3D Solitary Pattern Stability in a Multiscale Nonlinear Schrödinger Equation

Christopher Jones

Emmanuel Fleurantin

Jeremy Marzuola

Dmitro Golovanich

George Mason University and University of North Carolina at Chapel Hill

ELSEVIER

Physica D 124 (1998) 58–103

PHYSICA D

Bifurcation from the essential spectrum

Edge bifurcation

Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations

Todd Kapitula [a,1], Björn Sandstede [b,*]

[a] *Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA*

[b] *Weierstraß-Institute for Applied Analysis and Stochastics, Mohrenstraße 39, 10117 Berlin, Germany*

Received 2 July 1997; received in revised form 9 December 1997; accepted 16 March 1998

Abstract

The propagation of pulses in ideal nonlinear optical fibers without loss is governed by the nonlinear Schrödinger equation (NLS). When considering realistic fibers one must examine perturbed NLS equations, with the particular perturbation depending on the physical situation that is being modeled. A common example is the complex Ginzburg–Landau equation (CGL), which is a dissipative perturbation. It is known that some of the stable bright solitons of the NLS survive a dissipative perturbation such as the CGL. Given that a wave persists, it is then important to determine its stability with respect to the perturbed NLS. A major difficulty in analyzing the stability of solitary waves upon adding dissipative terms is that eigenvalues may bifurcate out of the essential spectrum. Since the essential spectrum of the NLS is located on the imaginary axis, such eigenvalues may lead to an unstable wave. In fact, eigenvalues can pop out of the essential spectrum even if the unperturbed problem has no eigenvalue embedded in the essential spectrum. Here we present a technique which can be used to track these bifurcating eigenvalues. As a consequence, we are able to locate the spectrum for bright solitary-wave solutions to various perturbed nonlinear Schrödinger equations, and determine precise conditions on parameters for which the waves are stable. In addition, we show that a particular perturbation, the parametrically forced NLS equation, supports stable multi-bump solitary waves. The technique for tracking eigenvalues which bifurcate from the essential spectrum is very general and should therefore be applicable to a larger class of problems than those presented here.

A major difficulty in analyzing the stability of solitary waves upon adding dissipative terms is that eigenvalues may bifurcate out of the essential spectrum. Since the essential spectrum of the NLS is located on the imaginary axis, such eigenvalues may lead to an unstable wave. In fact, eigenvalues can pop out of the essential spectrum even if the unperturbed problem has no eigenvalue embedded in the essential spectrum.

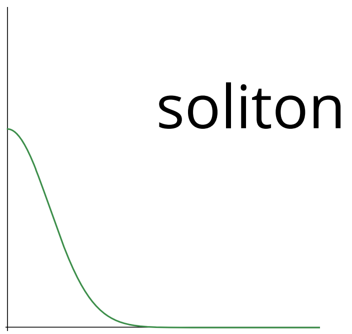
Nonlinear Schrödinger Equation

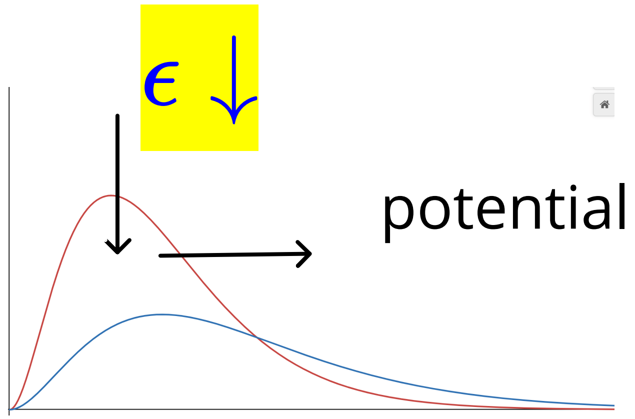
(Gross-Pitaevskii)

$$i\frac{\partial \varphi}{\partial t} = \triangle \varphi + |\varphi|^2 \varphi + V_\epsilon(|x|)\varphi$$

$$x \in \mathbb{R}^3$$

$$\triangle = \sum_{i=1}^{i=3} \frac{\partial^2}{\partial x_i^2} \qquad V_\epsilon(|x|) = \epsilon^2 V\left(\epsilon\,|x|\right)$$





Potential is NOT small

Schrödinger equation

$$p_{rr} + \frac{2}{r} p_r + \varepsilon^2 V(\varepsilon r) p = \lambda p$$

$$\rho = \varepsilon r \qquad \lambda = \varepsilon^2 \mu$$

$$\varepsilon^2 p_{\rho\rho} + \varepsilon^2 \frac{2}{\rho} p_\rho + \varepsilon^2 V(\rho) p = \varepsilon^2 \mu p$$

$$p_{\rho\rho} + \frac{2}{\rho} p_\rho + V(\rho) p = \mu p$$

Standing Wave

$$\varphi(x,t) = e^{-it}u(x)$$

$$u = \triangle u + |u|^2 u + V_\varepsilon(|x|)\,u$$

$$u_{rr} + \frac{2}{r}u_r + \left(|u|^2 - 1\right)u + V_\varepsilon(r)\,u = 0$$

where $r = |x|$

Golovanich and Marzuola:

$0 < \varepsilon \ll 1$ $\quad\quad u_\varepsilon(r) \to u_o(r)$

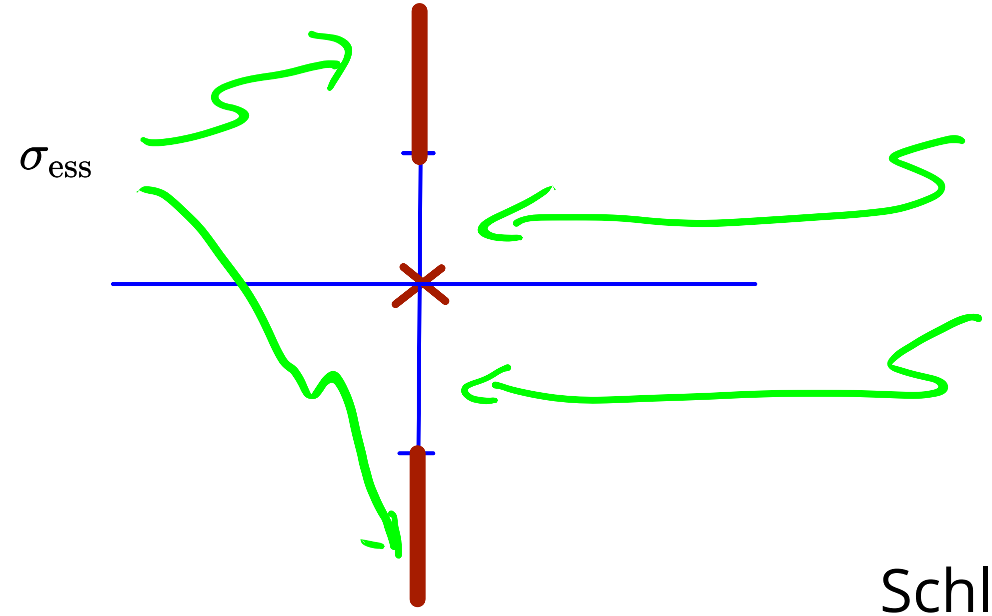
Linearization at wave and spectrum

Rotating frame --> Re and Im parts --> Linearize at $u_\varepsilon(r)$

$$JL = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$$

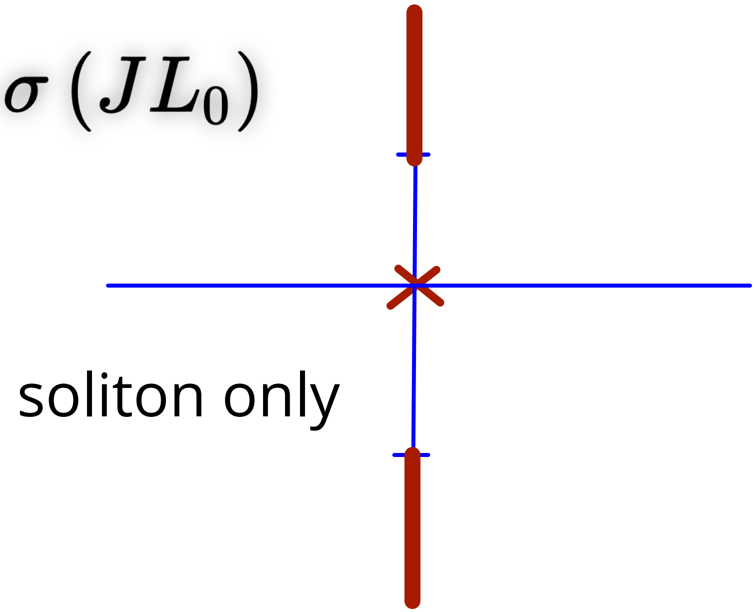
$$L_- = \Delta_r + u_\varepsilon^2 - 1 + V_\varepsilon$$

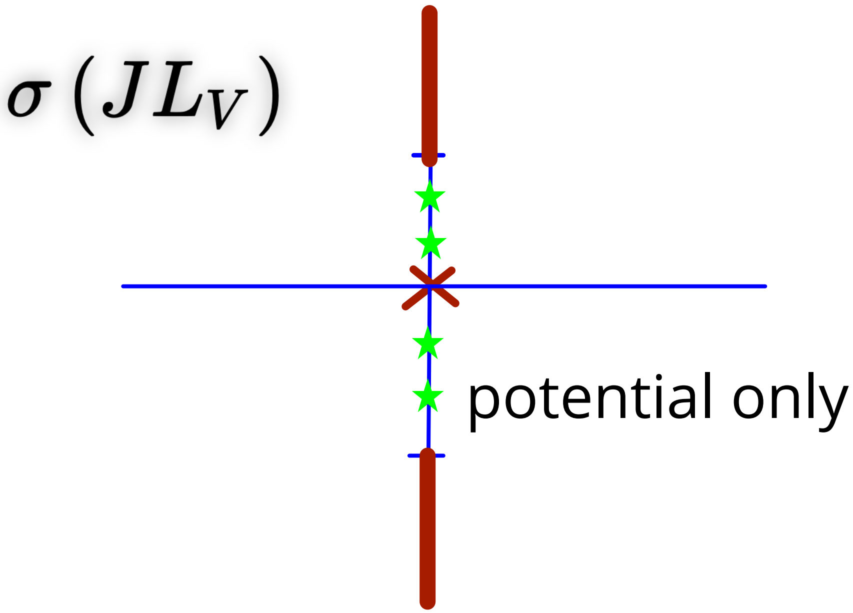
$$L_+ = \Delta_r + 3u_\varepsilon^2 - 1 + V_\varepsilon$$



GAP EIGENVALUES??

Schlag, Simpson-Marzuola $\varepsilon = 0$





- No gap eigenvalues
- No end-point resonance

- # gap eigenvalues = # positive eigenvalues of Schrödinger eq. = $m(V)$
- Gap eigenvalues are $O\left(\varepsilon^2\right)$ from end-points of the essential spectrum

Theorem (Fleurantin, Marzuola, J.) There are at least (exactly?) $m(V)$ gap eigenvalues in $\sigma\left(JL_0\right)$

Contrast with Kapitula and Sandstede (1998)

Interesting Aspects

1. Resetting into DS framework: compactification and desingluarization
2. Maslov Index: invariance of space of Lagrangian planes
3. Separation of soliton and potential: GSP
4. Estimates for gluing: blow-up at end-point of essential spectrum and bootsrapping estimates

see difference with Kapitula and Sandstede's work on bifurcation from the essential spectrum

$$\begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix} \qquad \lambda = i\omega$$

WLOG can take p real and q imaginary (note abuse of notation)

$$L_- q = \omega p$$
$$L_+ p = \omega q.$$

$$P = p + q \qquad Q = p - q$$

$$\begin{aligned} LP + 2u^2 P + u^2 Q &= \omega P \\ LQ + u^2 P + 2u^2 Q &= -\omega Q \end{aligned}$$

$$L = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - 1 + V_\varepsilon(r)$$

Setup

1. Write as a system and append equation for r
2. Campactify by introducing $\sigma = \frac{r}{r+1}$
3. Desingularize by changing independent variable to

$$s = r + \ln r$$

$$\begin{aligned}
\dot{P} &= \sigma R \\
\dot{Q} &= \sigma S \\
\dot{R} &= 2(\sigma - 1)R + \sigma \left[V_\varepsilon \left(\tfrac{\sigma}{1-\sigma} \right) P - 2u^2 P - u^2 Q + (\omega + 1) P \right] \\
\dot{S} &= 2(\sigma - 1)S + \sigma \left[V_\varepsilon \left(\tfrac{\sigma}{1-\sigma} \right) Q - u^2 P - 2u^2 Q + (1 - \omega) Q \right] \\
\dot{\sigma} &= \sigma(1 - \sigma)^2
\end{aligned}$$

$$(P, Q, R, S, \sigma) \in \mathbb{R}^4 \times [0, 1]$$

$$\{\sigma = 0\} \longleftrightarrow \{r \to 0\} \qquad \{\sigma = 1\} \longleftrightarrow \{r \to +\infty\}$$

$$
\begin{aligned}
\dot{P} &= \sigma R \\
\dot{Q} &= \sigma S \\
\dot{R} &= 2(\sigma-1)R + \sigma\left[V_\varepsilon\left(\tfrac{\sigma}{1-\sigma}\right)P - 2u^2P - u^2Q + (\omega+1)P\right] \\
\dot{S} &= 2(\sigma-1)S + \sigma\left[V_\varepsilon\left(\tfrac{\sigma}{1-\sigma}\right)Q - u^2P - 2u^2Q + (1-\omega)Q\right] \\
\dot{\sigma} &= \sigma(1-\sigma)^2
\end{aligned}
$$

BC for eigenvalue: ω is an eigenvalue if

$(R,S) \to 0$ as s $\to -\infty$ $(\sigma \to 0)$ regularity as $r \to 0$

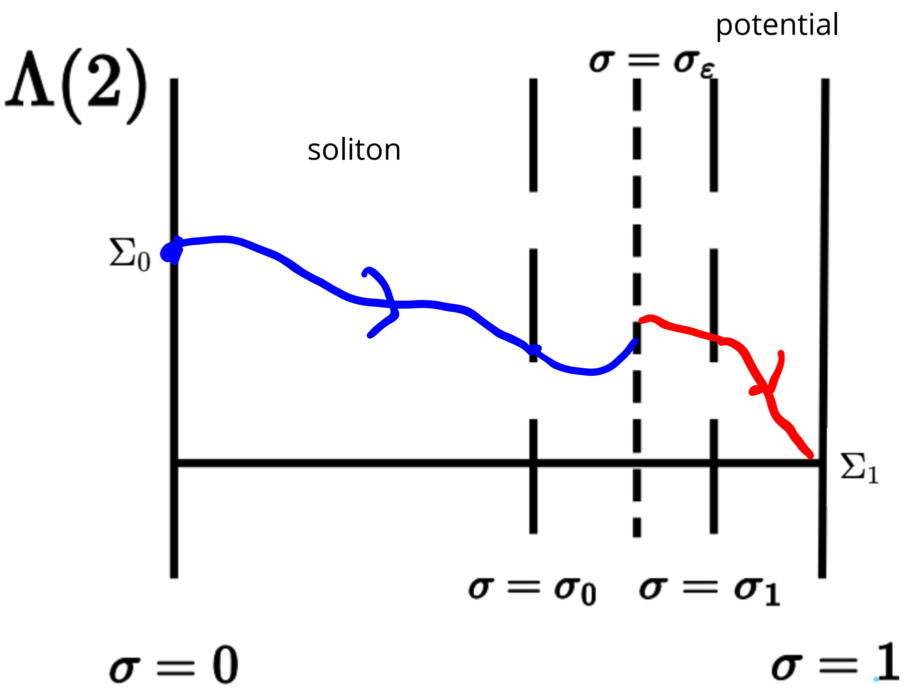
$(P,Q) \to 0$ as s $\to +\infty$ $(\sigma \to 1)$ decay as $r \to +\infty$

BC conditions correspond to 2D subspaces

Natural to consider flow induced on $G_{2,4} \times [0,1]$

The space of Lagrangian planes is an invariant submanifold

$$\Lambda(2) \times [0,1]$$



σ_0 finite but bdd away from 1

$$\sigma_1 = 1 - \varepsilon$$

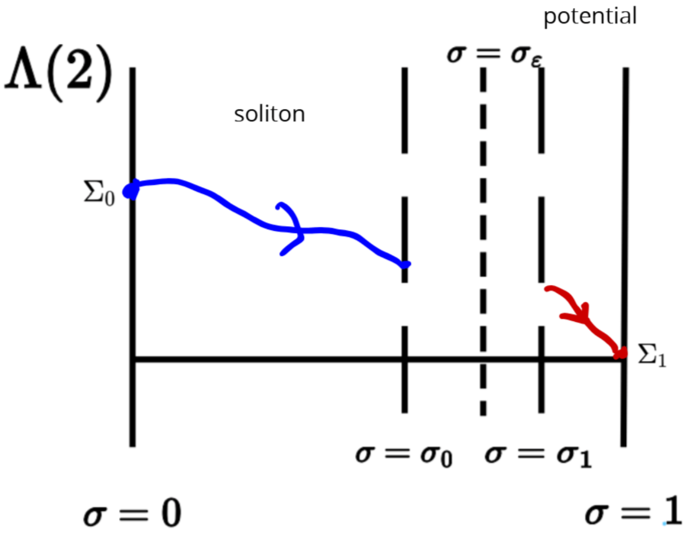
$$\sigma_\varepsilon = 1 - \varepsilon^\alpha$$
$$0 < \alpha < 1$$

$i\omega$ is an eigenvalue if there is a heteroclinic

$$\Sigma_0 = \{R = S = 0\} \in \{\sigma = 0\}$$

$$\Downarrow$$

$$\Sigma_1 = \{P = Q = 0\} \in \{\sigma = 1\}$$

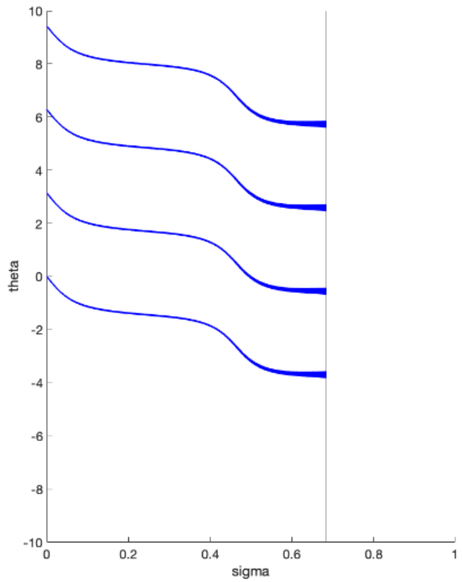


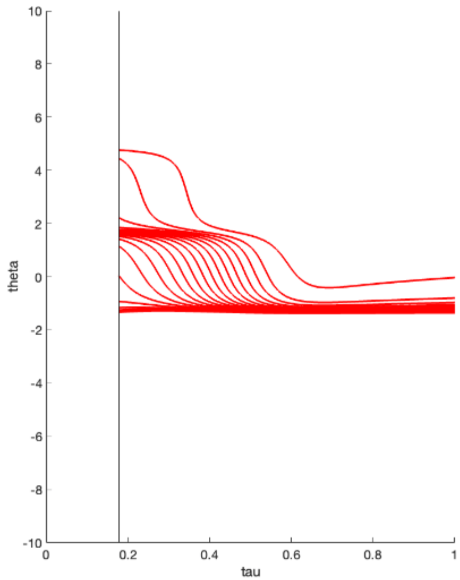
Rescale for potential part:

$$\rho = \varepsilon r$$

$$\tau = \frac{\rho}{\rho + 1}$$

covering space of $\Lambda(2)$ is $S^2 \times (0, 1)$





Coordinates on $\Lambda(2)$

2-plane determined by 2 linearly independent 4-vectors

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

$$\text{Plücker} : \mathrm{p}_{12} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \text{ etc.}$$

global but awkward as needs projectivizing and use of quadratic G-condition

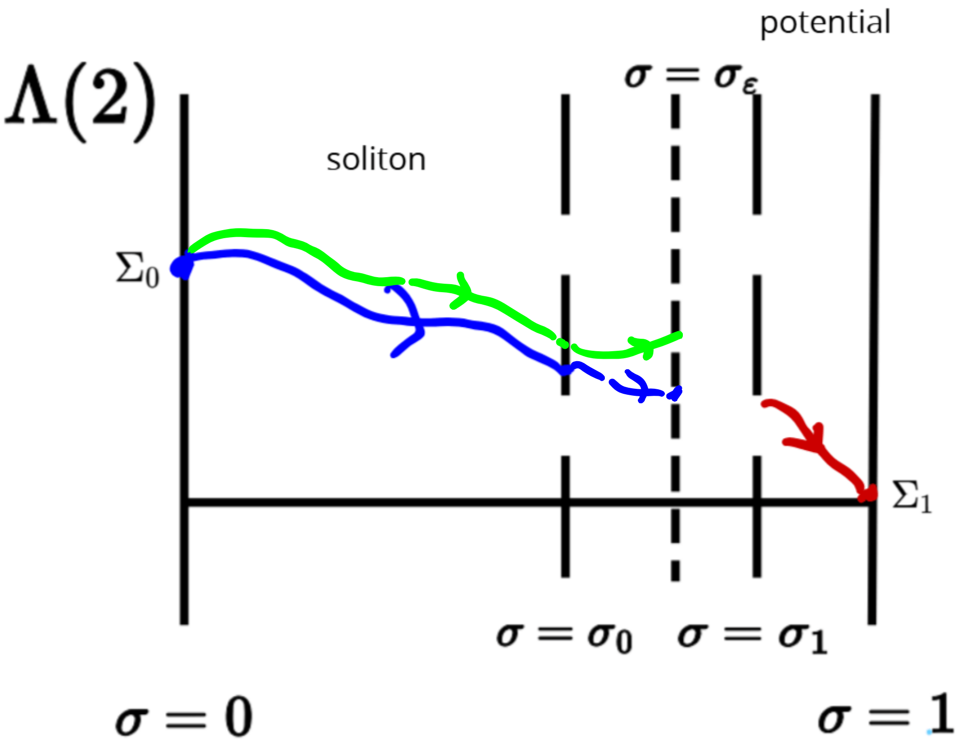
$$\text{Lagrangian} : \mathrm{p}_{13} + \mathrm{p}_{24} = 0$$

$$\text{Prüfer} : \begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \end{pmatrix} = \mathrm{A} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \text{ etc.}$$

easy to compute with but local

subspace is the graph of A

$$\text{Lagrangian} : \mathrm{A} \text{ symmetric}$$



Blue curve is governed by soliton only and so has nice limit as

$$s \to +\infty$$

$$\dot{A} = 2\zeta A + (\zeta+1)\left(\Psi_\varepsilon\left(-\frac{\zeta+1}{\zeta}\right) - A^2\right)$$

$$\zeta = \sigma - 1$$

$$\dot{\zeta} = \zeta^2(\zeta+1)$$

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

$$\dot{A} = 2\zeta A + (\zeta+1)\left(\Psi_\varepsilon\left(-\frac{\zeta+1}{\zeta},\omega\right) - A^2\right)$$

$$\dot{\zeta} = \zeta^2(\zeta+1)$$

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

set $\varepsilon = 0$

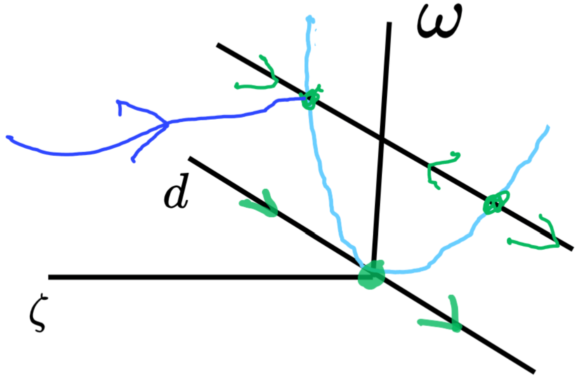
$$\dot{a} = 2\zeta a + (\zeta+1)\left[-2u_0^2 + 2 - \omega - \left(a^2+b^2\right)\right]$$
$$\dot{b} = 2\zeta b + (\zeta+1)\left[-u_0^2 - b(a+d)\right]$$
$$\dot{d} = 2\zeta d + (\zeta+1)\left[-2u_0^2 + \omega - \left(b^2+d^2\right)\right]$$
$$\dot{\zeta} = (\zeta+1)\zeta^2$$
$$\dot{\omega} = 0$$

$u = u_0$ (soliton only) and no potential $u_0 \to 0$ as $s \to +\infty$

From soliton's viewpoint, ω is small

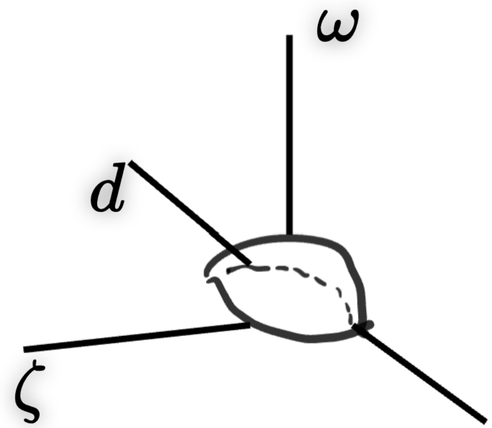
So, look in neighborhood of fixed point at $\left(\sqrt{2},0,0,0,0\right)$

$$W^c_{\text{loc}} \; : b = h_b\,(d, \zeta, \omega)$$



$$\dot{d} = 2\zeta d + (\zeta + 1)\left[-2u_0^2 + \omega - d^2 + h_b^2\right]$$
$$\dot{\zeta} = (\zeta + 1)\zeta^2$$
$$\dot{\mu} = 0$$

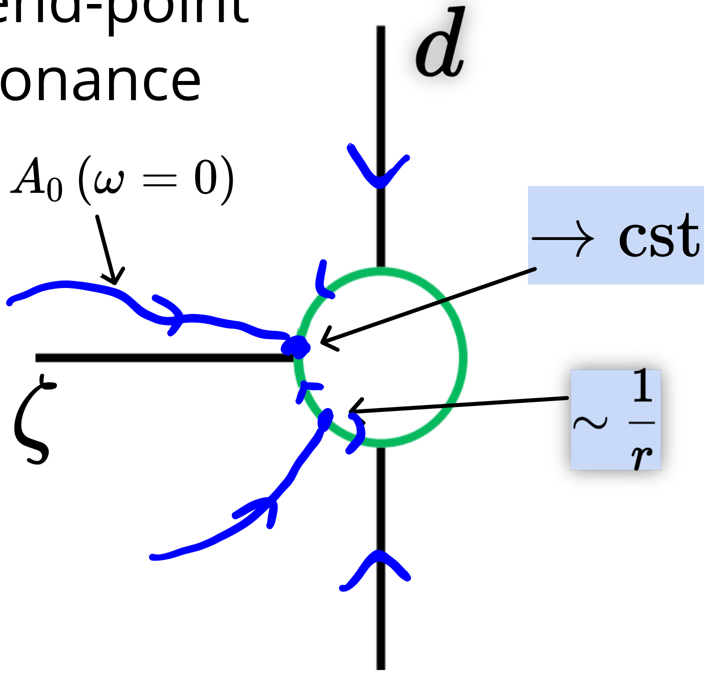
Blow up singularity at $(0, 0, 0, 0)$



Key is to understand flow on $\omega = 0$

No end-point resonance

$\implies A_0\,(\omega = 0)$



3D

Vielen Dank, Björn,

Alles Gute zum Geburtstag