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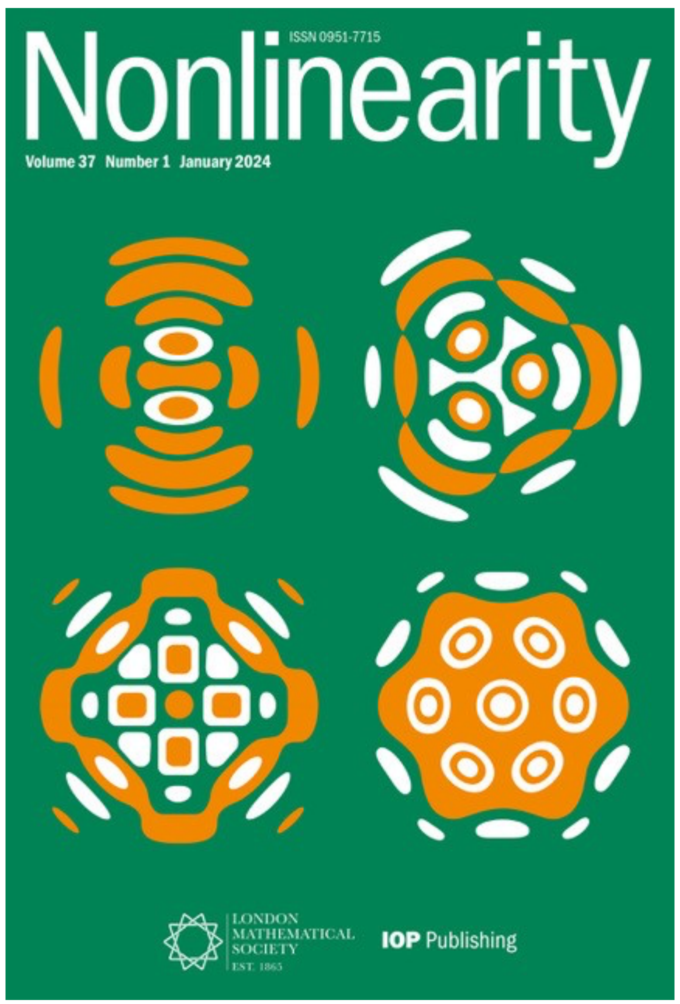
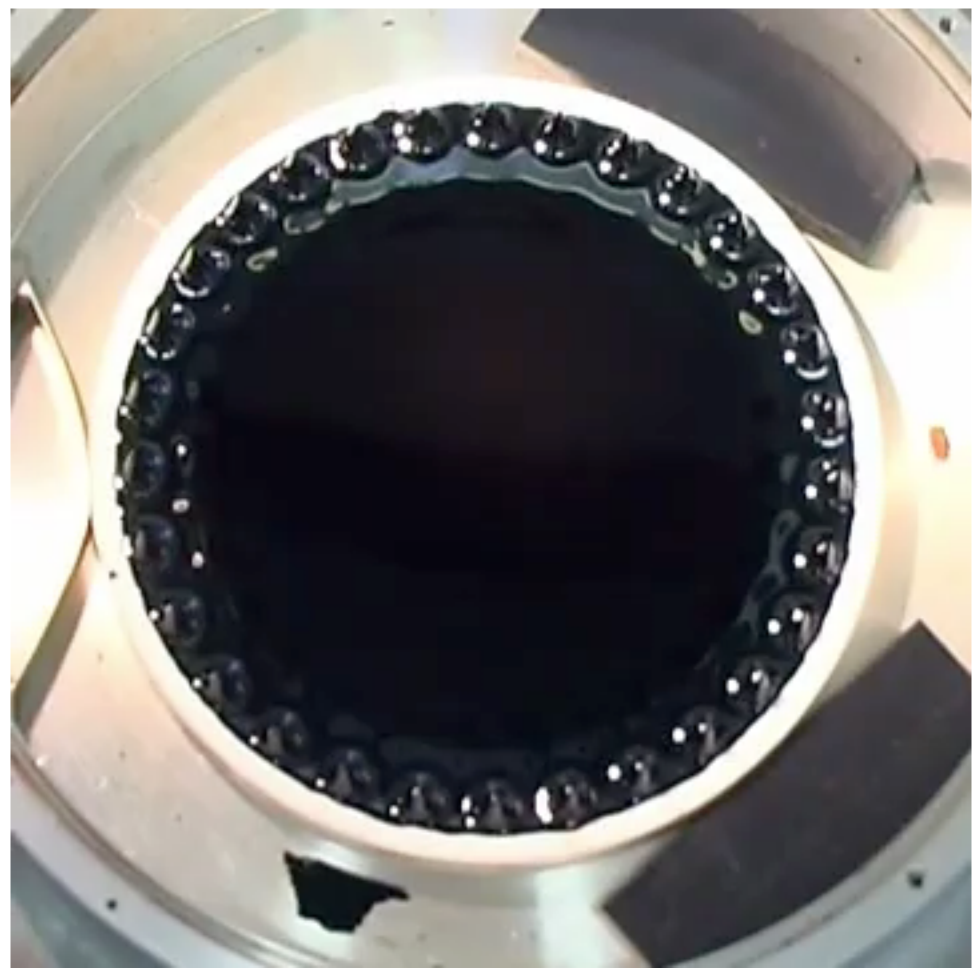
Emergence of Multi-Dimensional Localised Patterns

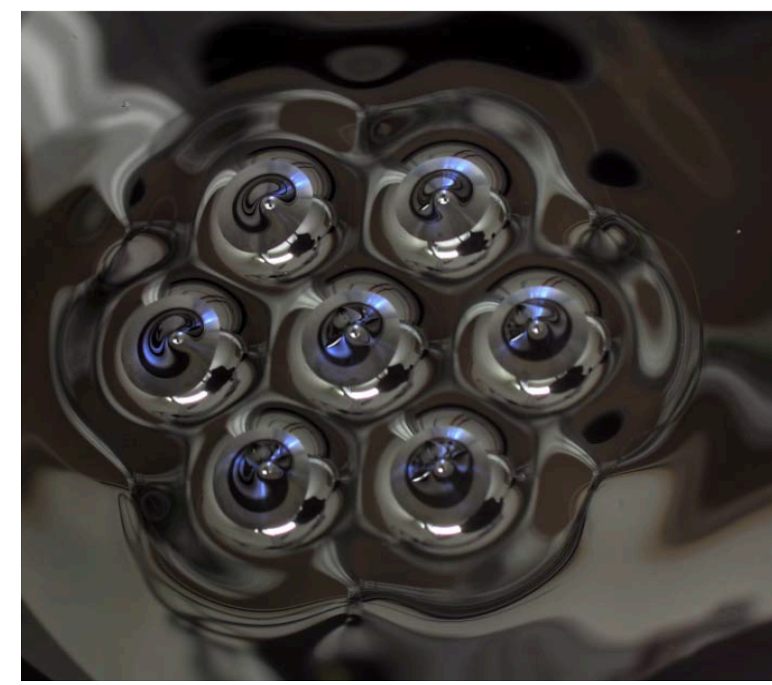
David J.B. Lloyd



[Hill, Bramburger, L., 2023]



[Hill, Bramburger, L. 2024]



[L., Gollwitzer, Rehberg, Richter, 2015]

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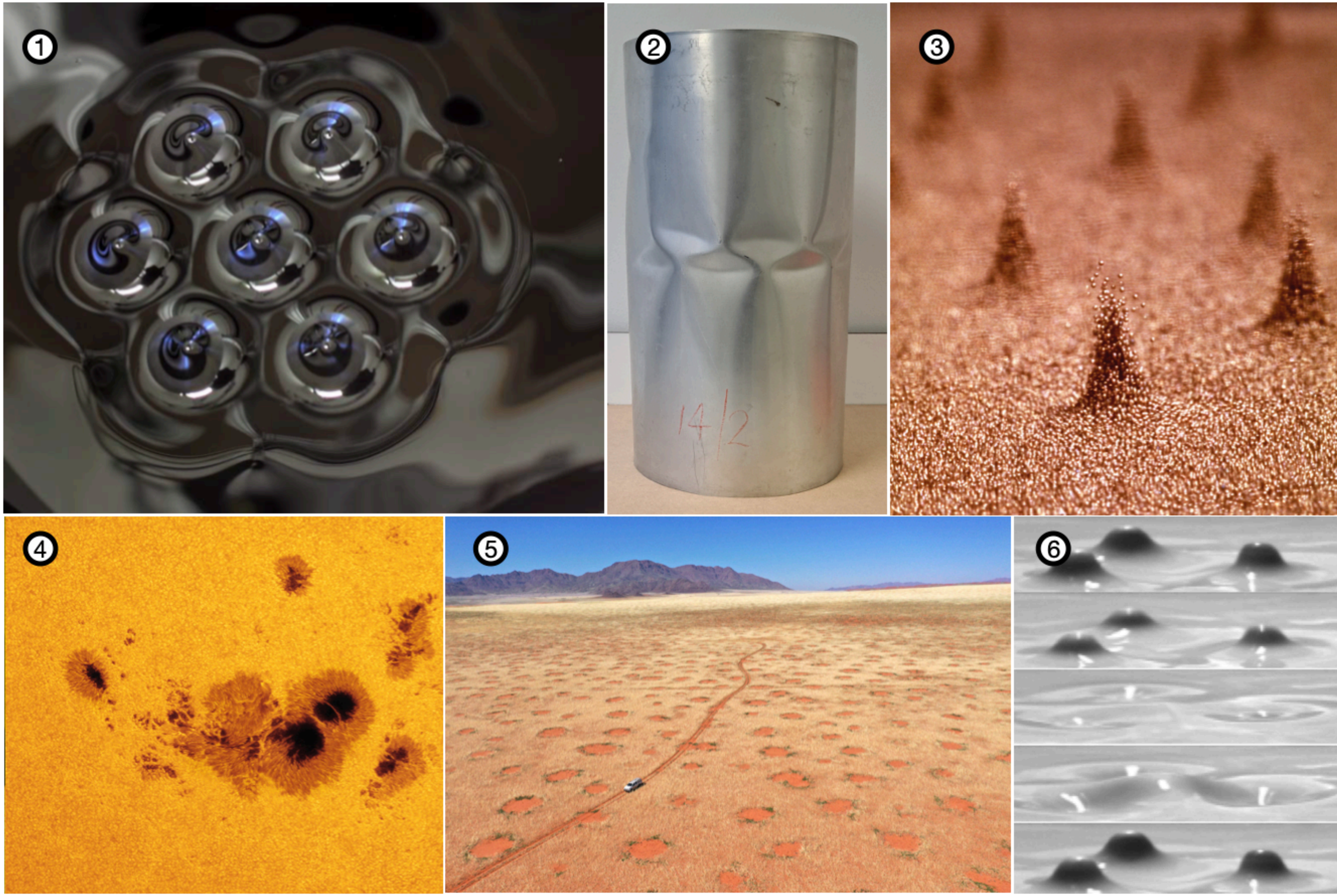
Jason Bramburger
Montreal

Daniel Hill
Saarlandes

Spatial Localisation



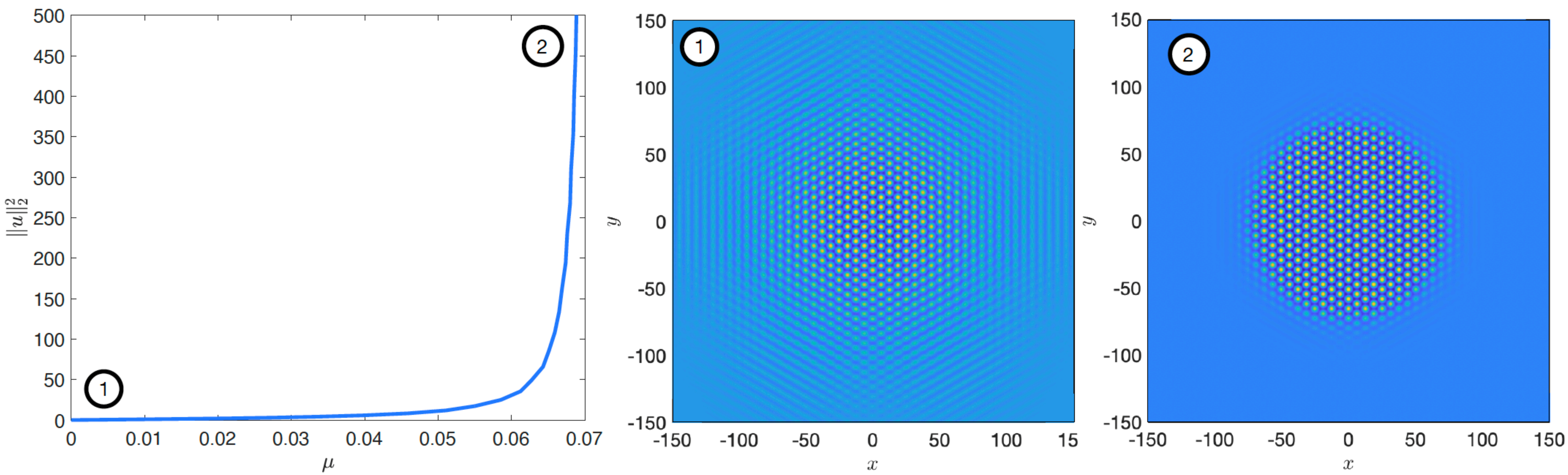
[Hill, Bramburger, L., 2024]

Emergence

Swift-Hohenberg equation:

$$u_t = -(1+\Delta)^2 u + \mu u + \nu u^2 - u^3$$

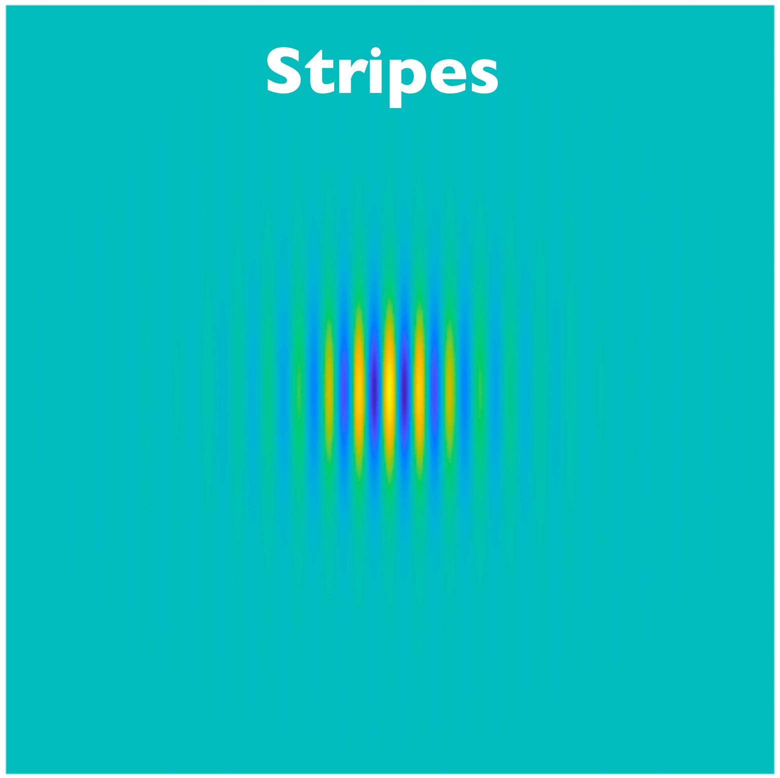
$\nu = 0.9$

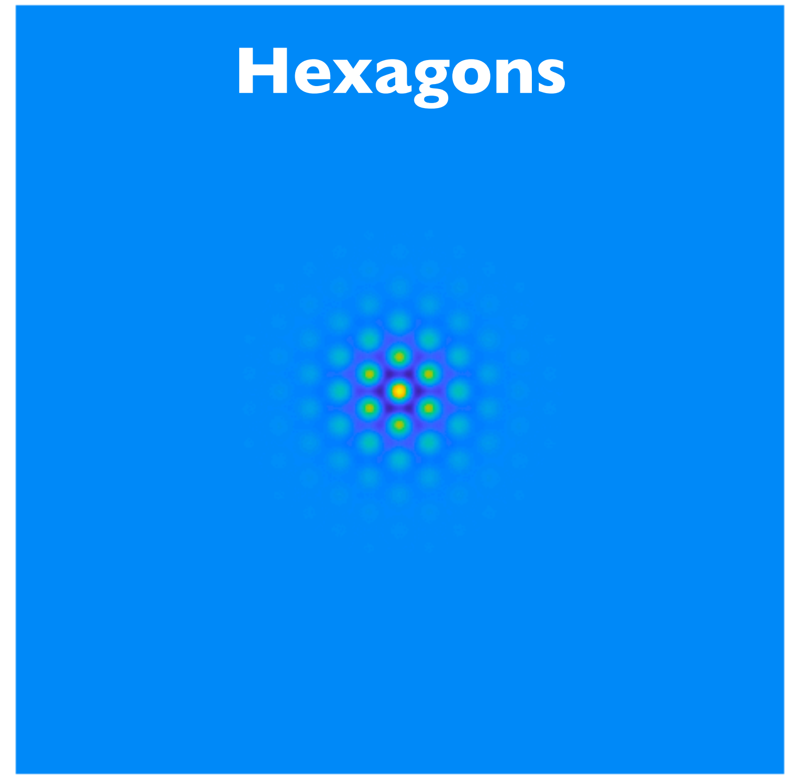


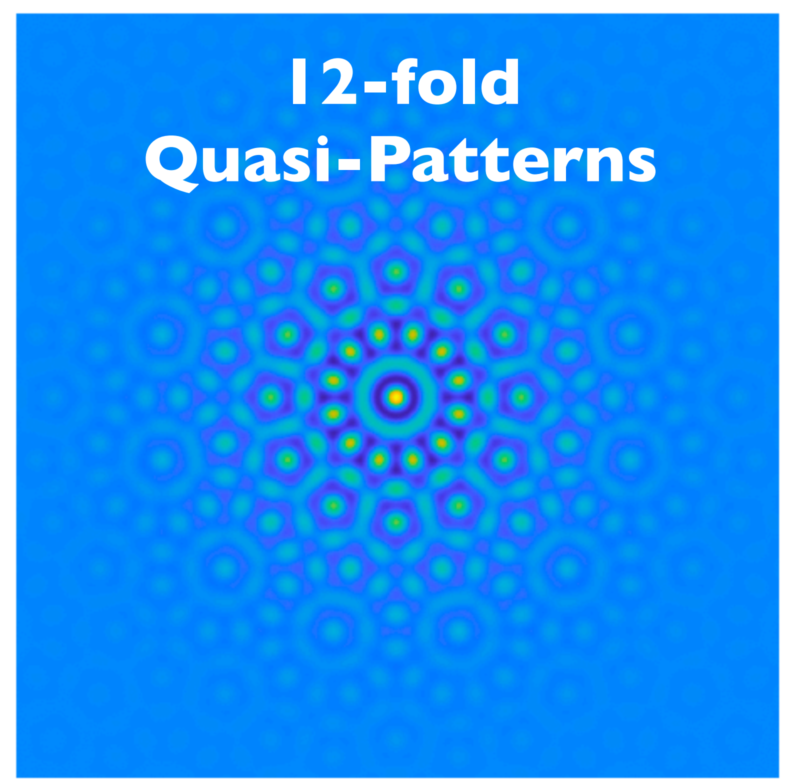
Aim: Prove emergence

[L., Sandstede, Avitabile Champneys 2008]

Different Types of Patches







Why is this difficult?

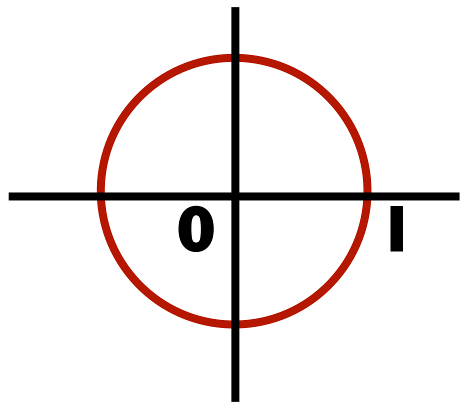
- **Key maths:** problem for patterns on the plane

$$U_t = -(1+\nabla^2)^2 U + \mu U, \qquad U = e^{\lambda t} e^{i\mathbf{k}\cdot\mathbf{x}} + c.c$$

- **Pattern forming instability at μ=0, $\Rightarrow$ critical circle of modes**

$$\Rightarrow \lambda = -(1-|\mathbf{k}|^2)^2 + \mu$$

- **No idea:** if the emerging pattern is a stripe/square/hexagon etc…



Approach I

- **Polar coordinates:** $u(r,\theta)$
- **Finite angular Fourier decomposition N<∞:**

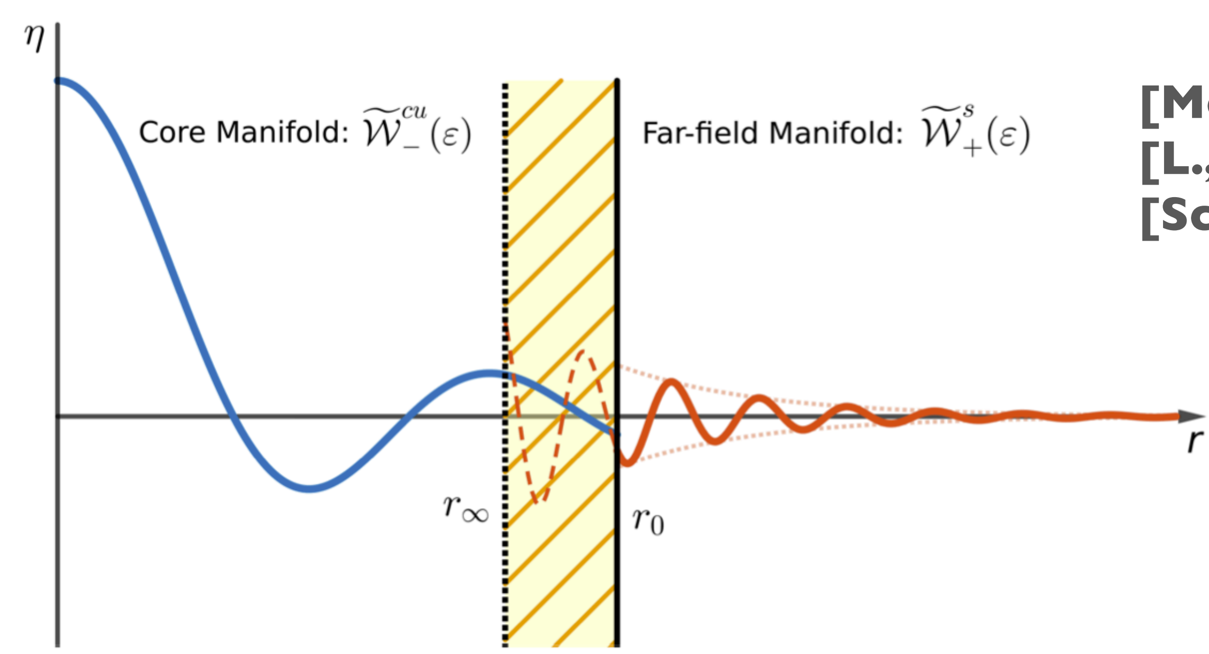
$$u(r,\theta) = u_0(r) + 2\sum_{n=1}^{N} u_n(r)\cos(mn\theta)$$

D_m solutions

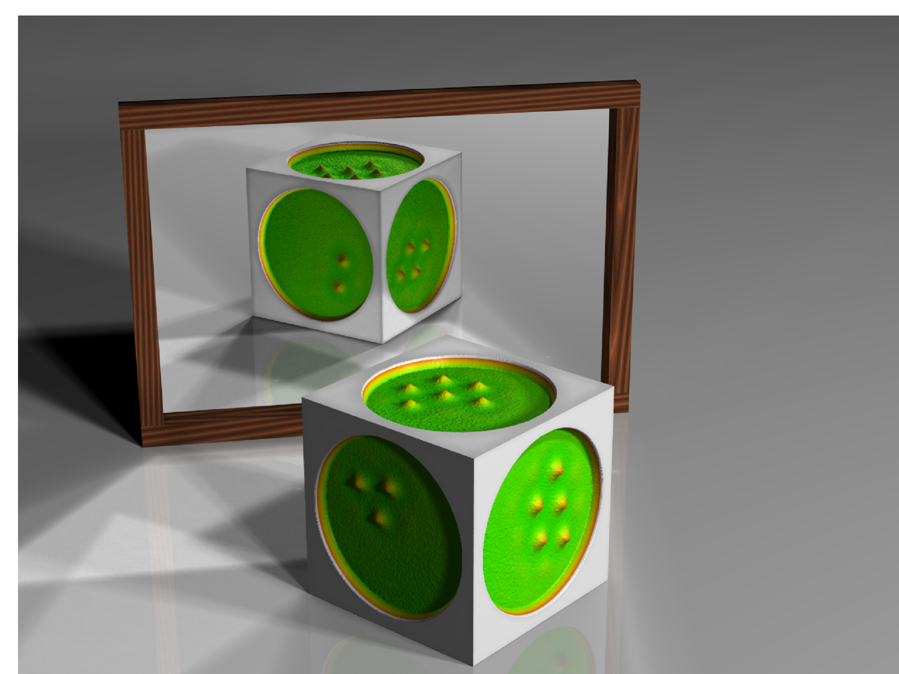
- **Radial Galerkin system:**

$$0 = L_n(r)u_n + F_n(\mathbf{u};\mu)\,, \qquad \forall n \in [0,N]$$

- **Radial normal form/centre manifold analysis for μ~0**



[McCalla, Sandstede]
[L., Sandstede]
[Scheel]



Patches: Matching conditions

Thm (Existence): [Hill, Bramburger, L. 2023]

Localised solutions exist for $\mu \in (0,\mu^*)$, fixed $\nu \neq 0$, where

$$u_n(r) = \mu^{\frac{1}{2}} a_n (-1)^{mn} \frac{\sqrt{3}}{\nu} J_{mn}(r) + \mathcal{O}(\mu)$$

on $r \in [0, r_0]$ and a_n satisfy the **matching condition:**

$$a_n = 2 \sum_{j=1}^{N-n} \cos\left(\frac{m\pi(n-j)}{3}\right) a_j a_{n+j} + \sum_{j=0}^{n} \cos\left(\frac{m\pi(n-2j)}{3}\right) a_j a_{n-j}$$

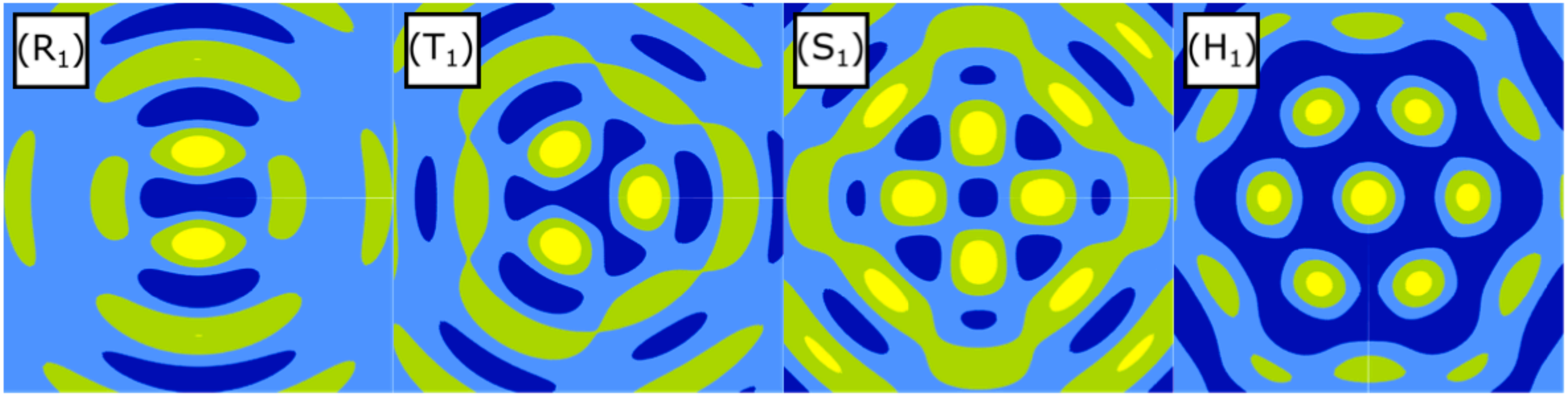
- Matching condition can be solved explicitly for N=1,2,3 for general lattices…
- Matching condition can be solved explicitly for N=4, D_{6m} lattices
- General N for m = 6p, p integer

- **Matching condition same for 2-component RD systems**
- **Predicts only certain types of patches can bifurcate…**

Matching Condition: N=1

$$a_0 = a_0^2 + C_m a_1^2, \qquad a_1 = C_m a_1 a_0, \qquad C_m := 2\cos\left(\frac{2m\pi}{3}\right)$$

4 Solutions:



m=2 **m=3** **m=4** **m=6**

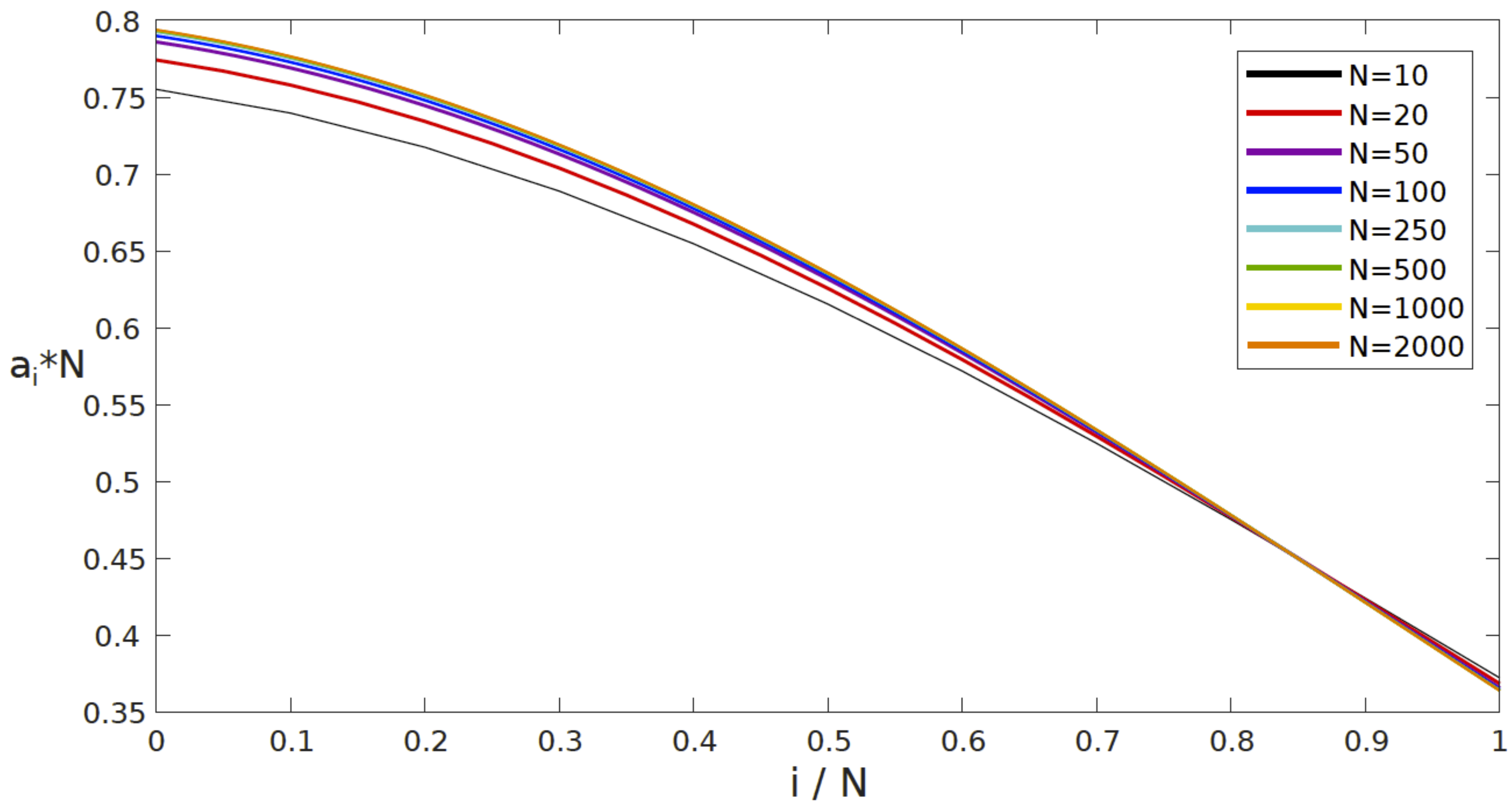
General N: D_{6m} case

m=6m$_0$:

$$a_n = 2\sum_{j=1}^{N-n} a_j a_{n+j} + \sum_{j=0}^{n} a_j a_{n-j}$$

Continuum approx:

$$a_i \approx \frac{1}{N} a\left(\frac{i}{N}\right)$$



Limit N->∞:

$$a(t) = 2\int_0^{1-t} \{a(s)\ a(t+s)\}\,\mathrm{d}s + \int_0^t \{a(s)\ a(t-s)\}\,\mathrm{d}s \qquad t \in [0,1]$$

Rigorous numerics solution -> exists a solution for arb. large N discrete

[Hill, Bramburger, L. 2023]

D_{6m} Patches: large N

Thm (Existence): [Hill, Bramburger, L. 2023]

Fix m = $6m_0$, then there exists an $\varepsilon > 0$ such that for all $N > N_\varepsilon$

$$\sup_{n\in[0,N]} \left| a_n - \frac{1}{N+1}\alpha^*\left(\frac{n}{N+1}\right)\right| < \varepsilon$$

Where $\alpha^*(t)$ is a positive, continuous solution of

$$\alpha(t) - 2\int_0^{1-t} \alpha(s)\alpha(s+t)\mathrm{d}s - \int_0^t \alpha(s)\alpha(t-s)\mathrm{d}s = 0$$

Proof:

1. Rigorous numerics (Newton–Kantorovich thm) on space of regulated functions: (Closure of space of real-valued step functions on [0,1] with respect to the sup norm)

2. Newton-Kantorovich argument on $\mathbf{F}_N : \mathbb{R}^N \to \mathbb{R}^N$

$$\mathbf{F}_N(a)_n = a_n - \frac{2}{N+1}\sum_{j=1}^{N-n} a_j a_{n+j} - \frac{1}{N+1}\sum_{j=0}^{n} a_j a_{n-j}, \quad n = 0, 1, \ldots, N.$$

Rings: Matching conditions

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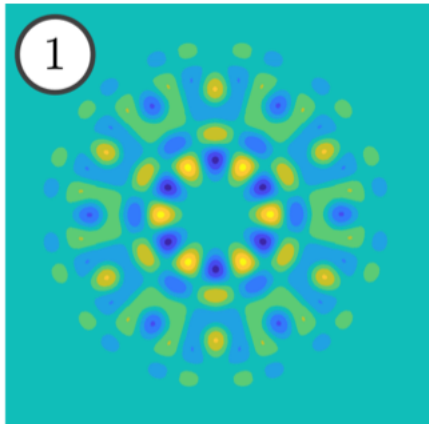


Thm (Existence): [Hill, Bramburger, L. 2024]

Localised solutions exist for μ∈(0,μ*), fixed *ν*>√(27/38), where

$$u_n = 2\mu^{3/4} a_n q_0 \sqrt{\frac{\pi}{2}} r J_{mn+1} + \mathcal{O}(\mu)$$

on r∈[0,r_0] and a_n satisfy the matching condition:

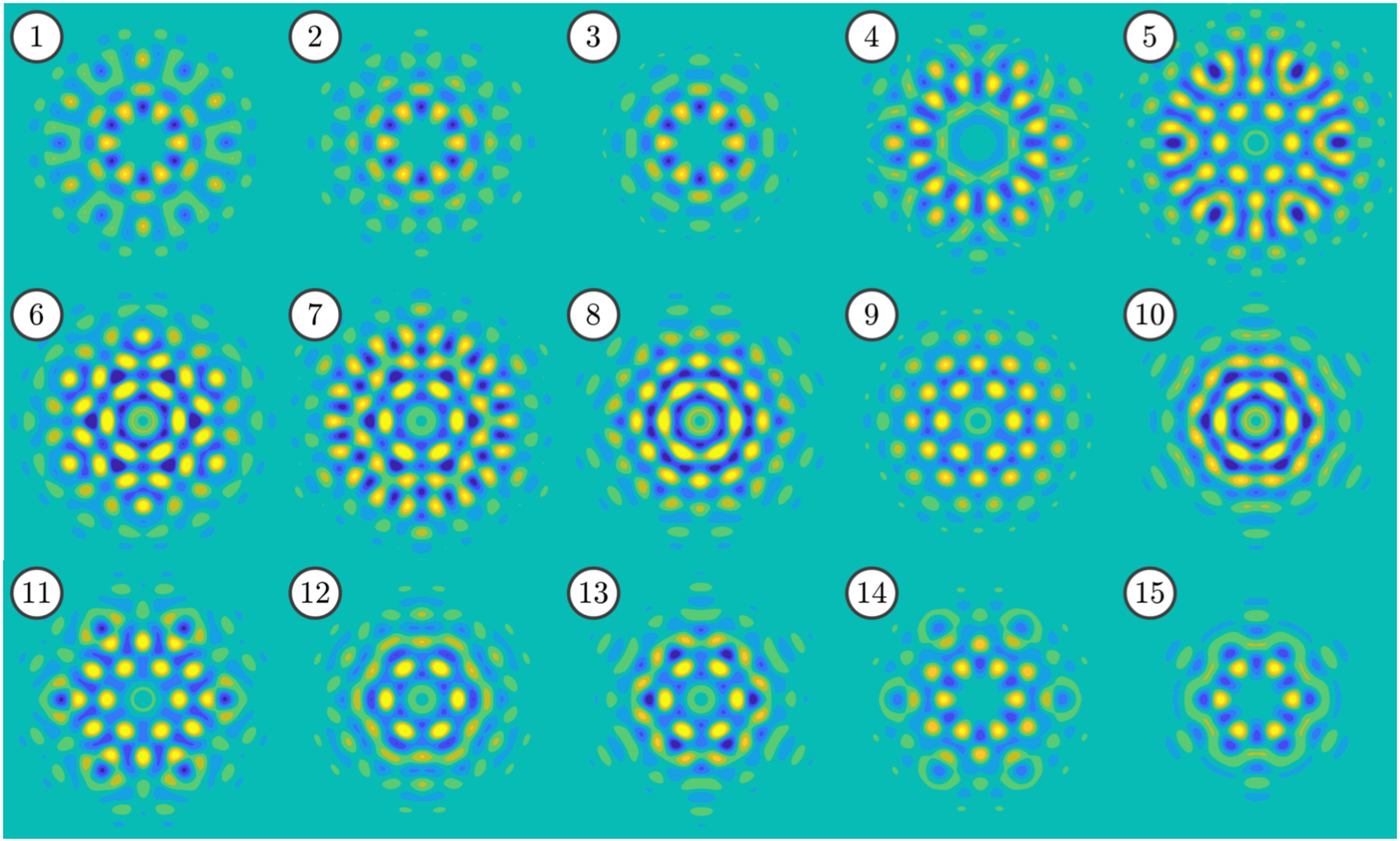
$$a_n = \sum_{i+j+k=n} (-1)^{\frac{m(|i|+|j|-|k|-n)}{2}} a_{|i|} a_{|j|} a_{|k|}, \quad n = 0, 1, \ldots, N, \quad -N \leq i, j, k \leq N$$

and q(s) solves

$$\left(d_s + \frac{1}{2s}\right)^2 q(s) = c_0 q(s) + c_3 q(s)^3, \quad q(s) = \begin{cases} q_0 s^{\frac{1}{2}} + \mathcal{O}(s^{\frac{3}{2}}), & s \to 0, \\ (q_+ + \mathcal{O}(e^{-\sqrt{c_0}s}) s^{-\frac{1}{2}} e^{-\sqrt{c_0}s}, & s \to \infty \end{cases}$$

- Matching condition can be solved explicitly for N=1,2,3 for general lattices…
- Matching condition can be solved explicitly for N=4, D_{2m} lattices
- General N for m = 2p, p integer
- **Matching condition same for 2-component RD systems**
- **Predicts only certain types of rings can bifurcate…**

Matching Condition: N=3



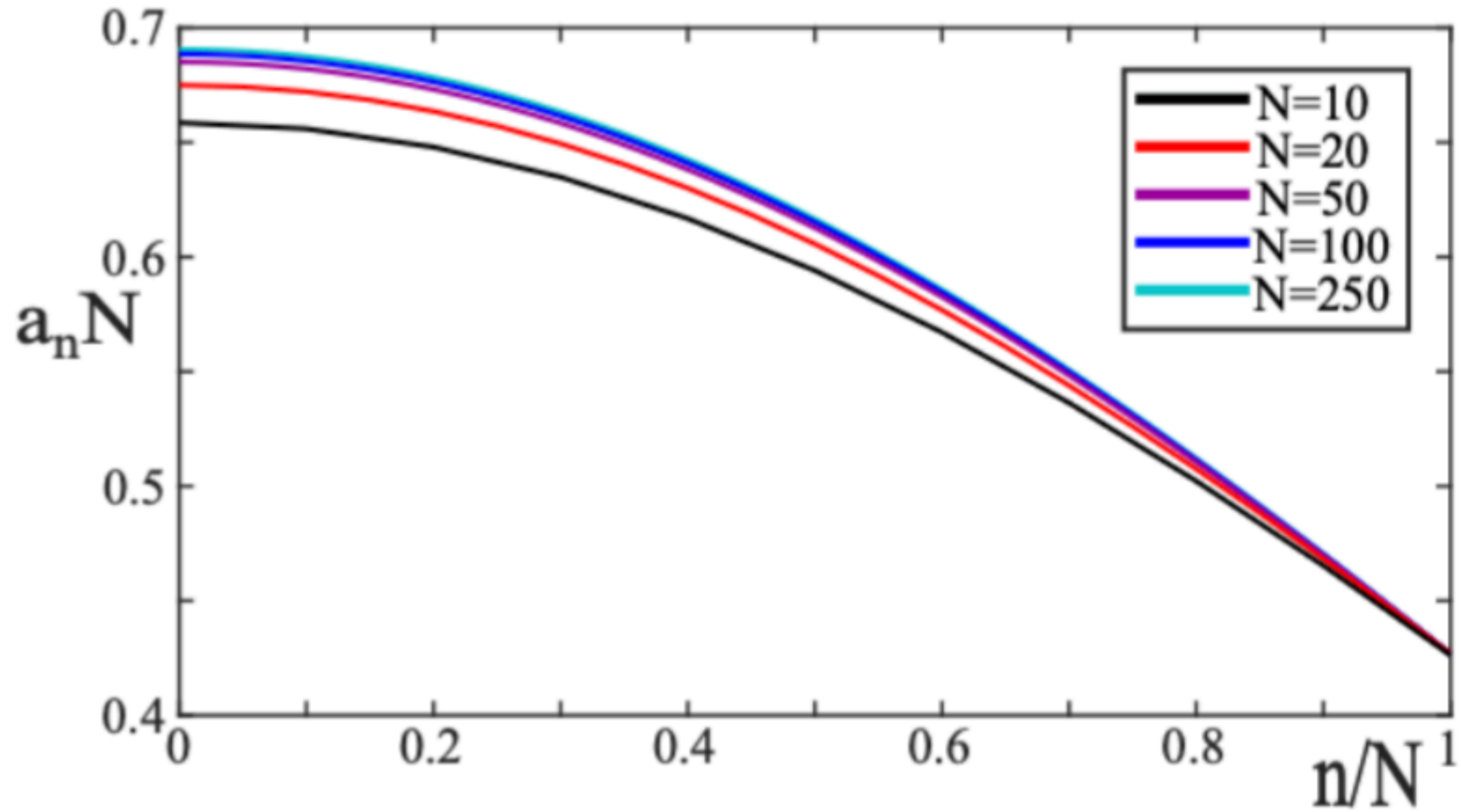
General N: Even case

m=2m₀:

$$a_n = \sum_{i+j+k=n} a_{|i|}a_{|j|}a_{|k|}$$

Continuum approx:

$$a_i \approx \frac{1}{N} a\left(\frac{i}{N}\right)$$



Limit N->∞:

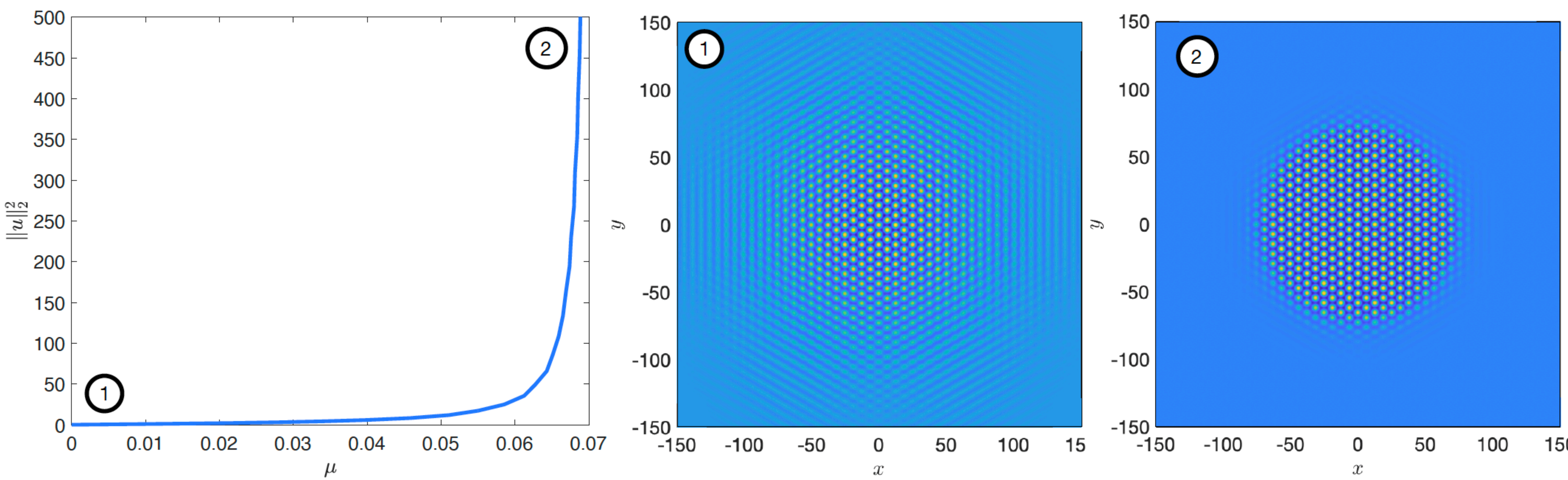
$$\begin{aligned}\alpha(t) = {} & 2\int_0^{1-t}\int_0^{1-s} \alpha(t+s)\alpha(x)\alpha(s+x)\,\mathrm{d}x\,\mathrm{d}s + 2\int_0^{1-t}\int_0^{1-(s+t)} \alpha(s)\alpha(x)\alpha((s+t)+x)\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_0^{1-t}\int_0^{s} \alpha(t+s)\alpha(x)\alpha(s-x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^{1-t}\int_0^{(s+t)} \alpha(s)\alpha(x)\alpha((s+t)-x)\,\mathrm{d}x\,\mathrm{d}s \\ & + 2\int_0^{t}\int_0^{1-s} \alpha(t-s)\alpha(x)\alpha(s+x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^{t}\int_0^{s} \alpha(t-s)\alpha(x)\alpha(s-x)\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_0^{t}\int_0^{1-s} \alpha(s+1-t)\alpha(x+s)\alpha(1-x)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

$$t \in [0, 1]$$

[Hill, Bramburger, L. 2024]

Problems with Approach I

- **Validity of existence region goes to zero as N goes to infinity…**
- **Guess is that patch =** 2D cellular pattern **X** radial envelope
- **Technical derivation**



Approach II

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- **Find an approximation:** 2D Pattern X radial envelope
- **Fourier polar decomposition:**

$$u(t,x,y) = u(t,r,\theta) = \sum_{n\in\mathbb{Z}} u_n(t,r)e^{in\theta}$$

- **Projected Swift-Hohenberg equation:**

$$\partial_t u_n = -(1+\Delta_n)^2 u_n - \mu u_n + \nu \sum_{i+j=n} u_i u_j - \sum_{i+j+k=n} u_i u_j u_k$$

$$\Delta_n = \partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} \qquad \mu = \epsilon^2\hat{\mu}, \quad (T,R) = (\epsilon^2 t, \epsilon r)$$

- **Aim:**

$$u_n(t,r) = \epsilon\left(A(T,R)a_n J_n(r) + A(T,R)a_n J_{-n}(r)\right) + \mathcal{O}(\epsilon^2),$$

[Hill, L., 2024]

Radial Differential Operators

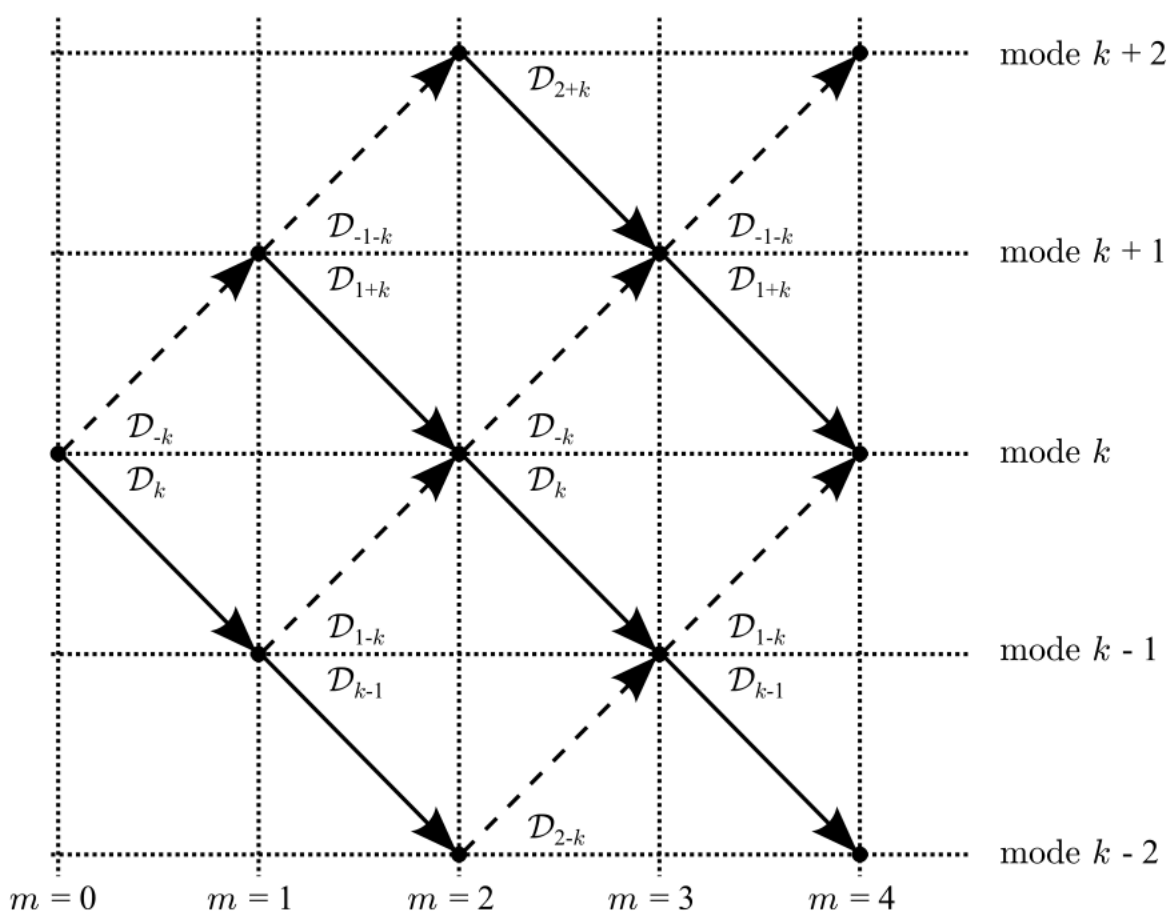
- **Problem with Bessel functions:**

$$\partial_r J_n(r) = \frac{1}{2}(J_{n-1}(r) - J_{n+1}(r)) = J_{n-1}(r) - \frac{n}{r}J_n(r) = -J_{n+1}(r) + \frac{n}{r}J_n(r)$$

- **Solution - Bessel Differential operators:**

$$\mathcal{D}_n := \partial_r + \frac{n}{r}, \quad n \in \mathbb{Z}$$

$$\mathcal{D}_n J_n(r) = J_{n-1}(r), \quad \mathcal{D}_{-n} J_n(r) = -J_{n+1}(r)$$



$$\mathcal{D}_{n-1}\mathcal{D}_n = \partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} = \Delta_n$$

[Hill, Groves, 2024]
[Hill, L., 2024]

Radial Differential Operators II

- **Multiple scales expansion:**

$$u_n = u_n(r, R), \quad R = \epsilon r$$

- **Slow scale Bessel operators:**

$$\widehat{\mathcal{D}}_{\pm n} := \partial_R \pm \frac{n}{R}, \qquad \widehat{\Delta}_n := \widehat{\mathcal{D}}_{1-n}\widehat{\mathcal{D}}_n$$

- **Expand Laplacian:**

$$\Delta_n u_n = \mathcal{D}_{1-n}\mathcal{D}_n u_n + \epsilon(\mathcal{D}_n + \mathcal{D}_{-n})\widehat{\mathcal{D}}_0 u_n + \epsilon^2 \widehat{\mathcal{D}}_1 \widehat{\mathcal{D}}_0 u_n$$

- **Expand Laplacian:**

$$(1+\Delta_n)^2 u_n = (1+\Delta_n)^2 u_n + \epsilon[4\partial_r(1+\Delta_n)\partial_R u_n] + \epsilon^2[2(1+\Delta_n)\widehat{\Delta}_0 u_n + 4\Delta_n \partial_R^2 u_n] + \mathcal{O}(\epsilon^2)$$

[Hill, L., 2024]

Bessel Convolutions

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- **Projecting on to each Bessel function mode**

$$e^{ir}e^{i4r}e^{-3ir} = e^{2ir} \quad \sim \quad \sum_{i+j+k=n} J_i(r)J_j(4r)J_{-k}(3r) = J_n(2r)$$

- **General result provided** $n \in \mathbb{Z}$

$$\sum_{i+j+k=n} J_i(ar)J_j(br)J_{\pm k}(cr) = J_n((a+b\pm c)r)$$

[Hill, L., 2024]

Radial Amplitude equation

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Expansion

$$u_n = \epsilon v_n^{(0)}(T,r,R) + \epsilon^2 v_n^{(1)}(T,r,R) + \epsilon^3 v_n^{(2)}(T,r,R) + \mathcal{O}(\epsilon^4)$$

Collect at orders of epsilon

$$\mathcal{O}(\varepsilon^1) \qquad 0 = -\left(1+\Delta_n\right)^2 v_n^{(0)}$$

$$\mathcal{O}(\varepsilon^2) \qquad 0 = -\left(1+\Delta_n\right)^2 v_n^{(1)} - 4\partial_r\left(1+\Delta_n\right)\partial_R v_n^{(0)} + \nu \sum_{i+j=n} v_i^{(0)} v_j^{(0)}$$

$$\mathcal{O}(\varepsilon^3) \qquad \partial_T v_n^{(0)} = -\left(1+\Delta_n\right)^2 v_n^{(2)} - 4\partial_r\left(1+\Delta_n\right)\partial_R v_n^{(1)} - 2\left(1+\Delta_n\right)\hat{\Delta}_0 v_n^{(0)} - 4\,\Delta_n\,\partial_R^2 v_n^{(0)} - \hat{\mu} v_n^{(0)} + 2\nu \sum_{i+j=n} v_i^{(0)} v_j^{(1)} - \sum_{i+j+k=n} v_i^{(0)} v_j^{(0)} v_k^{(0)}$$

Choice of $v_n^{(0)}$ leads to different amplitude equations

Stripes: $$v_n^{(0)} = A(T,R)J_n(r) + \overline{A}(T,R)J_{-n}(r)$$

Hexagons: $$v_{3n}^{(0)} = A(T,R)J_{3n}(r) + \overline{A}(T,R)J_{-3n}(r)$$

Rhomboids: $$v_n^{(0)} = 3A(T,R)J_n(r) + 3\overline{A}(T,R)J_{-n}(r)$$

Amplitude Equations

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- **Localised Stripes:**

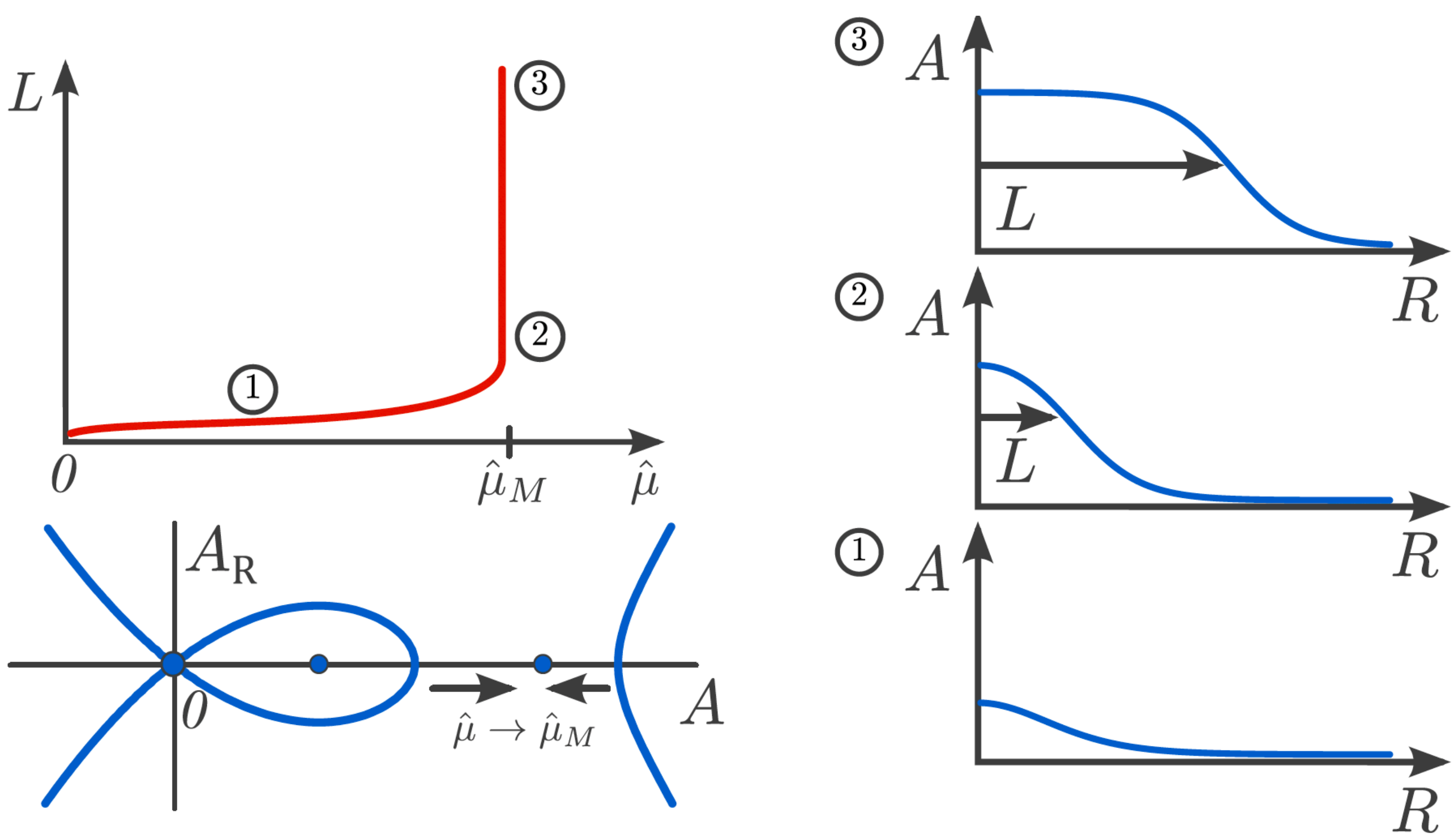
$$A_T = 4A_{RR} - \hat{\mu}A + 4\left(\frac{19\nu^2}{18} - \frac{3}{4}\right)|A|^2 A$$

- **Localised hexagons/rhomboids:**

$$A_T = 4A_{RR} - \hat{\mu}A + 2\hat{\nu}\overline{A}^2 - 15|A|^2 A$$

- **Localised quasi-patterns 12 fold**

$$A_T = 4A_{RR} - \hat{\mu}A + 2\hat{\nu}\overline{A}^2 - 33|A|^2 A$$



[Hill, L., 2024]

Going 3D…

- **Spherical harmonic expansion**

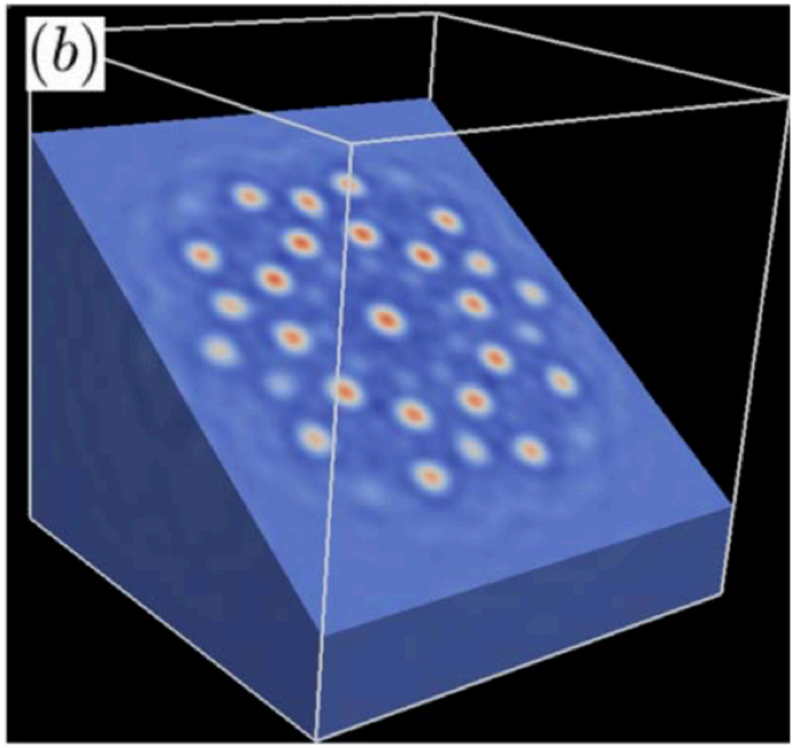
$$u(t,r,\varphi,\theta) = \sum_{(\ell,m)\in\mathcal{I}_1} u_{\ell,m}(t,r) i^{\ell} Y_{\ell}^{m}(\theta,\varphi), \quad u_{\ell,m} = (-1)^{\ell-m}\overline{u}_{\ell,-m}$$

$$\mathcal{I}^d = \{(\mathbf{p},\mathbf{q}) \in \mathbb{N}_0^d \times \mathbb{Z}^d : \quad |q_i| \leq p_i, \quad i = 1,\ldots,d\}$$

- **Look for simple cubic, face-centred cubic, body centred-cubic**

- **Extend Approach I and II…**

[Hill, Bramburger, L. In prep]



[Subramanian, Archer, Knobloch, Rucklidge]

Conclusions

- **Can get a long way with just approximations…**
- **Approach I**
  - Based on numerical method
  - Small truncations most useful
- **Approach II**
  - Formal multiple scales method
  - Rigorous validity (Hill and Groves)
  - Exponential asymptotics extension?
  - Gronwall estimates?
- **Extensions**
  - 3D in progress…
  - Spatial heterogeneity
  - Different differential operators and special functions
  - Turing-Hopf bifurcation