\mathbb{Q}(\zeta_{16})\big),\ \big(\{2,7\},\mathbb{Q}(\zeta_8,\sqrt[4]{2})\big),$$
$$\big(\{2,7\},\mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),\ \big(\{2,3,5\},\mathbb{Q}(\zeta_{16})\big),\ \big(\{2,3,7\},\mathbb{Q}(\zeta_{16})\big),\ \big(\{2,3,7\},\mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),$$
$$\big(\{2,5,7\},\mathbb{Q}(\zeta_{16})\big),\ \big(\{2,5,7\},\mathbb{Q}(\zeta_8,\sqrt[4]{2})\big),\ \big(\{2,3,5,7\},\mathbb{Q}(\zeta_4)\big),\ \big(\{2,3,5,7\},\mathbb{Q}(\sqrt{-2})\big),$$
$$\big(\{2,3,5,7\},\mathbb{Q}(\sqrt{2})\big),\ \big(\{2,3,5,7,11,13\},\mathbb{Q}\big).$$

- If you can find any more curves, you'll win ~~£100~~ \$300. (T&Cs apply)

Gross lattices of supersingular elliptic curves

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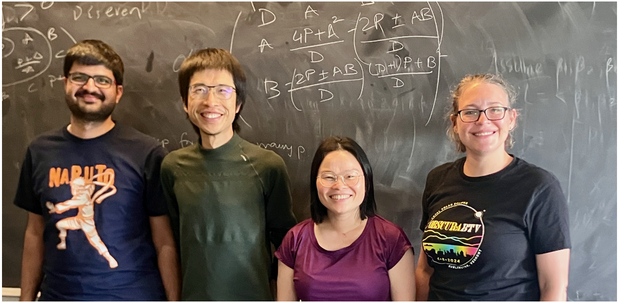
joint work with Chenfeng He, Gaurish Korpal, and Christelle Vincent



Gross lattices

Let $B_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ that ramifies exactly at p & ∞.

- The **Gross lattice** of a maximal order $\mathcal{O}$ in $B_{p,\infty}$ is the rank 3 lattice

$$\mathcal{O}^T = \{2x - \mathrm{Tr}(x) : x \in \mathcal{O}\}.$$

- In case E is supersingular, $\mathrm{End}(E) \cong \mathcal{O}$ for some maximal order $\mathcal{O}$. We also say $\mathcal{O}^T$ **is the Gross lattice of** E.
- Define the **inner product** in $B_{p,\infty}$ as $(x, y) = \frac{1}{2}\mathrm{Tr}(x\bar{y})$.
- $\mathcal{O}^T$ always admits a basis that attains its successive minima, i.e., there exist $\beta_i \in \mathcal{O}^T$ such that $\|\beta_i\|^2 = \mathrm{N}(\beta_i) = D_i$, here D_i is the smallest value D_i such that the rank $\langle x \in \Lambda : \mathrm{N}(x) \leq D_i \rangle_{\mathbb{Z}} \geq i$ for $i \in \{1, 2, 3\}$

Why Gross lattices?

Theorem (Chevyrev-Galbraith (2014) and Goren-Love (2024))

Let $\mathcal{O}$ and $\mathcal{O}'$ be two maximal orders of $B_{p,\infty}$.
Suppose that $\mathcal{O}^T$ and $\mathcal{O}'^T$ have the same successive minima.
Then $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic.

Lemma (Chevyrev-Galbraith 2014)

Let $\mathcal{O} \cong \mathrm{End}(E)$ be a maximal order in $B_{p,\infty}$ and D_1 and D_2 the first two successive minima of $\mathcal{O}^T$.
If $j(E) \in \mathbb{F}_p$, then $D_1 D_2 \leq \frac{16p}{3}$.

Our main results

Theorem (HKTV 2025)

Let E be a supersingular elliptic curve defined over $\overline{\mathbb{F}}_p$.
Then $j(E) \in F_p$ if and only if one of the following conditions holds.

1. *Its Gross lattice $\mathcal{O}^T$ has a rank 2 sublattice of determinant $4p$.*
2. *A rank 2 sublattice of $\mathcal{O}^T$ with a basis consisting of two elements that attain the first two successive minima of $\mathcal{O}^T$ is of determinant $4p$.*
3. *The third successive minimum of its Gross lattice $\mathcal{O}^T$ is bounded by $p \leq D_3 \leq \frac{8p}{7} + \frac{7}{4}$.*

Our main results (cont.)

Theorem (HKTV 2025)

Let E be a supersingular elliptic curve defined over $\overline{\mathbb{F}}_p$.
Then $j(E) \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$ if and only if one of the following conditions holds.

1. *Any rank 2 sublattice of its Gross lattice $\mathcal{O}^T$ has determinant $4np$ where $n \geq 2$ is an integer.*
2. *The third successive minimum of its Gross lattice $\mathcal{O}^T$ is $D_3 \leq \frac{3}{5}p + 5$.*

Homomorphisms from Rational Points on Elliptic Curves to Ideal Class Groups

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LuCaNT 2025
Tuesday, July 8th, 2025

Past Work

In the past, people have found homomorphisms

Rational Points on EC's $\to$ Ideal Class Groups of Quadratic Fields

Buell, 1977: $$\left\{ \begin{array}{c} \text{Rational Points on} \\ y^2 = 4(x+a)(x^2+bx+c)+\mathcal{D} \end{array} \right\} \to \mathrm{Cl}(\mathbb{Q}(\sqrt{\mathcal{D}}))$$

- Noticed when finding quadratic class groups with large 3-rank
- Later found (2016) to correspond to the *ideal class pairing*.

Soleng, 1992: $$\left\{ \begin{array}{c} \text{Subgroup of the Rational Points on} \\ y^2 = x^3 + a_2x^2 + a_4x + a_6 \end{array} \right\} \to \mathrm{Cl}(\mathbb{Z}[\sqrt{a_6}])$$

- Used to proved that ideals in class groups of quadratic fields can have arbitrarily large order
- Find infinitely many quadratic class groups with certain subgroups.

Quadratic Orders

Using E given by $y^2 = \overbrace{x^3 + a_2x^2 + a_4x + a_6}^{f(x)}$:

Integral Points (x_0, y_0) on E

$$\xrightarrow{x=n} (n - x_0, y - y_0) \text{ in } \frac{\mathbb{Z}[y]}{(y^2 - f(n))}$$
$$\longrightarrow (n - x_0, \sqrt{f(n)} - y_0) \text{ in } \mathbb{Z}[\sqrt{f(n)}]$$

and this corresponds to the ideal classes that Buell and Soleng found.

This is an invertible ideal in $\mathbb{Z}[\sqrt{f(n)}]$ if and only if

$$\gcd\left(n - x_0, 2y_0, n^2 + nx_0 + x_0^2 + a_2(n + x_0) + a_4\right) = 1$$

Fix $n = 0$ and extend to rational points to obtain a homomorphism

$$E(\mathbb{Q})_{\text{quadratic primitive}} \to \text{Cl}(\mathbb{Z}[\sqrt{a_6}])$$
$$\left(\frac{a}{c^2}, \frac{b}{c^3}\right) \mapsto [(a, -kb + \sqrt{a_6})], \text{ with } kc^3 = 1 \mod a.$$

Cubic Orders

Using E given by $x^3 = \overbrace{y^2 + a_3 y - a_6}^{g(y)}$:
Integral Points (x_0, y_0) on E

$$\xrightarrow{y=m} (x - x_0, m - y_0) \text{ in } \frac{\mathbb{Z}[x]}{(x^3 - g(m)))}$$

$$\longrightarrow (\sqrt[3]{g(m)} - x_0, m - y_0) \text{ in } \mathbb{Z}[\sqrt[3]{g(m)}]$$

This is an invertible ideal in $\mathbb{Z}[\sqrt[3]{g(m)}]$ if and only if

$$\gcd\left(m - y_0, 3x_0^2, m + y_0 + a_3\right) = 1$$

Fix $m = 0$ and extend to rational points to obtain

$$E(\mathbb{Q})_{\text{cubic primitive}} \to \mathrm{Cl}(\mathbb{Z}[\sqrt[3]{a_6}])$$

$$\left(\frac{a}{c^2}, \frac{b}{c^3}\right) \mapsto [(b, -ka + \sqrt[3]{a_6})], \text{ with } kc^2 = 1 \mod b.$$

Thank You!

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Wild Conductor Exponents of Plane Quartics

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LuCaNT: 08/07/25

Joint work with Elvira Lupoian (University College London)

Conductors of Abelian Varieties

Definition

The *conductor* of an abelian variety, $A/\mathbb{Q}$, is $\prod_p p^{n_p}$, where $n_p = n_{p,tame} + n_{p,wild}$.
For any $\ell \neq p$, the tame part is given by $n_{p,tame} = V_\ell(A)^{I_p}$.
Let $G = \mathrm{Gal}(\mathbb{Q}_p(A[\ell])/\mathbb{Q}_p)$, then $n_{p,wild} = \sum_{i=1}^{\infty} \frac{\mathrm{codim}\, A[\ell]^{G_i}}{[G_0 : G_i]}$

Wild conductor exponents at odd primes	$\leftrightsquigarrow$	2-torsion
Wild conductor exponent at 2	$\leftrightsquigarrow$	3-torsion

Notation

The conductor of a curve is the conductor of its Jacobian.

Odd Conductor Exponents

For hyperelliptic curves, $y^2 = f(x)$, 2-torsion is generated by differences of roots of f. This gives an explicit formula for the conductor, see Dokchitser-Dokchitser-Maistret-Morgan 2018.

Theorem

The two-torsion of a plane quartic is generated by differences of bitangents

Algorithm for computing wild exponents at $p \neq 2$:

1. Determine equations for bitangents from a choice of model [Gröbner basis]
2. Find field of definition of all bitangents over $\mathbb{Q}_p$ and its Galois group
3. Compute a basis for 2-torsion from the bitangents [linear algebra]
4. Calculate Galois representation on 2-torsion
5. Evaluate conductor formula

Even Conductor Exponents

Is there an analogue for bitangents for 3-torsion? No! Even hard for hyperelliptic curves (for $g = 2$, see Dokchitser-Doris 2017, and $g = 3$, see Lupoian 2022).

Theorem (Lupoian-Rawson 2024/5)

Let C be a plane quartic with $P \in C(\mathbb{Q})$, then 3-torsion corresponds to plane cubics through P, meeting the curve with multiplicity (a multiple of) 3 at every intersection point.

Algorithm for computing wild exponents at 2:

1. Compute scheme of such cubics
2. Calculate complex approximations to the roots [homotopy continuation]
3. Recover the defining polynomials [continued fractions or LLL]
4. Find the field of definition its Galois group over $\mathbb{Q}_2$
5. Determine the Galois action and associated representation

Numerical Results

For the first 100 curves in Drew Sutherland's database of genus 3 curves, we computed the wild conductor exponents at odd primes. There were 7 curves with non-zero wild exponent, all at 3.

Discriminant	Conductor
$15957 = 3^4 \times 197$	$3^{2+2} \times 197^{1+0}$
$15957 = 3^4 \times 197$	$3^{2+2} \times 197^{1+0}$
$17307 = 3^3 \times 641$	$3^{2+1} \times 641^{1+0}$
$20331 = 3^4 \times 251$	$3^{2+2} \times 251^{1+0}$
$22707 = 3^3 \times 29^2$	$3^{2+1} \times 292 + 0?$
$25029 = 3^5 \times 103$	$3^{2+3} \times 103^{1+0}$
$27702 = 2 \times 3^6 \times 19$	$2^{1+0} \times 3^{4+2} \times 19^{1+0}$

We also computed two examples at 2. These had discriminant $7376 = 2^4 \times 461$ and $12368 = 2^4 \times 773$, and both had wild conductor 2 at 2.

Wild conductor exponents of curves

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8 July 2025

Wild conductor exponents

We can provably (and practically) compute $n_{C,p,\text{wild}}$ in the following cases:

- C an elliptic curve.
- C hyperelliptic, $p > 2$ or low genus.
- C superelliptic, p prime to exponent.
- C non-hyperelliptic of genus $3, 4$ or 5, $p > 2$.
- C plane quartic with a rational point.

Theorem (M^{2}D^2)

Let $C/\mathbb{Q} : y^2 = f(x)$ be a hyperelliptic curve, $p > 2$:

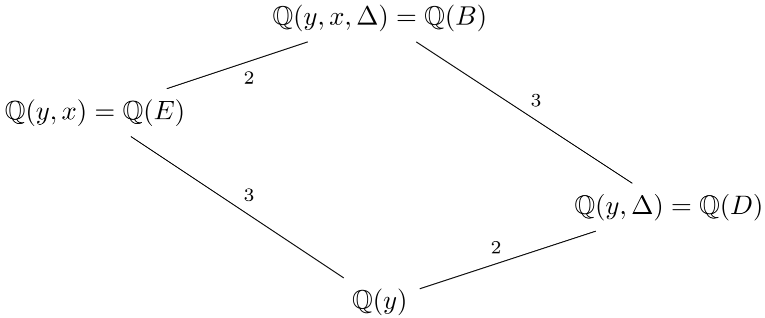
$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

where R is the set of roots of f over $\bar{\mathbb{Q}}_p$.

Idea: generalise this formula to degree d covers of $\mathbb{P}^1$, $p > d$.

Toy example

Consider an elliptic curve $E/\mathbb{Q} : y^2 = x^3 + ax + b$ with $a \neq 0$. We consider the degree 3 cover to $\mathbb{P}^1$ and fill in the Galois diagram:



where $\Delta^2 = \text{disc}_x(x^3 + ax + (b - y^2))$. It turns out $E[3] \cong \text{Jac}(D)[3]$ and E and D have the same wild conductor exponents at $p \neq 3$.

Replacing $E \to \mathbb{P}^1$ by a degree 3 simply branched cover $C \to \mathbb{P}^1$, $\text{Jac}(D)[3] \cong \text{Jac}(C)[3] \oplus$ stuff as wild inertia reps for $p \neq 3$.

Main theorem

Theorem

Let $C \to \mathbb{P}^1$ be a degree d simply branched cover. For $p > d$,

$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

where R is the set of $\bar{\mathbb{Q}}_p$-branch points.

Idea of proof

Unramified cyclic covers + Galois theory + representation theory

Perturbations

For $g \in \mathbb{Q}_p[t]$, write

$$w_p(g) = \sum_{r \in R/G_{\mathbb{Q}_p}} m(r) \cdot (v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p}),$$

where R is the set of $\bar{\mathbb{Q}}_p$-roots of g and r is a root with multiplicity $m(r)$.

Lemma

For p^{th}-power free g, the quantity $w_p(g)$ is locally constant.

Likewise, wild conductor exponents are locally constant. If we can perturb a degree d cover $\pi : C \to \mathbb{P}^1$ to obtain $\tilde{C} \to \mathbb{P}^1$ simply branched, then we can read off wild conductor exponents from the branch locus of π.

Example

Suppose $C : f(x, y) = 0$ is a smooth affine model for a curve and $\deg_x f = d$. Then, for $p > d$, we have $n_{C,p,\text{wild}} = w_p(\text{disc}_x f)$.

Learning Fricke signs from Maass Form Coefficients

Joanna Bieri, Giorgi Butbaia, Edgar Costa, **Aly Deines**, Kyu-Hwan Lee, David Lowry-Duda, Tom Oliver, Yidi Qi, and **Tamara Veenstra** [1]

LuCaNT 2025 Lightning Talk

[1] Special thanks to the Mathematics and Machine Learning Program at Harvard University's Center of Mathematical Sciences and Applications. Paper to appear in the Proceedings from this program in *Advances in Theoretical and Mathematical Physics*.

What's up with Maass forms?

- $\approx$ 35K Maass forms recently added to LMFDB. (see David Lowry-Duda's talk).
- but with Fricke'n sign problems
- For about half of the Maass forms in the dataset the Fricke sign is unknown.
- Its complicated to compute. In general, need trace formula, Hejhal's algorithm, and modularity properties.

Can we predict it with machine learning techniques instead?

Experiments on training and validation data

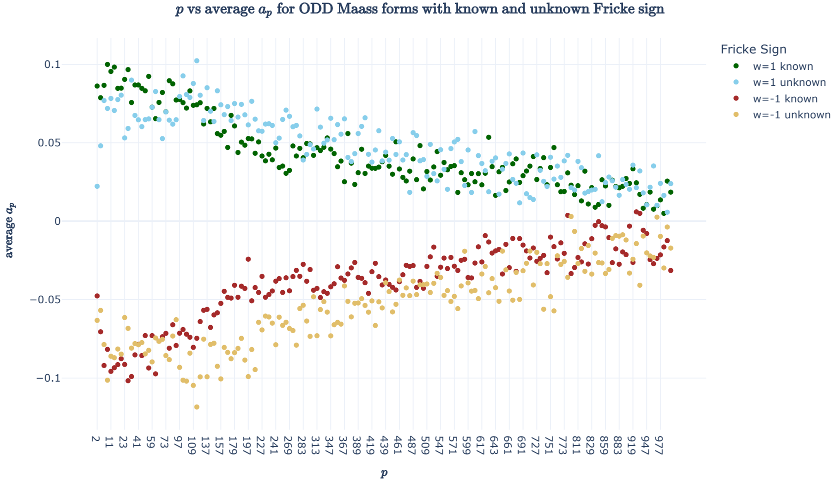
- Performed LDA on all the training data (12,795 observations) normalized by symmetry. Tested accuracy on validation data (3199 observations).
- Compared to heuristic Hejhal's algorithm for computing Fricke sign (try all possible combinations and see if consistent with other information: requires precision in Fourier coefficients.)

LDA method	all symmetry (normalized)	even symmetry	odd symmetry
LDA using all a_n	0.9612	0.9488	0.9633
LDA using only a_p	0.8625	0.8261	0.8800
LDA vs Heuristic Hejhal	0.9545	0.9461	0.9609

Notes:

- Surprisingly, a_n is better than a_p.
- Interestingly, unsupervised clustering algorithms such as K-means and PCA were not successful at distinguishing the Fricke sign.

Examined average value of $(-1)^{\sigma} a_p$ for known and unknown Fricke signs separated by known and predicted Fricke sign. First for ODD Maass forms.



Many more experiments

- How many a_n do we need?
- Is LDA using information about the Fricke sign embedded in the Fourier coefficients (when the prime divides the level)?