Families of p-adic fields

Jordi Guàrdia-Rubies, John W. Jones, Kevin Keating,
Sebastian Pauli, David P. Roberts, David Roe

1. Pre-tour focused on an example
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1. Pre-tour focused on an example

The improved p-adic section of the LMFDB tabulates all degree n extensions of $\mathbb{Q}_p$ for $n \leq 23$ and $p < 200$. The number of such extensions up to isomorphism is

n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$p=2$	1	7	2	59	2	47	2	1823	3	158	2	5493	$\cdots$
$p=3$	1	3	10	5	2	75	2	8	795	6	2	785	$\cdots$
$p=5$	1	3	2	7	26	7	2	11	3	258	2	17	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

For the example, we take one of the 795 nonic 3-adic fields $L = \mathbb{Q}_3[x]/f(x)$, the one defined by

$$f(x) = x^9 + 18x^4 + 3x^3 + 18x^2 + 18x + 3.$$

$f(x) = x^9 + 18x^4 + 3x^3 + 18x^2 + 18x + 3$ is 3-Eisenstein, hence irreducible over $\mathbb{Q}_3$, so $L = \mathbb{Q}_3[x]/f(x)$ really is a field. An analysis of L from first principles would include the following points:

A. The *degree* $n = ef$ is 9, with *residue field degree* $f = 1$ and *ramified index* $e = 9$.

B. $\mathrm{disc}(f(x)) = -3^{18} \cdot 19 \cdot 43 \cdot 12979 \cdot 18313$ and so the *discriminant exponent* is $c = 18$.

C. There is a unique element $y \in L$ with $y^3 + 3y + 3 = 0$. The cubic field $L_1 = \mathbb{Q}_3(y)$ is the unique intermediate field. Its discriminant exponent is 3.

D. The theory of *Artin slopes* decomposes c as

$$18 = 0 + \frac{3}{2} + \frac{3}{2} + \frac{5}{2} + \frac{5}{2} + \frac{5}{2} + \frac{5}{2} + \frac{5}{2} + \frac{5}{2}.$$

We summarize this by saying the **family invariant** of L in Artin-language is $I = [\frac{3}{2}, \frac{5}{2}]^f_\epsilon$ with $\epsilon = f = 1$. There are 35 other fields L' in the same family $I/\mathbb{Q}_3$.

E. The extension $L/\mathbb{Q}_3$ is far from Galois because $|\mathrm{Aut}(L/\mathbb{Q}_3)| = 1$.

F. The splitting field L^{gal} is determined by factoring various resolvents. Its Galois group is $C_3^3.D_6 = 9T24$, which has order $324 = 3^4 2^2$. The family invariant of L^{gal} is $[\frac{3}{2}, \frac{3}{2}, \frac{5}{3}, \frac{5}{2}]_2^2$.

Three ways to index families. If $\mathbb{Q}_3 \subset L_1 \subset L$ were tame then the discriminants of L_1 and L up to units would be 3^2 and 3^8. So the contribution of wild ramification to the two discriminant exponents is $(3, 18) - (2, 8) = (1, 10)$. We call $\langle m_1, m_2 \rangle = \langle \frac{1}{3}, \frac{10}{9} \rangle$ the *means*. Similarly $[s_1, s_2] = [\frac{3}{2}, \frac{5}{2}] - [1, 1] = [\frac{1}{2}, \frac{3}{2}]$ are the *(Swan) slopes*. There are also *rams* $(r_1, r_2) = (\frac{1}{2}, \frac{7}{2})$.

In general, we can write the family invariant of an extension in terms of slopes (=upper numbering), means, or rams (= lower numbering). In our example of $L/\mathbb{Q}_3$ this becomes

$$I = \left[\frac{1}{2}, \frac{3}{2}\right] = \left\langle \frac{1}{3}, \frac{10}{9} \right\rangle = \left(\frac{1}{2}, \frac{7}{2}\right).$$

There is also a fourth way, via Herbrand functions.

2. Tour of improved p-adics in the LMFDB

Two tour-stops:

1. The example $L/\mathbb{Q}_3$ is one of 36 fields in its family $I/\mathbb{Q}_3$, all coming from a common *generic polynomial*,

$$x^9 + 9b_{13}x^4 + 3a_3x^3 + 9b_{11}x^2 + 9a_{10}x + 3.$$

The fields are obtained by specializing $a_j \in \{1, 2\}$ and $b_j \in \{0, 1, 2\}$.

2. Generic polynomials in general can be visualized by *Eisenstein diagrams*. Slopes, means, and rams all have their place on these diagrams.

Commented main references

The original database from which the LMFDB database grew:

J. W. Jones and D. P. Roberts, *A database of local fields*, J. Symbolic Comput. 41(1) (2006) 80–97.

The most important reference for the concept of families:

M. Monge, *A family of Eisenstein polynomials generating totally ramified extensions, identification of extensions and construction of class fields*. Int. J. Number Theory 10 (2014), no. 7, 1699–1727.

Our LuCaNT paper:

GJKPRR, *Families of local fields*, Arxiv. Database at `https://www.lmfdb.org/padicField/`

Further references are available in the paper.

A sequel going into more detail on a number of points:

GJKPRR, *Distinguished defining polynomials for extensions of p-adic fields*, in preparation.