# Localized Patterns and Modeling Dynamics on Networks

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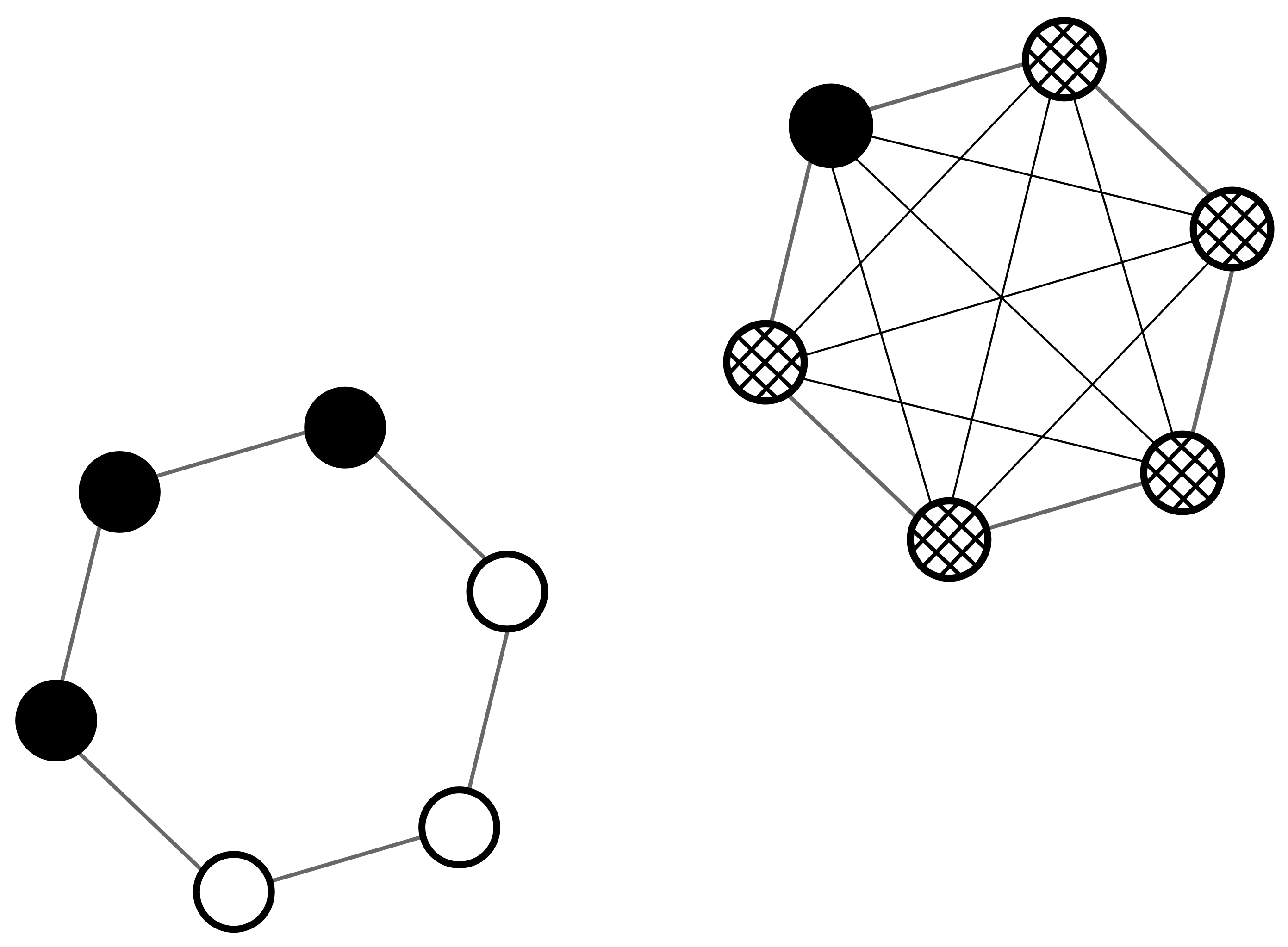
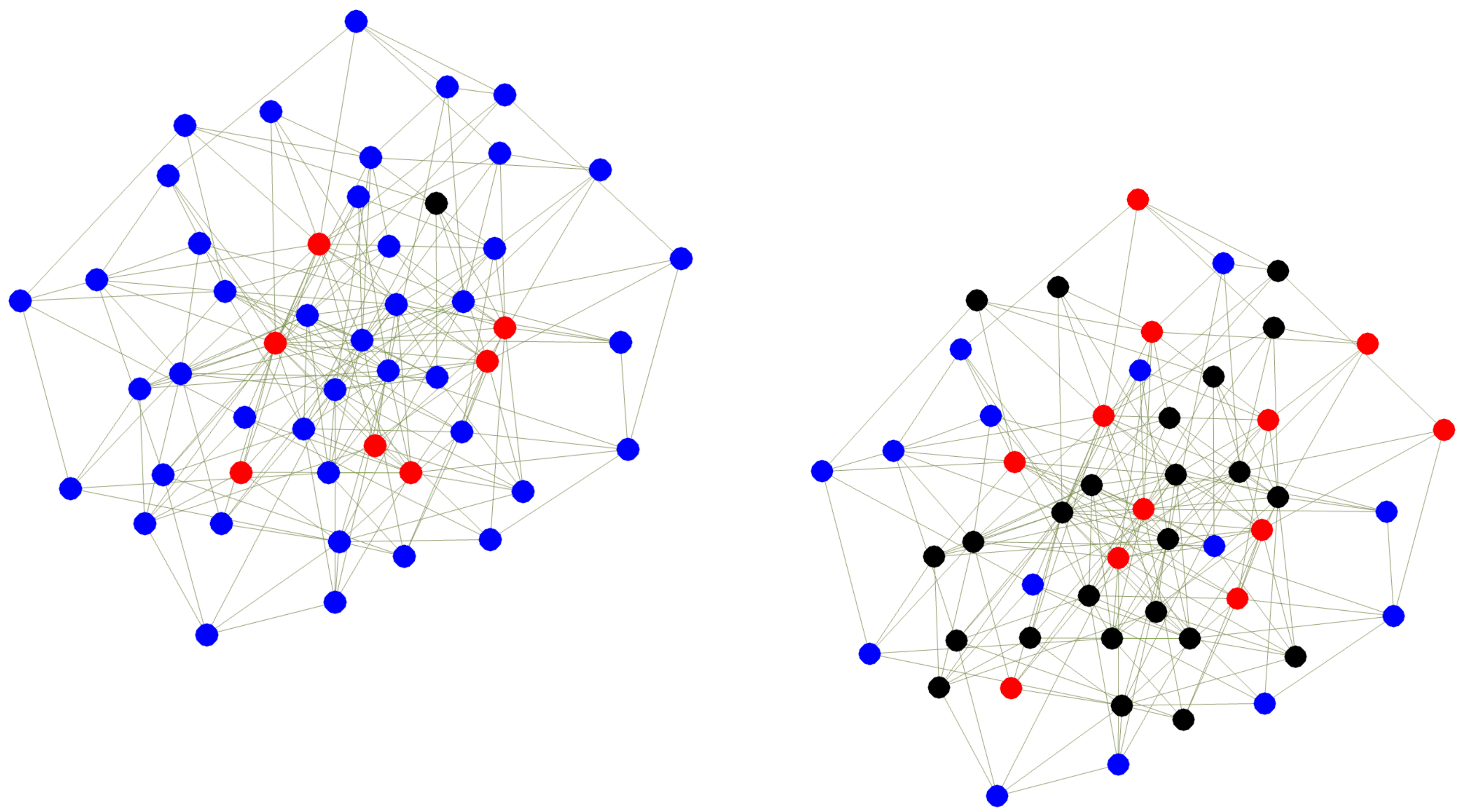
## Localized Patterns on Graphs

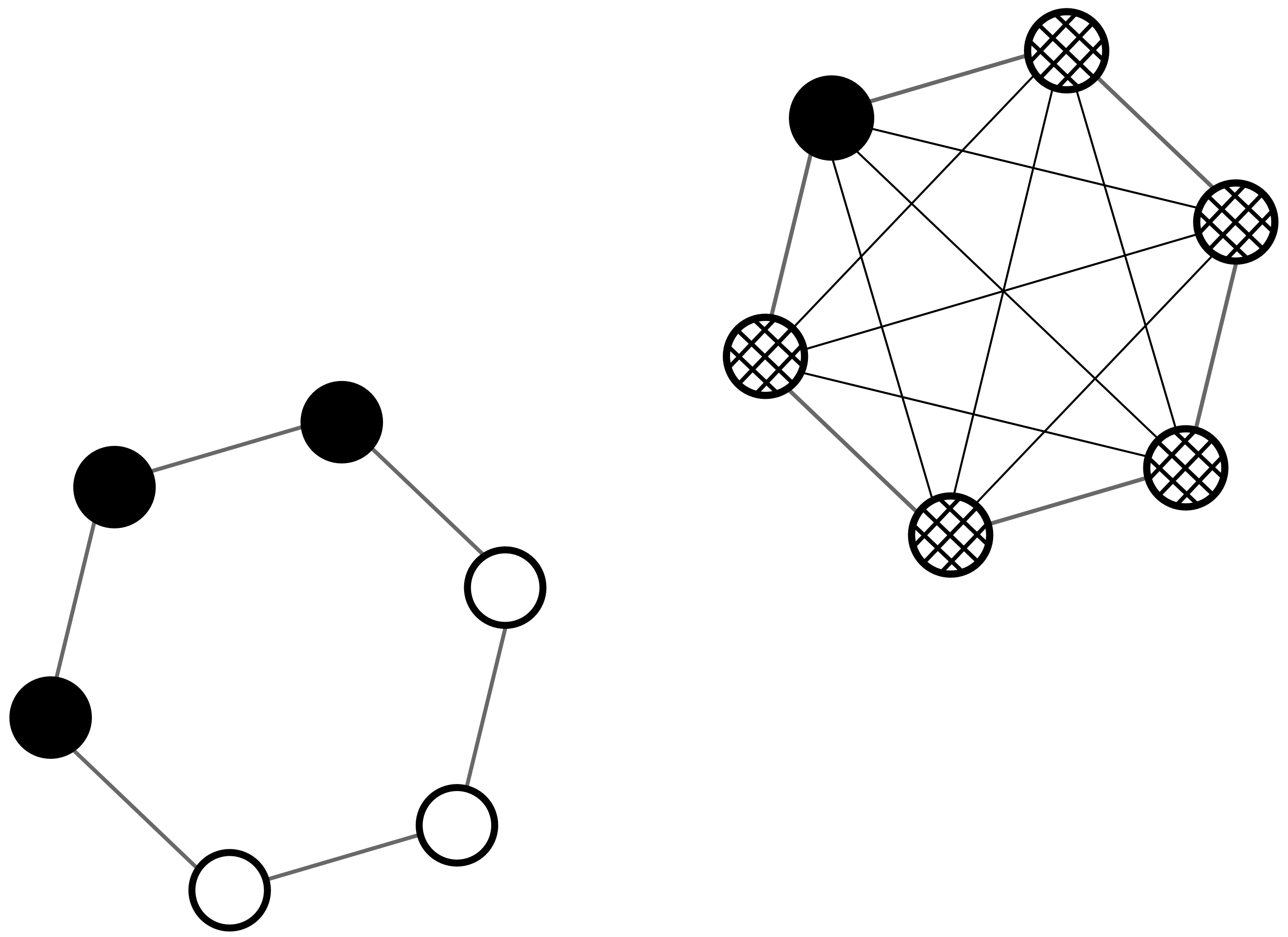


## Modeling Coupled Online & Offline Dynamics of Protesting Activity on Networks

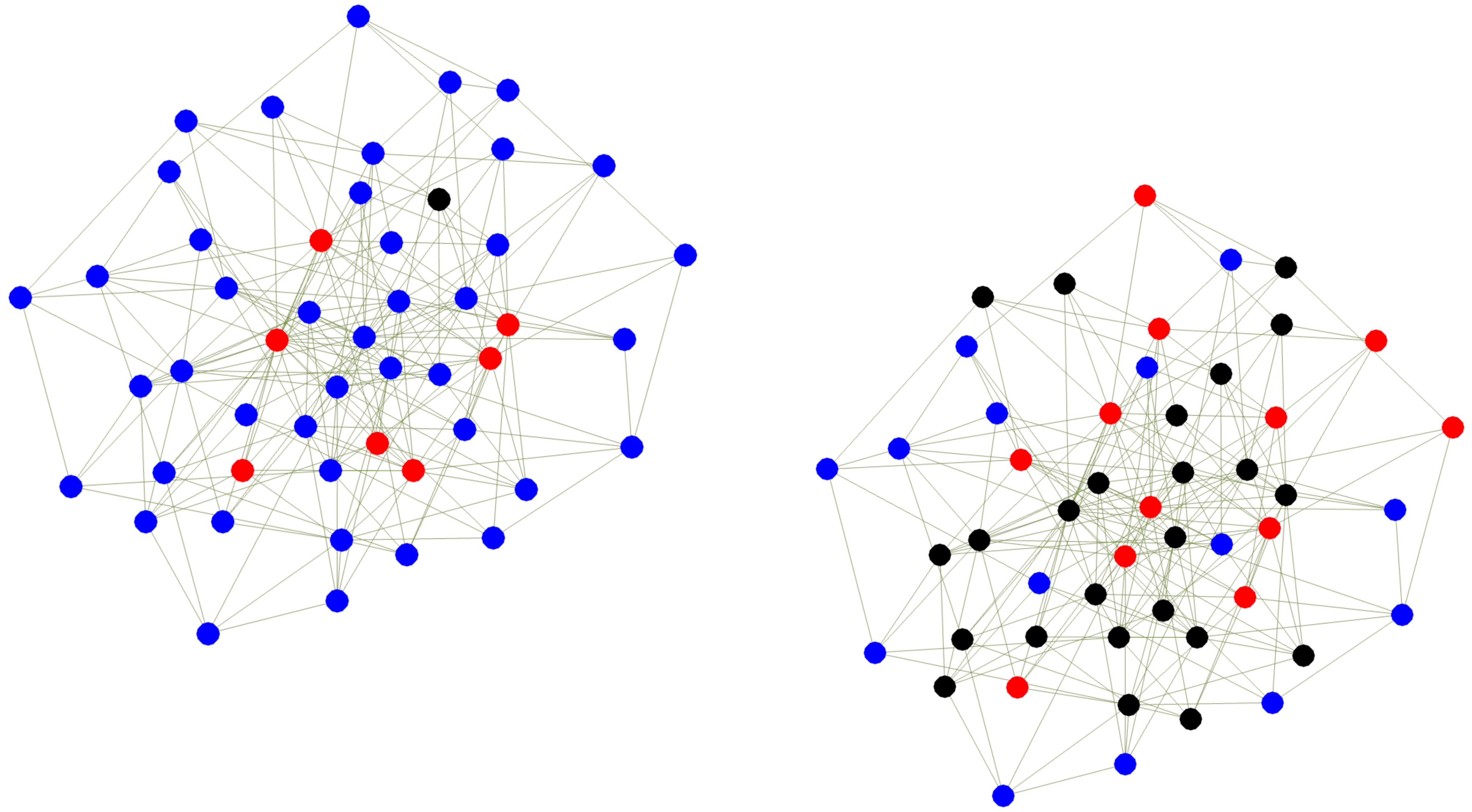


# How to incorporate network structure?

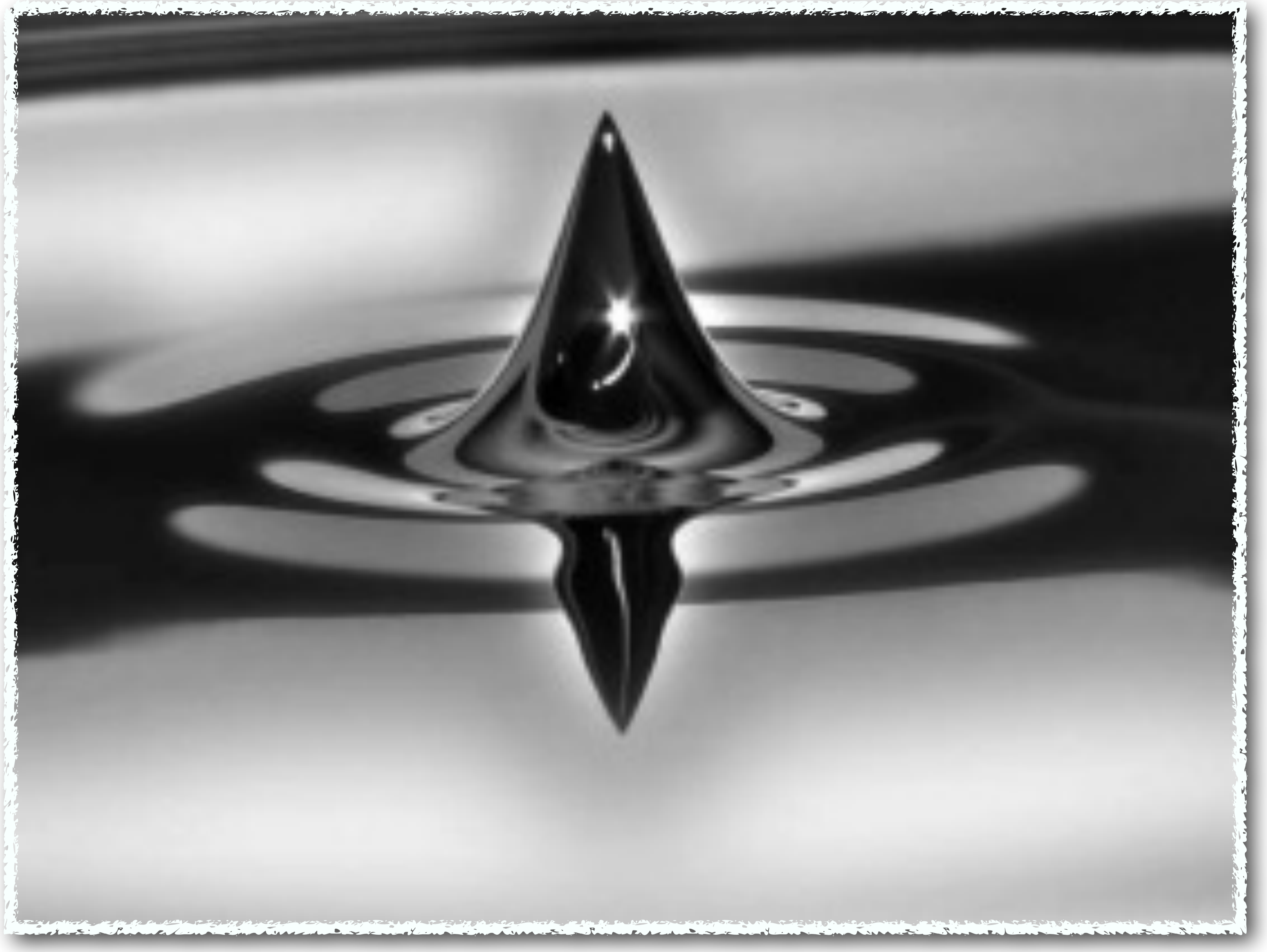
## Localized Patterns on Graphs



## Modeling Coupled Online & Offline Dynamics of Protesting Activity on Networks

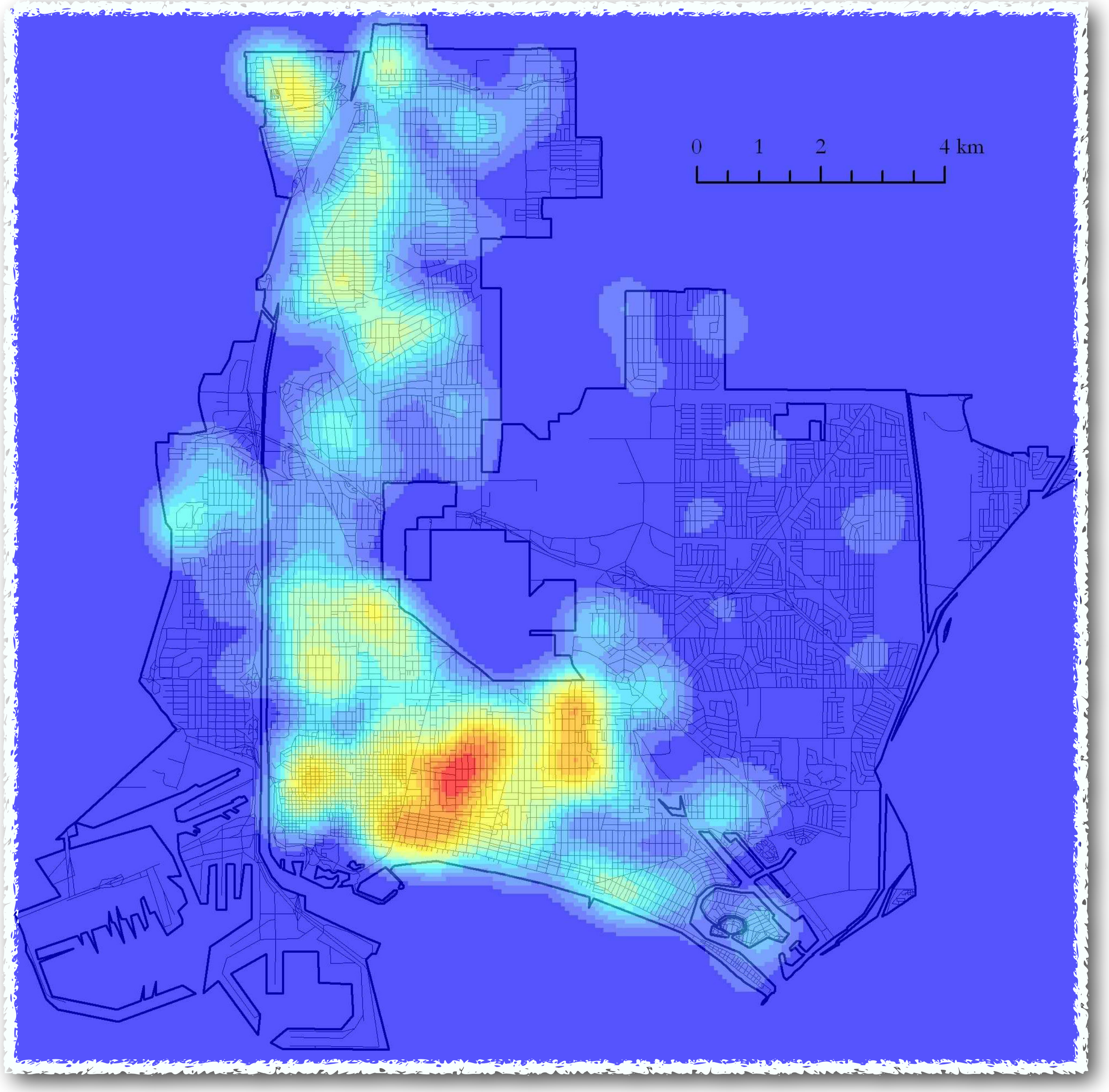


# Localized patterns observed in nature and experiments



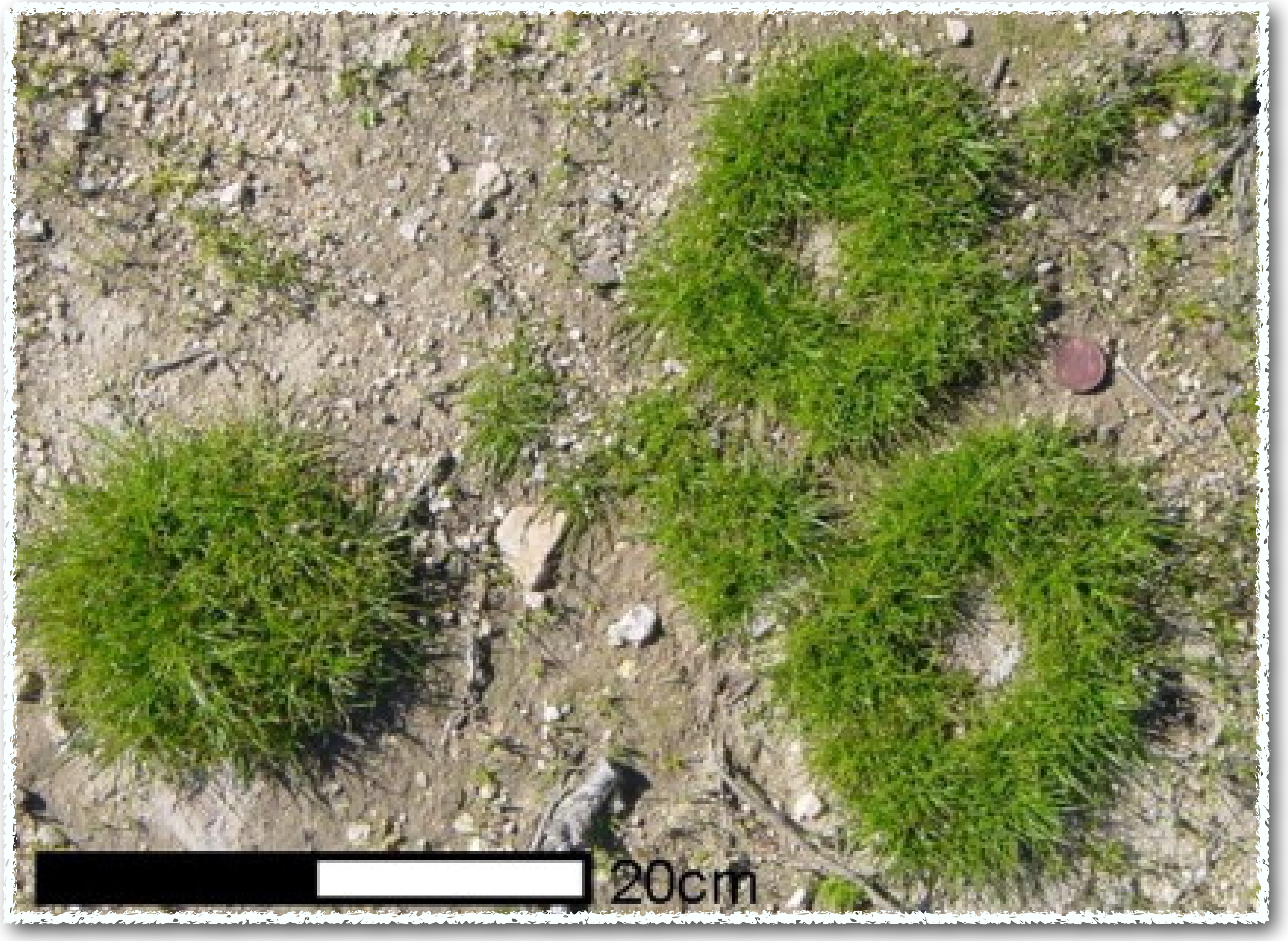
ferrosoliton in ferrofluid

[Richter, Europhys. News (2011)]




burglary hotspots

[Short, D'Orsogna, et al., Math. Models Methods Appl. Sci. (2008)]




patterned grass

[Sheffer, Yizhaq, et al., Ecol. Complex. (2007)]

**★Reaction-diffusion systems with bistability**

- **Question:** How does graph structure impact the connection of localized patterns?
- **Setup:** Bistable reaction-diffusion systems on graph $G = (V, E)$

Stationary Pattern

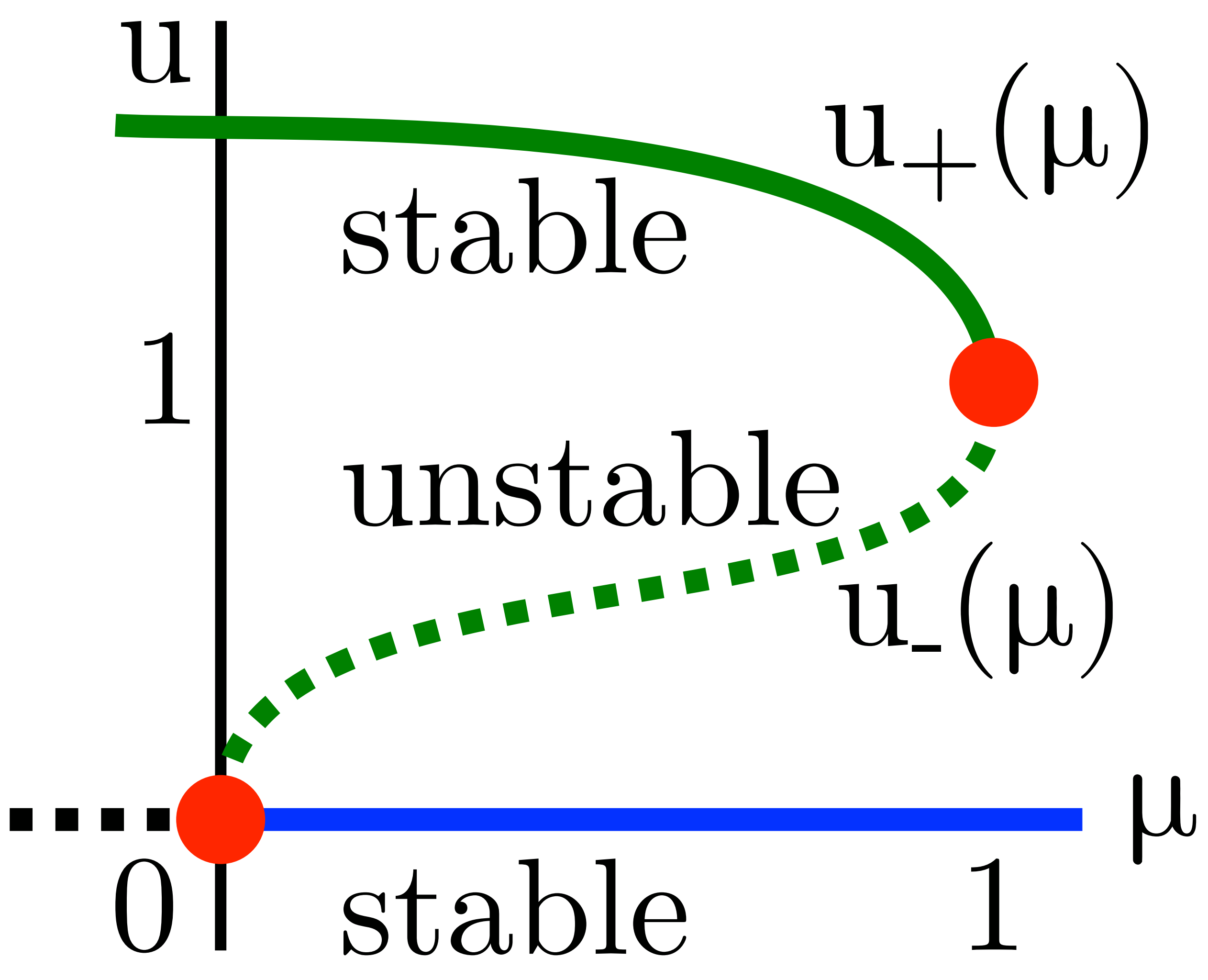
0

$$\dot{u}_n = d(\Delta U)_n + f(u_n, \mu), \quad n \in V, U = (u_1, u_2, \ldots, u_{|V|}) \in \mathbb{R}^{|V|}$$

Coupling Strength
(focus on $0 < d \ll 1$)

Graph Laplacian

Bistable Nonlinearity



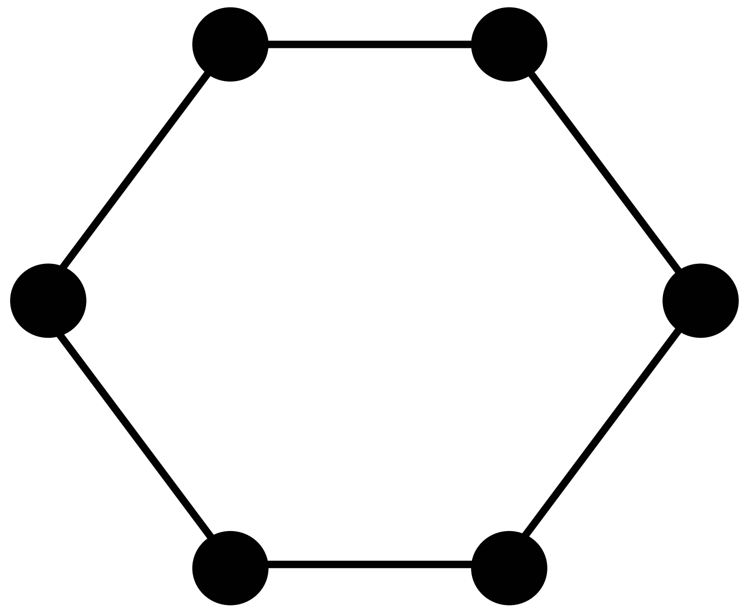
- **Setup (continued):** We focus on ring systems with $N \geq 5$

0

$$\dot{u}_n = d(\Delta_m U)_n + f(u_n, \mu), \qquad 1 \leq n \leq N, U = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$$

Coupling Operator

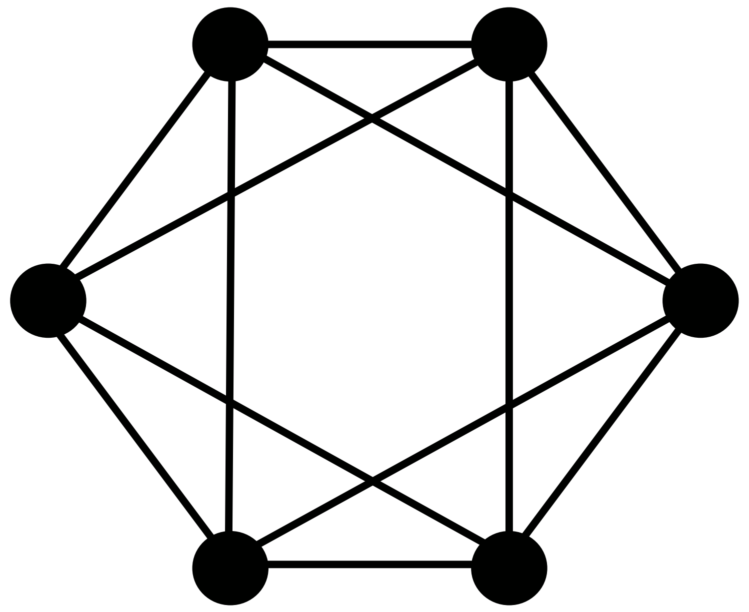
Note: $1 \leq m \leq \left\lfloor \frac{N}{2} \right\rfloor$



Sparse

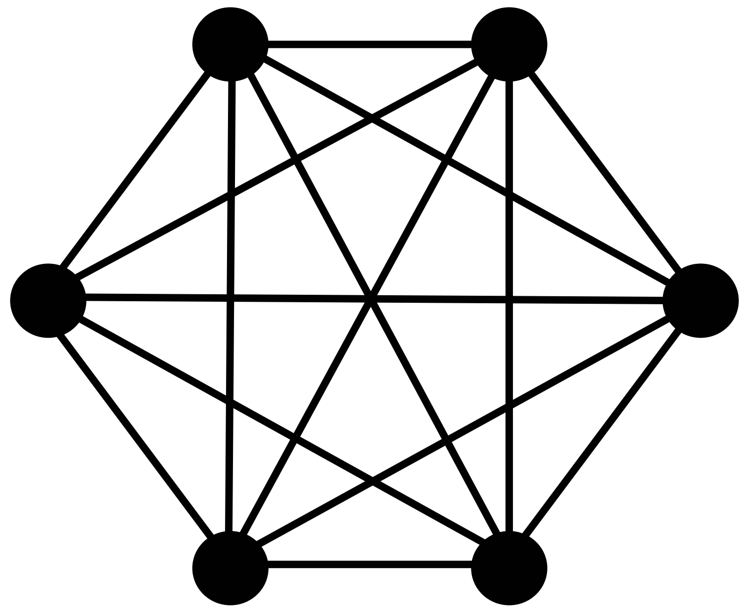
$m = 1 (N \geq 5)$

or $1,2 (N \geq 7)$



Almost All-to-all

$N$ even, $m = \frac{N}{2} - 1$



All-to-all

$m = \left\lfloor \frac{N}{2} \right\rfloor$

- We use cubic-quintic nonlinearity for demonstration: $f(u_n, \mu) = -\mu u_n + 2u_n^3 - u_n^5$

# Main Bifurcation Results

## in different coupling regimes


Sparse Coupling
*N* folds Snaking

Almost all-to-all Coupling
Still Snakes but Complicated

All-to-all Coupling
Closed curve with 6 folds always



[Tian, Bramburger & Sandstede, IMA J Appl Math (2021)]

- **Why is the all-to-all coupling case so different from the others?**

$$d(\Delta_m U)_n + f(u_n, \mu) = 0$$

$$\Rightarrow \quad d\sum_{j=1}^{N}(u_j - u_n) + f(u_n, \mu) = 0$$

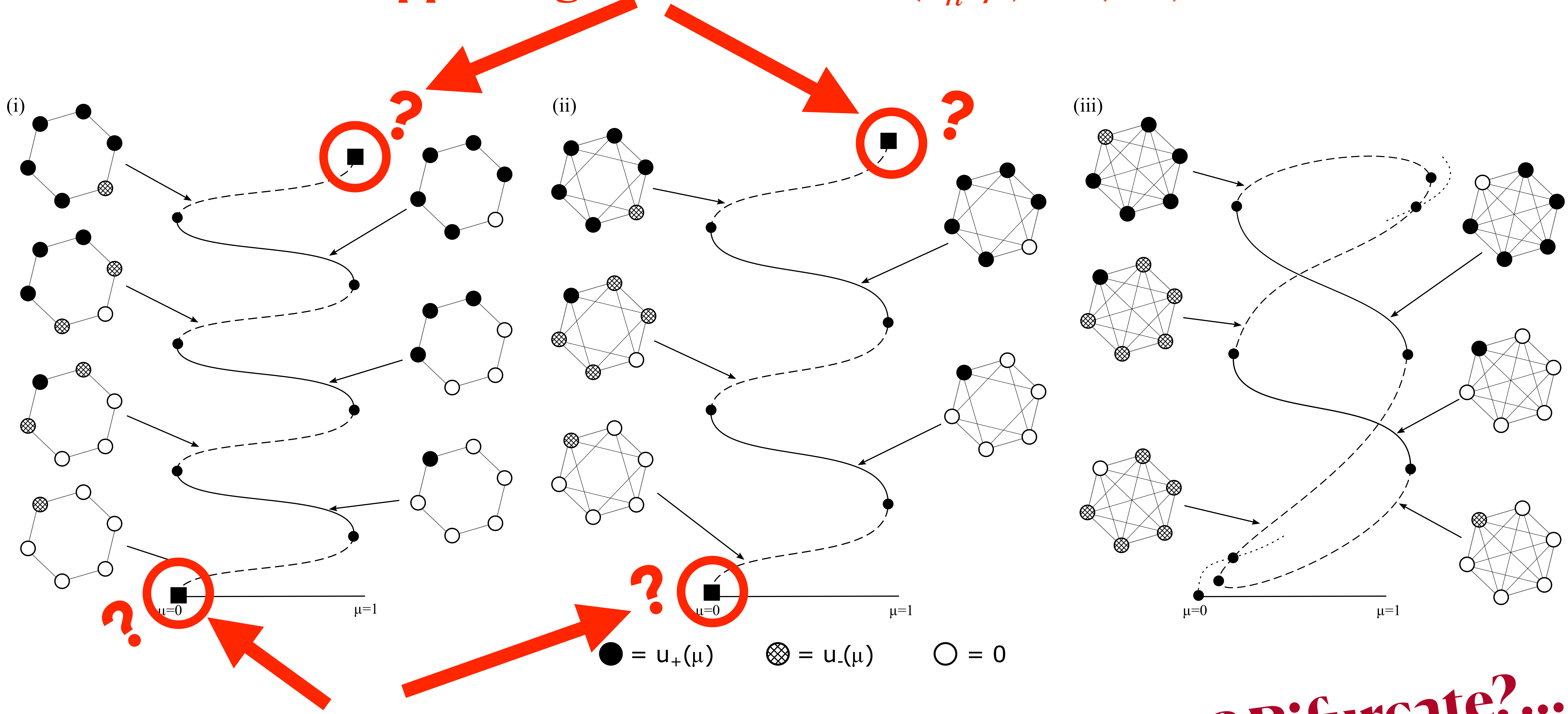
$$\Rightarrow \begin{cases} d(N-k)(v_2 - v_1) + f(v_1, \mu) = 0 \\ dk(v_1 - v_2) + f(v_2, \mu) = 0 \end{cases}$$

$v_1$ denotes the value of first $k$ nodes,

$v_2$ denotes the value of last $N-k$ nodes

★ Solutions are restricted to the $S_k \times S_{N-k}$-invariant subspace

## Upper-Right Corner near $(u_n, \mu) = (1,1)$




## Lower-Left Corner near $(u_n, \mu) = (0,0)$

**Terminate? Bifurcate?…**

# Analytical Procedures

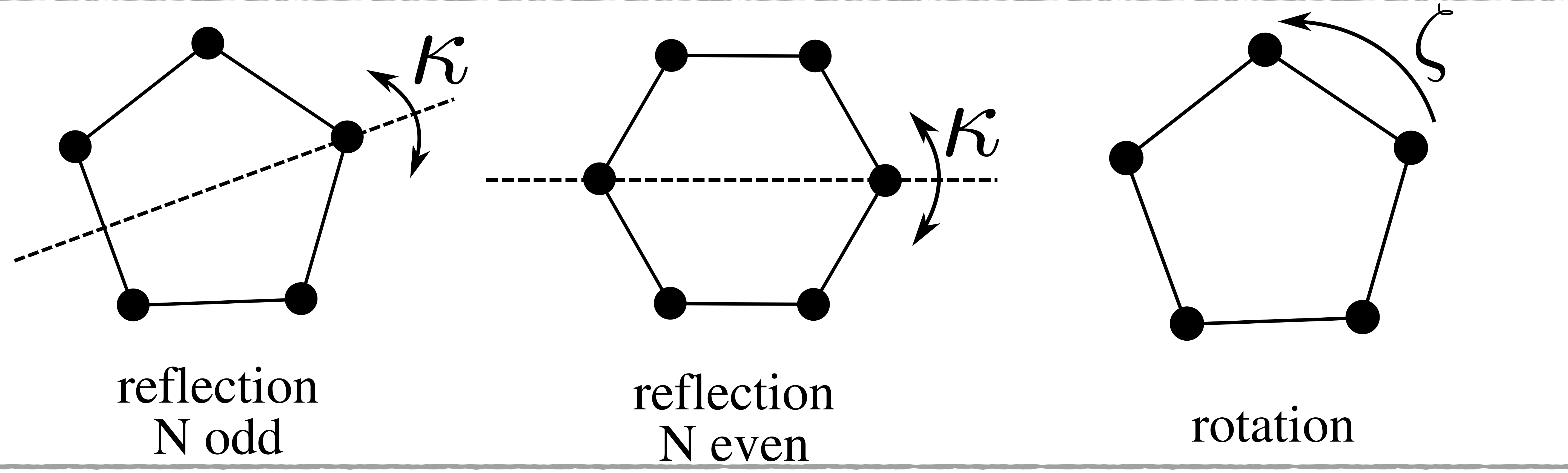
**for proving bifurcation from homogeneous solution**

- Consider homogeneous solution branch $u_{-}(\mu)\mathbf{e}$ where $\mathbf{e} = (1,1,\ldots,1) \in \mathbb{R}^N$

- Find bifurcation points $\left(U_*^{(j)}(d), \mu_*^{(j)}(d)\right)$ along the homogeneous branch where $j = 0,1,\ldots,N-1$

- Focus on the $j = 1$ case (since it aligns with the numerical simulation)

## Group Actions




- Reduce our system to normal form with respect to the lower-left (or upper-right) regime by rescaling (state variable $u_n \to v_n$, bifurcation parameter $\mu \to \lambda$)

- Perform center manifold reduction and apply bifurcation theory with dihedral symmetry $D_N$ †, which allows us to understand the existence of bifurcation branches as well as their criticality



† [Golubitsky *et al., Singularities and Groups in Bifurcation Theory* (1988)]

**Assumption:** The null space of the Jacobian at the bifurcation $\left(U_*^{(1)}(d), \mu_*^{(1)}(d)\right)$, i.e., the center manifold, is $2$-dimensional. **[proved for $m = 1,2$]**

**Theorem:**

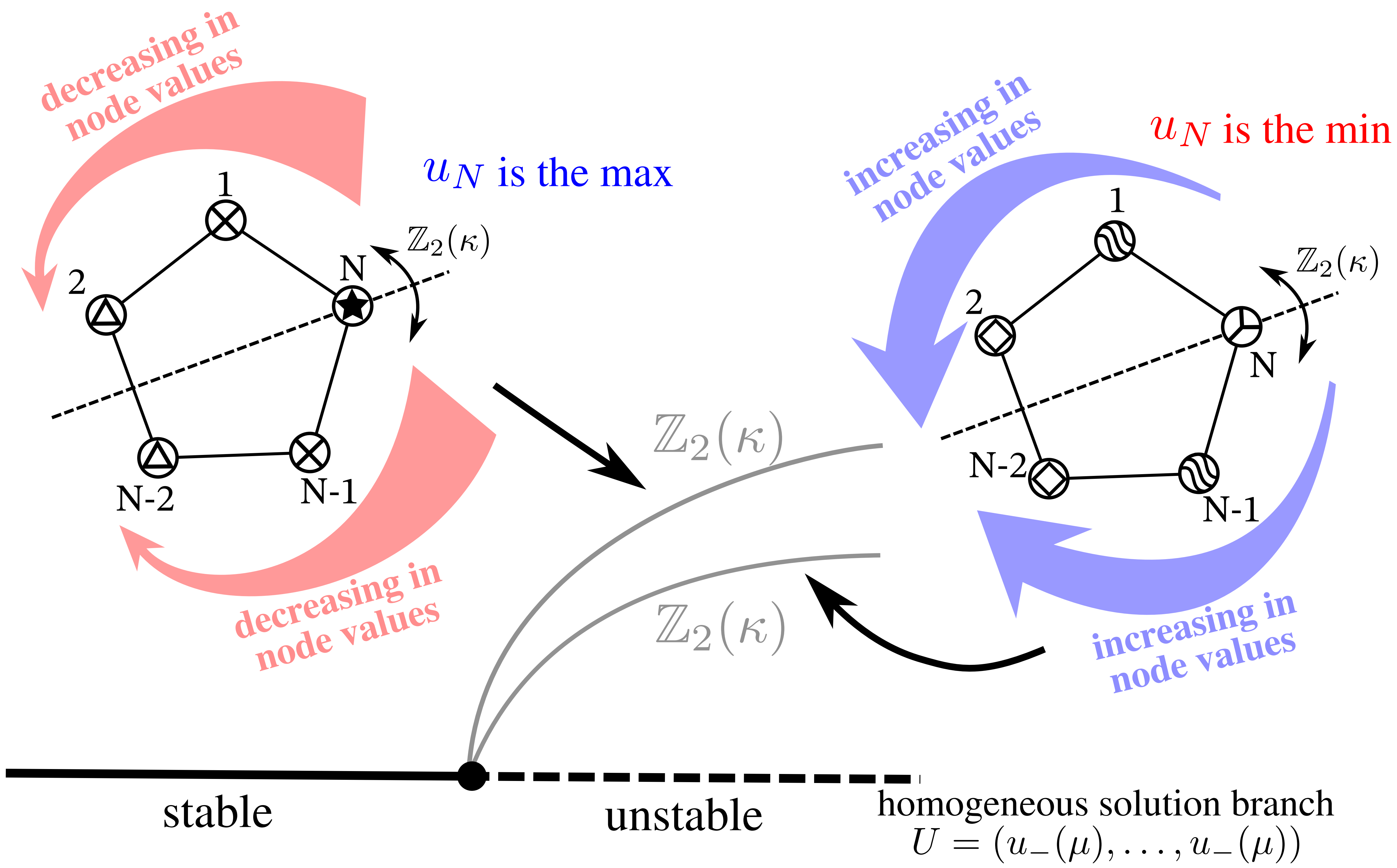
1. There exist two distinct bifurcation branches from the homogeneous solution $u\_(\mu)\mathbf{e}$. [Detailed illustrations on next page]

2. For any given $N, m$, we can predict the bifurcation criticality by explicit calculation of a formula that depends only on $N$ & $m$.

3. As $N \to \infty$, the bifurcation criticality changes from supercritical to subcritical at $\dfrac{m}{N} \approx 0.39$ in the lower-left corner regime, and $\dfrac{m}{N} \approx 0.41$ in the upper-right corner regime.

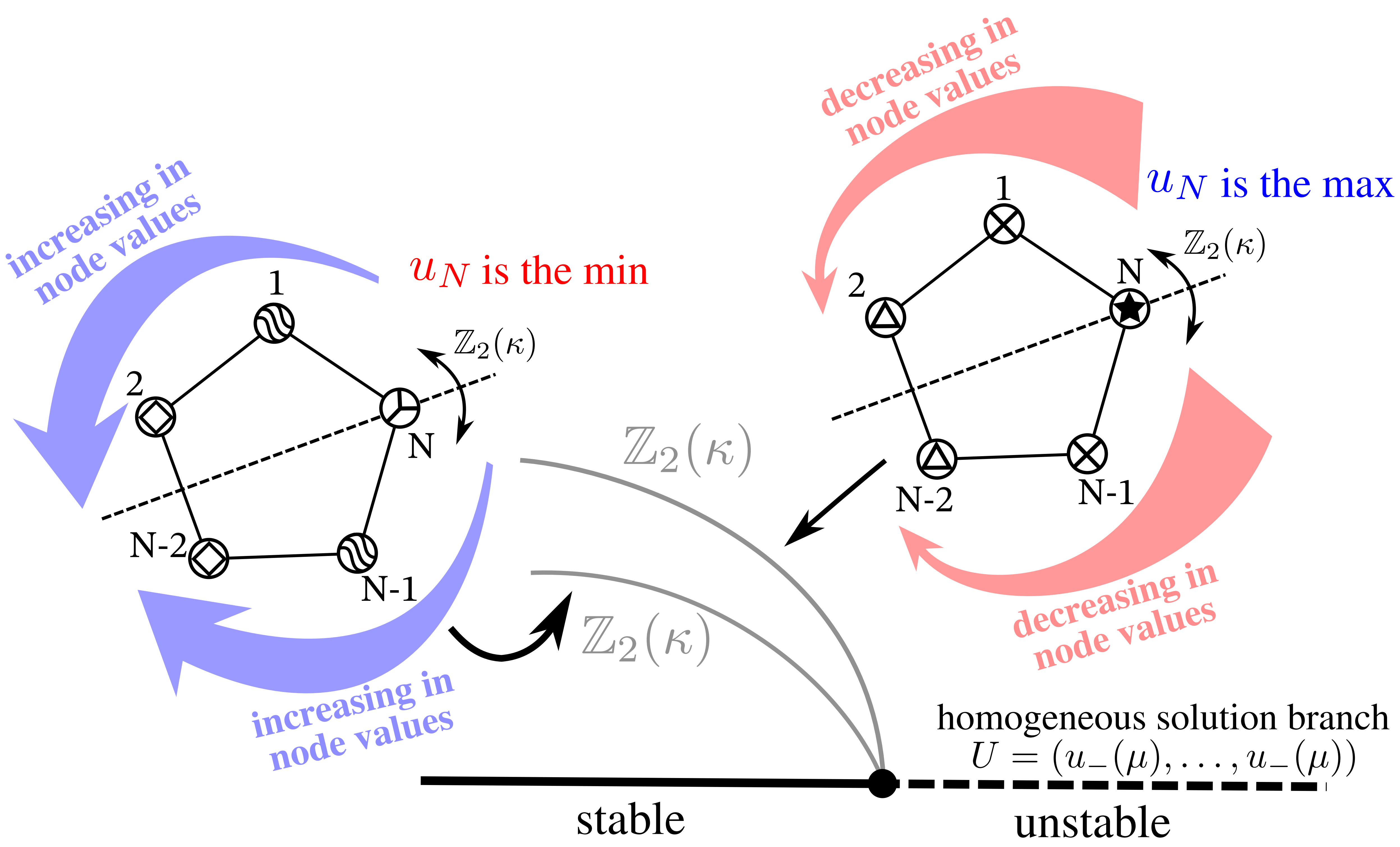
# Illustration of Bifurcation Branches & Patterns

**[$N$ odd]**




**supercritical**

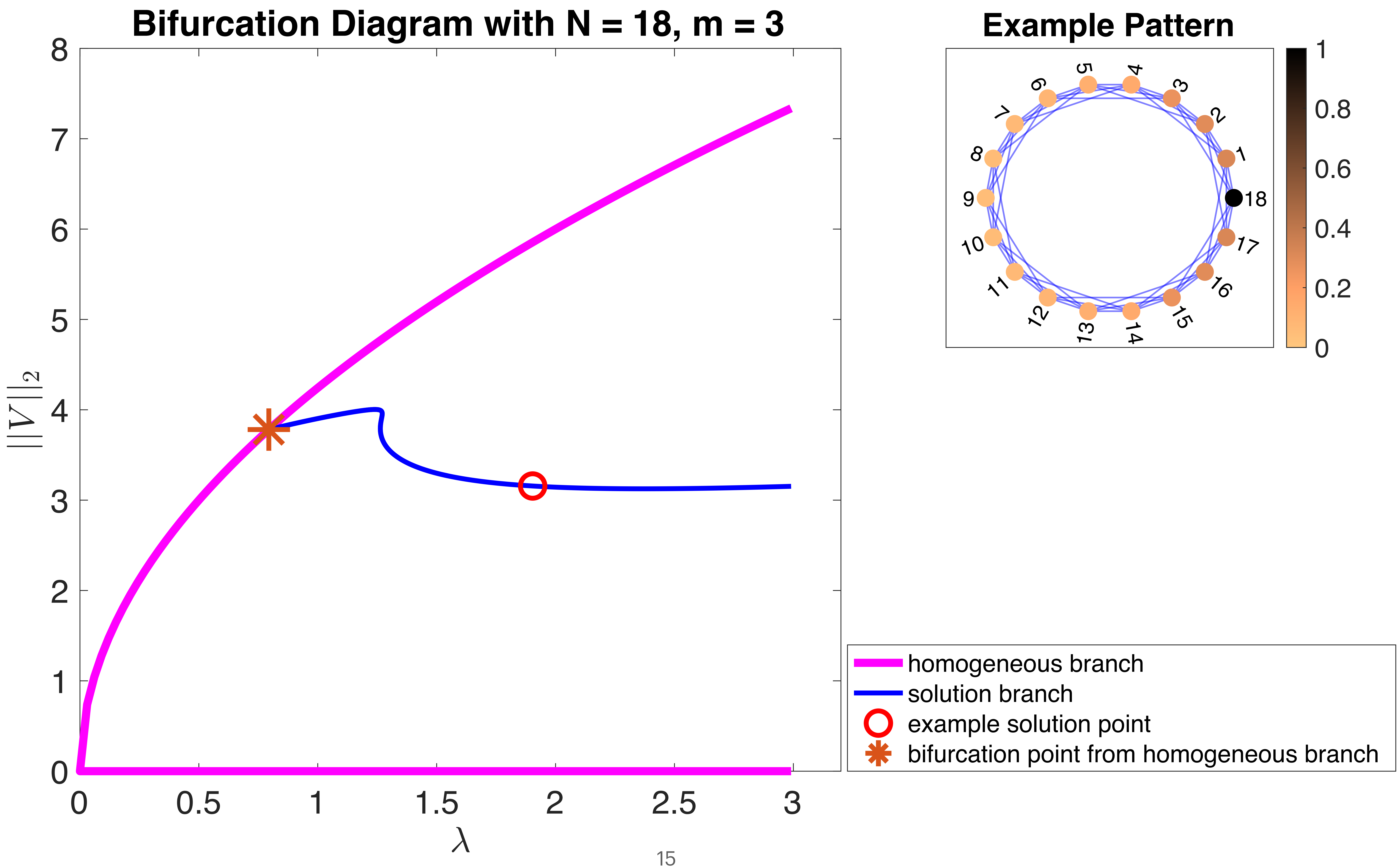
**subcritical**



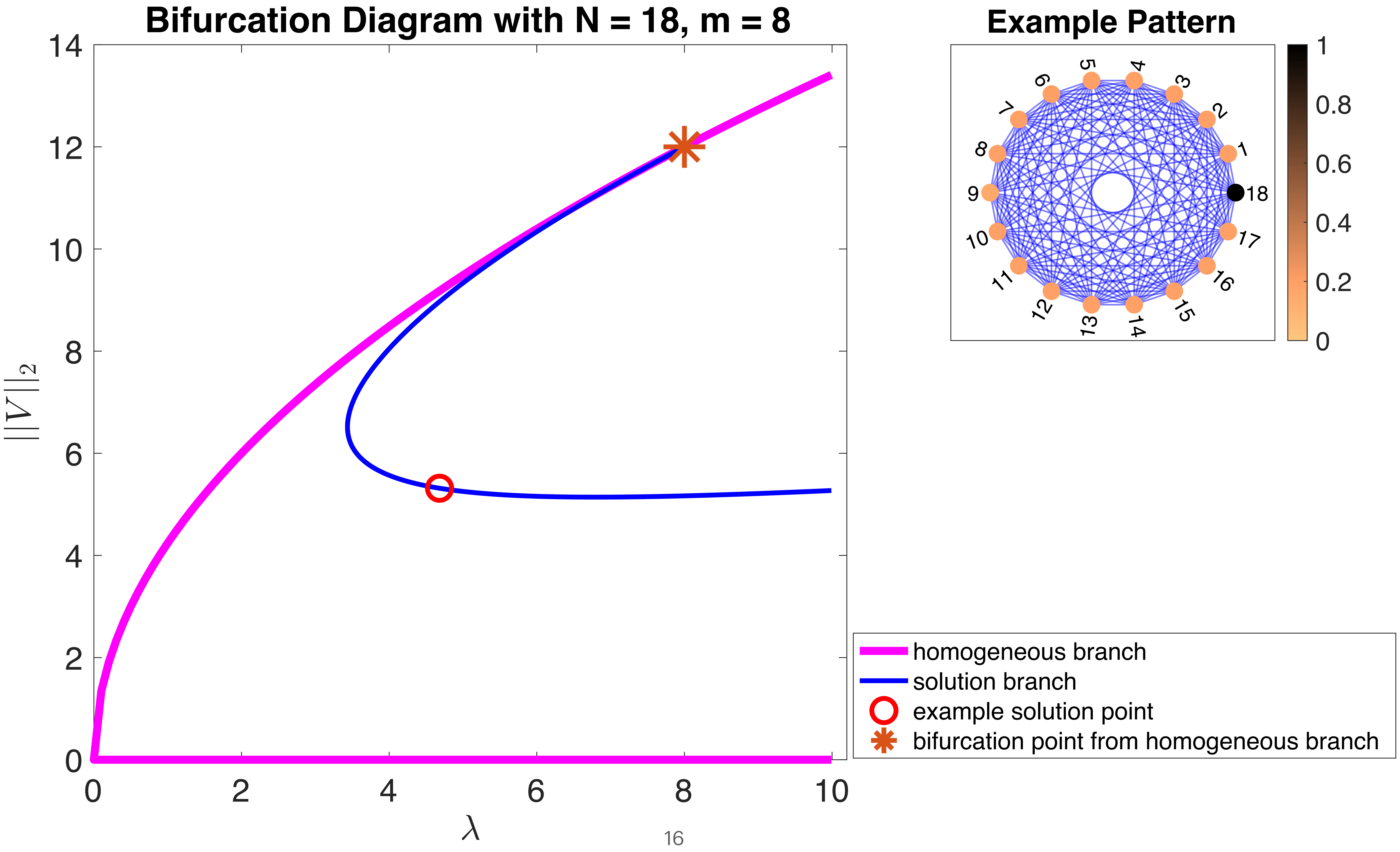
# Illustration of Bifurcation Branches & Patterns

**[$N$ even]**

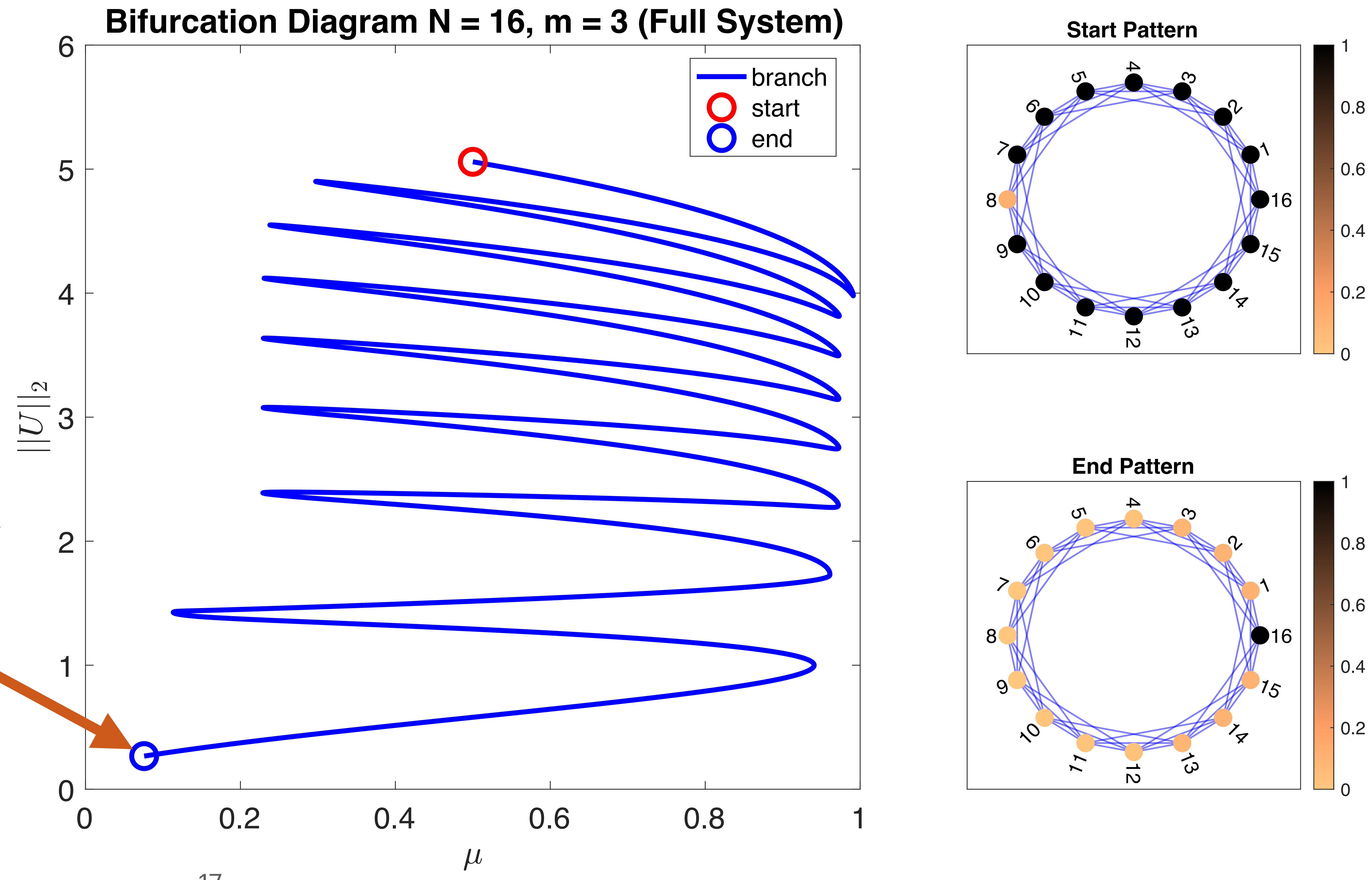
# Numerical Simulations



# Bifurcation location and criticality align with the analytical predictions!

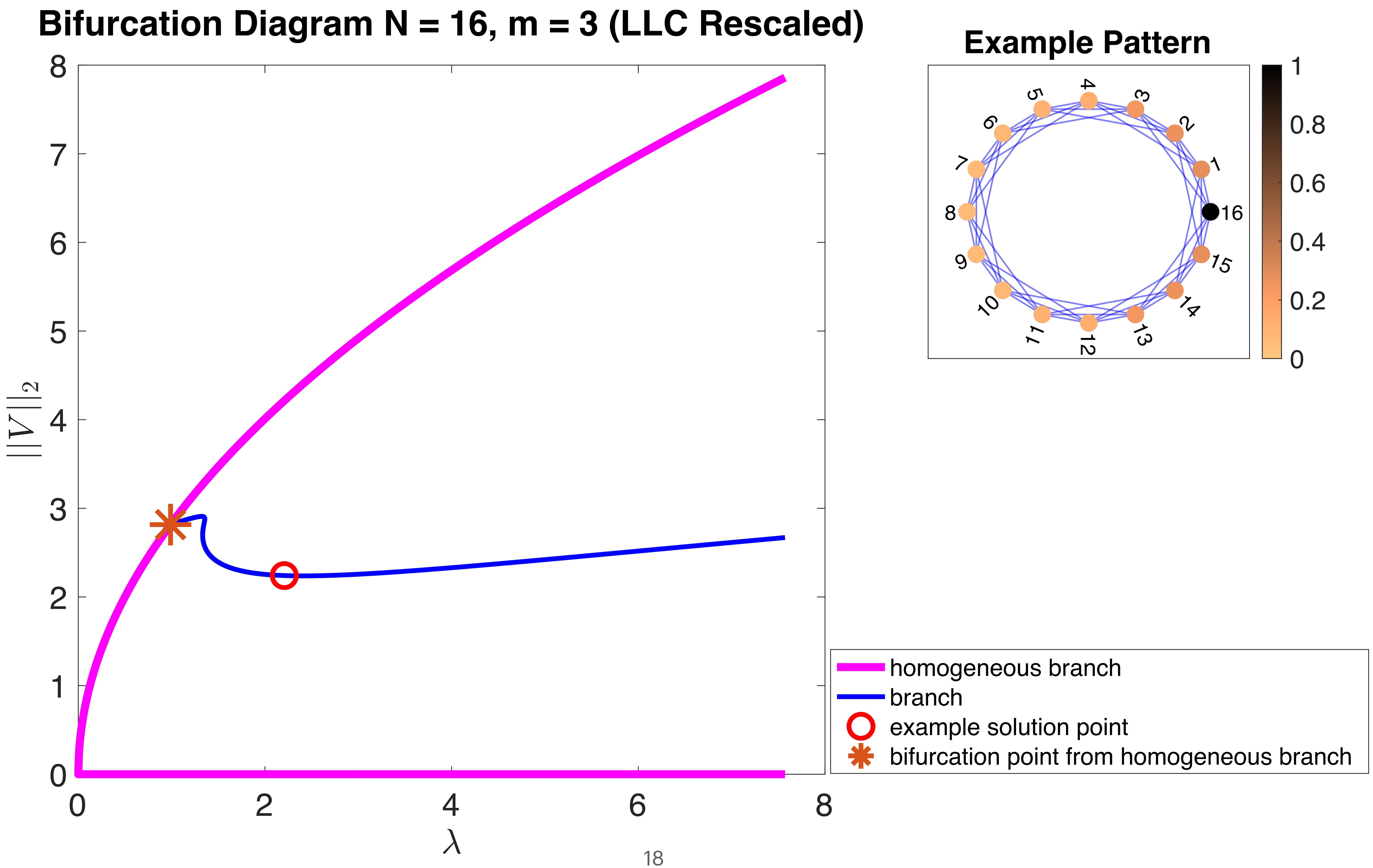


We verify numerically that the bifurcation branch of emergent patterns from the homogeneous solutions connect with the snaking branches of patterns in the original system.

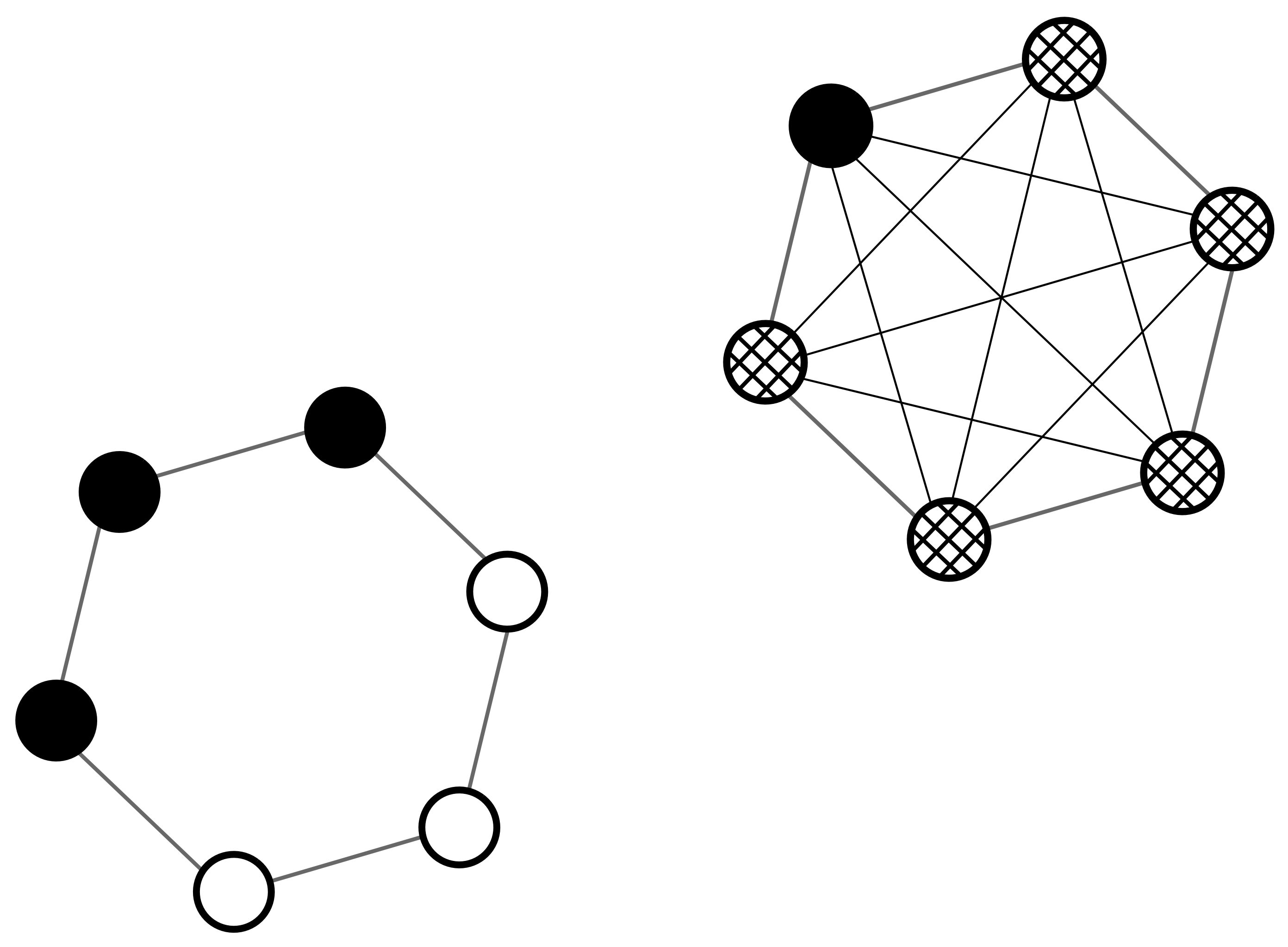



Then continue in the rescaled system of the lower-left corner from here

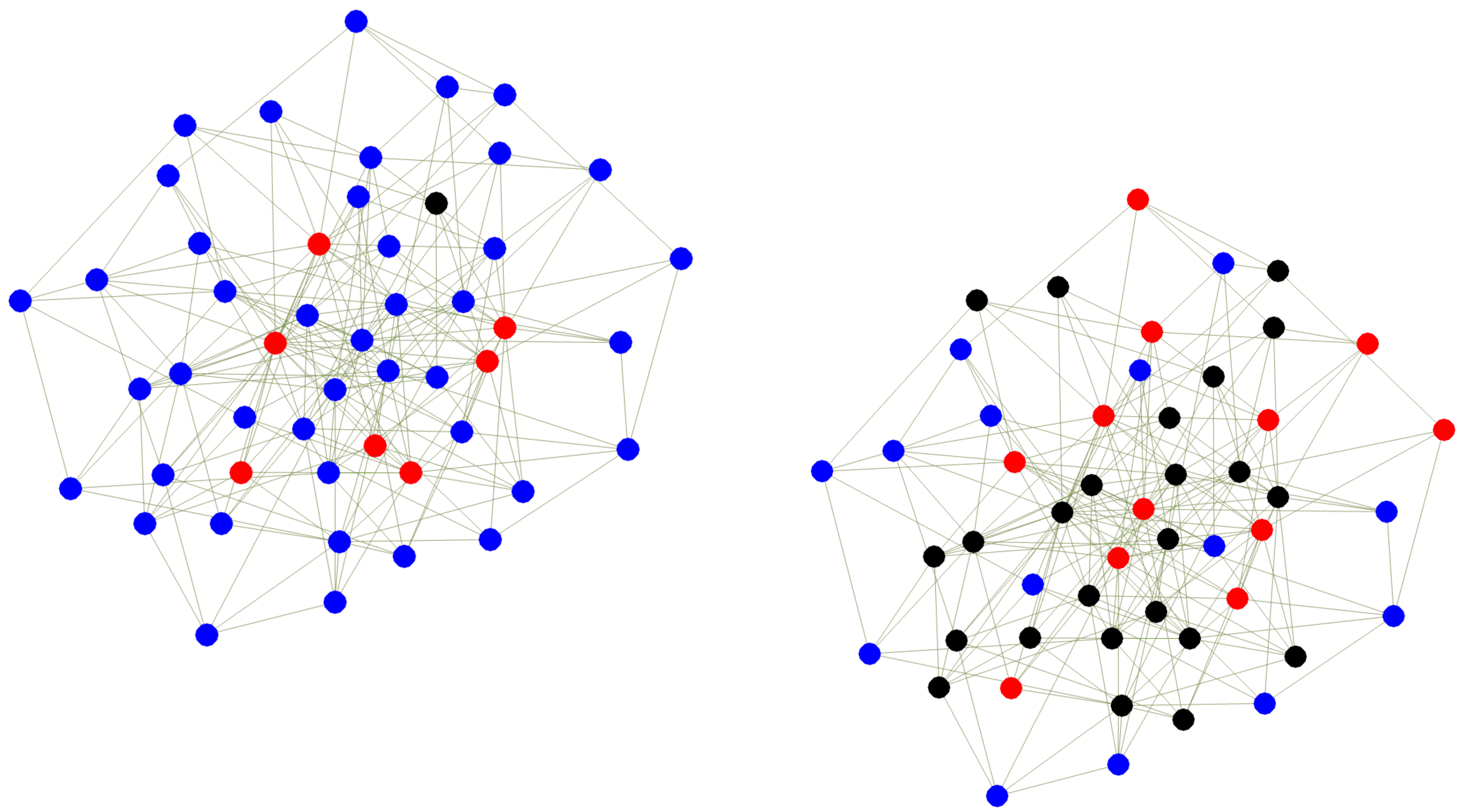
We verify numerically that the bifurcation branch of emergent patterns from the homogeneous solutions connect with the snaking branches of patterns in the original system.



## Localized Patterns on Graphs



## Modeling Coupled Online & Offline Dynamics of Protesting Activity on Networks



- **Motivating Question:** How do protests spread?

- **Fact:**

Online-offline spillovers: Over 60% of the global population uses social media (according to Digital 2024 Global Overview Report by DataReportal). Social media is transforming how, when, and where conflicts occur.

  - We need a framework that incorporates the online-offline connection.

Involves Network!

- We use multi-layer network compartmental model to incorporate the coupled effects of online engagement and offline protests.

U: Uninterested

E: Engaged

D: Disengaged

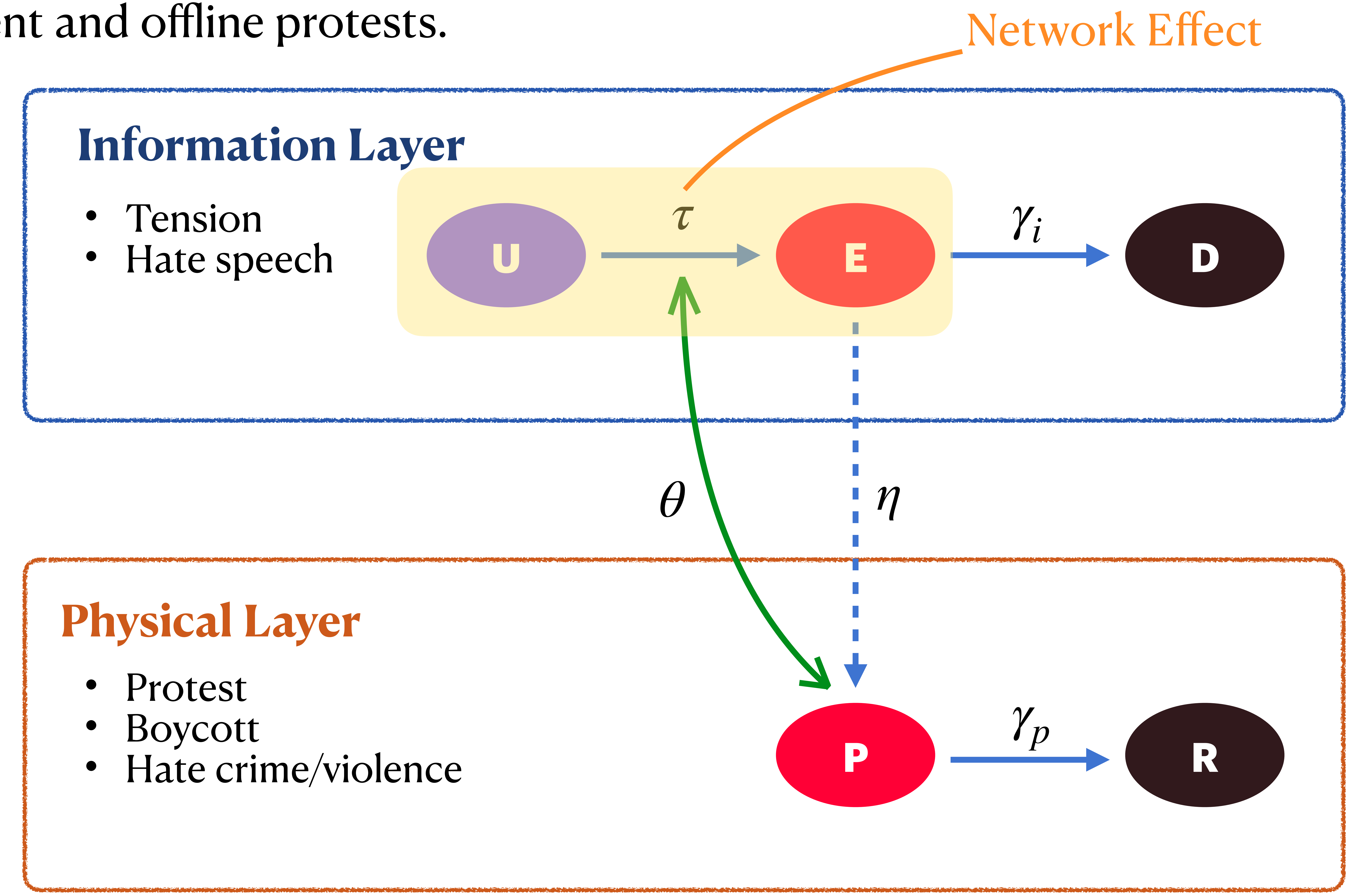
| |
|---|
| $\tau$ : transmission rate |
| $\gamma_i$ : recovery rate |
| $\eta$ : transmission rate |
| $\gamma_p$ : recovery rate |
| $\theta$ : self-excitement |

NP: Non-protesting

(Not shown; default state)

P: Protesting

R: Recovered from Protesting



[X]: Expected # of individuals in state X

[XY]: Expected # of Y neighbors of all Xs

- Markovian Stochastic Process -> ODE Model

**Online**

$$\begin{cases} \dfrac{d[U](t)}{dt} = -\tau[UE] - \dfrac{\theta}{N}[U][P] \\ \dfrac{d[E](t)}{dt} = \tau[UE] + \dfrac{\theta}{N}[U][P] - (\eta + \gamma_i)[E] \\ \dfrac{d[D](t)}{dt} = (\eta + \gamma_i)[E] \end{cases}$$

**Offline**

$$\begin{cases} \dfrac{d[P](t)}{dt} = \eta[E] - \gamma_p[P] \\ \dfrac{d[R](t)}{dt} = \gamma_p[P] \end{cases}$$

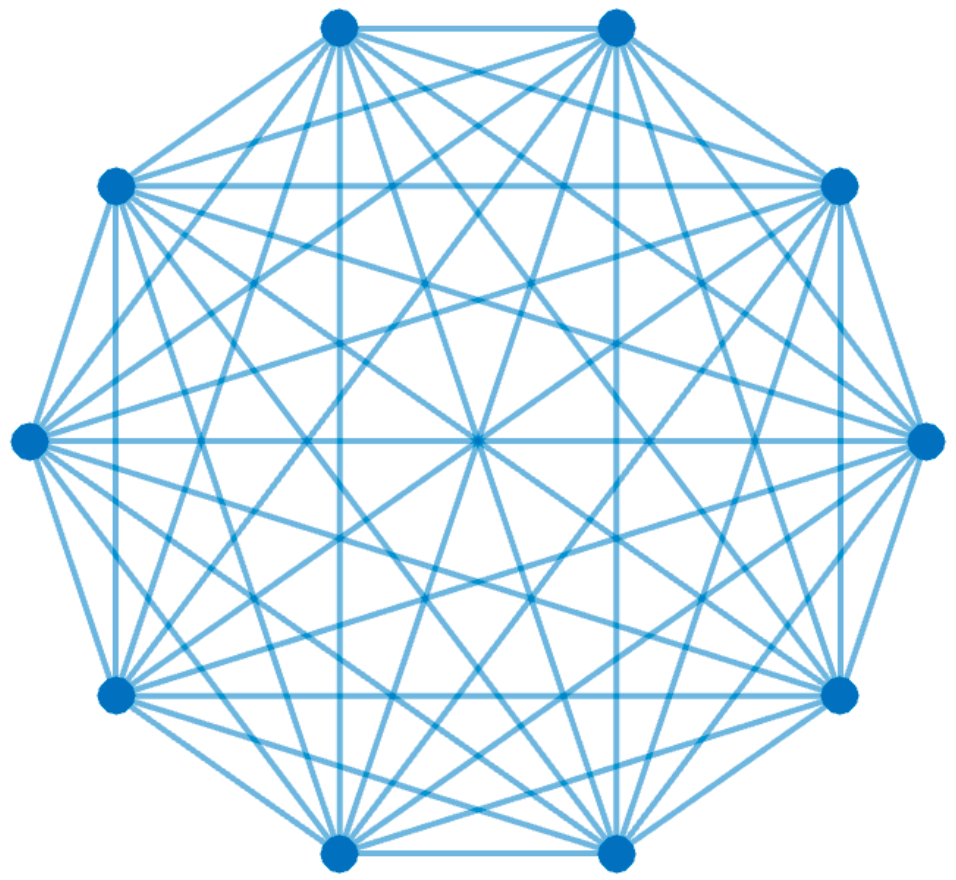
- **Main task**:

  We aim to approximate the [UE] term in the model to capture the network topology, ensuring it is “good enough” to achieve relatively accurate proportions without excessive computational cost.

- **Method**:

  Mean-field approximations according to network types

# Fully-Connected Network



An average engaged individual has approximately $[U]/N$ uninterested neighbors.

-> approximate [UE] by $\frac{[U]}{N}[E]$



N: total # of nodes

[XYZ]: Expected # of triples with Y having both X&Z neighbors

- For other types of networks, it may be necessary to capture finer-grained details by accounting for edge evolution:

$$\begin{cases} \dfrac{d[UE](t)}{dt} = -(\gamma_i + \eta + \tau)[UE] + \tau([UUE] - [EUE]) + \dfrac{\theta[P]}{N}[UU] - \dfrac{\theta[P]}{N}[UE] \\ \dfrac{d[UU](t)}{dt} = -\tau([UUE] + [EUU]) - \dfrac{2\theta[P]}{N}[UU] \end{cases}$$

- This involves **pairwise approximation** through triplets $[UUE]$ and $[EUE]$

## Homogeneous (k-regular graph)

**Single-Level Approximation**

Assume that engaged individuals are distributed randomly, and close the pair $[UE]$ accordingly.

**Pairwise Approximation**

Assume that $[UU]$ and $[UE]$ connections originating from an uninterested person are uniformly distributed, respectively, in triplet closure.

## Heterogeneous

Group node states based on their degrees

**Single-Level Approximation**

It follows a similar approach to the homogeneous single-level model, with the added differentiation by degrees.

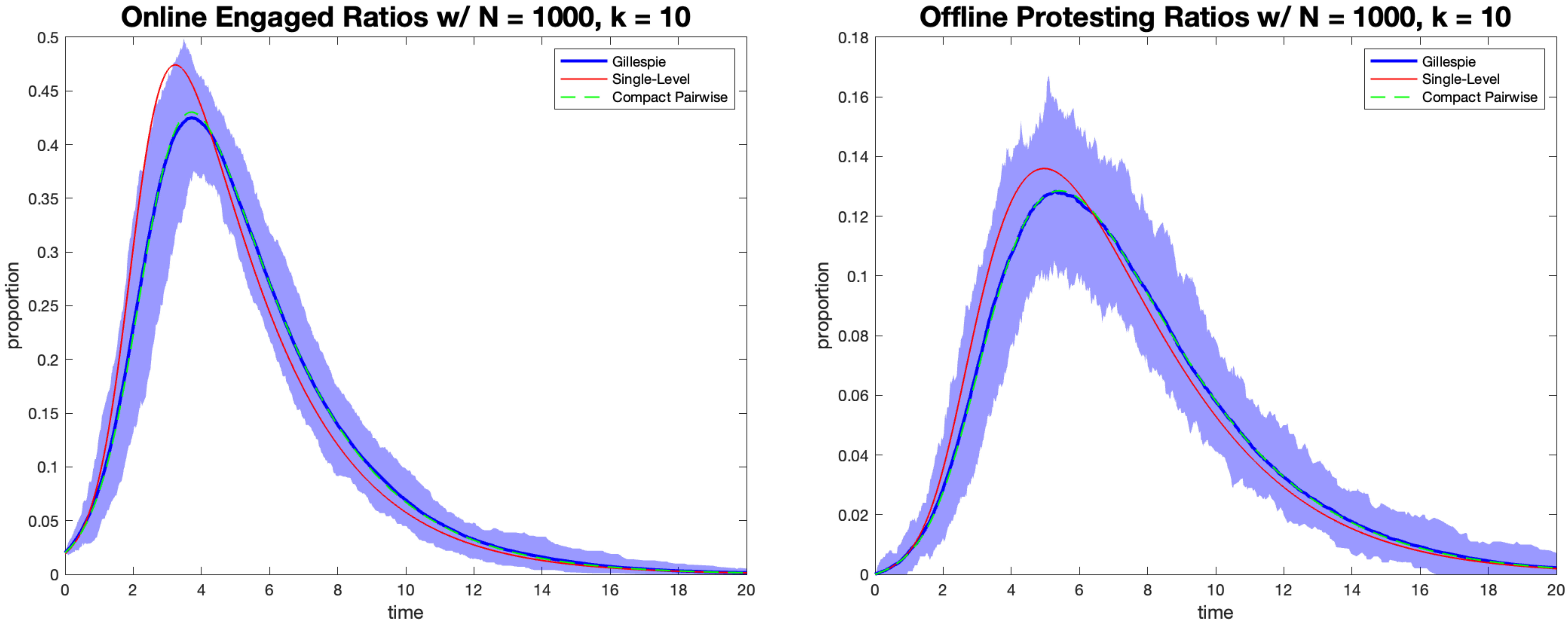
**Compact Pairwise Approximation**

Assume that the neighbors of all uninterested individuals are interchangeable when closing triplets, with respect to different degrees.



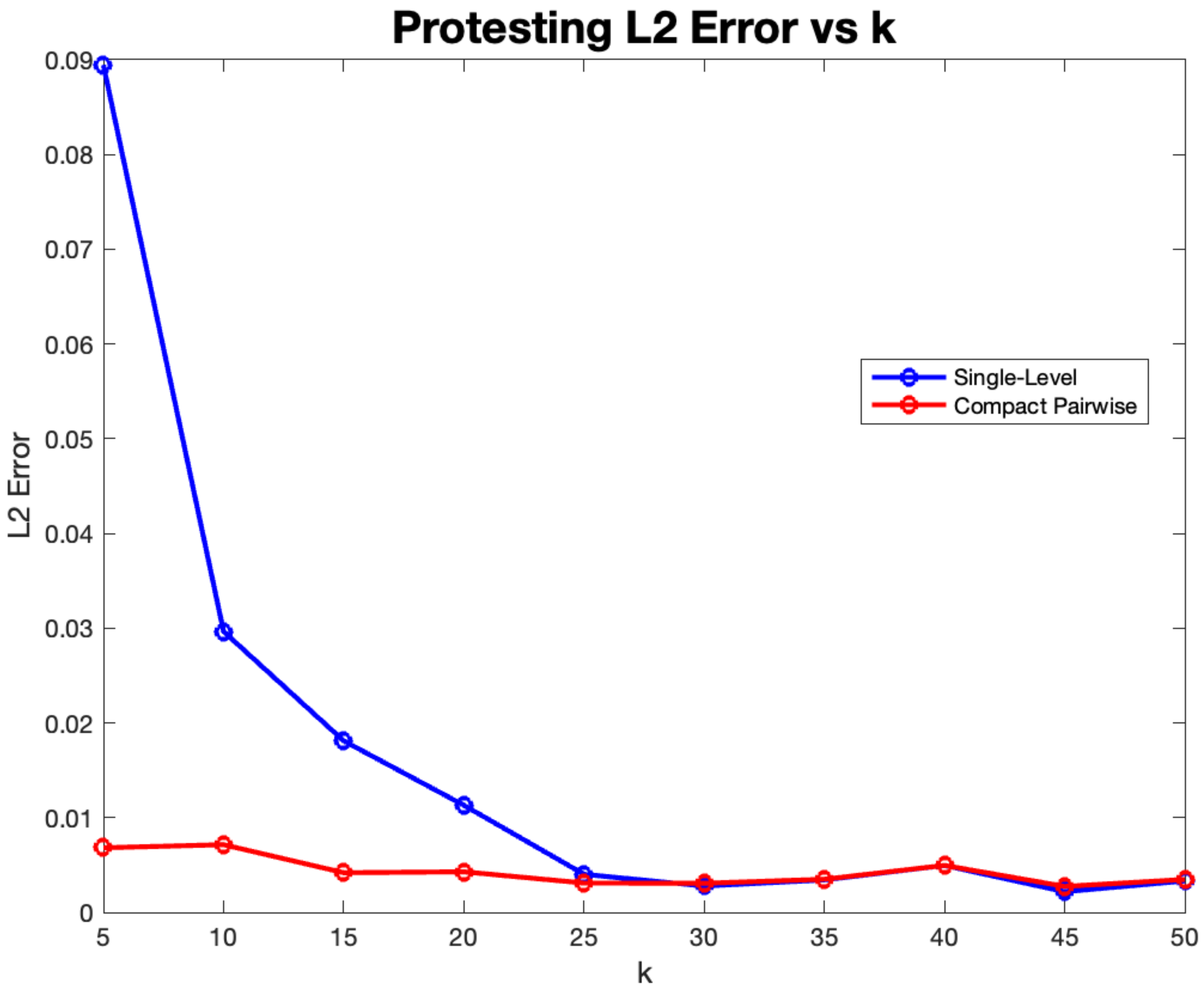
[Kiss, Miller & Simon, *Mathematics of Epidemics on Networks: From Exact to Approximate Models* (2017)]

# Stochastic vs. ODE Approximation




k: average degree

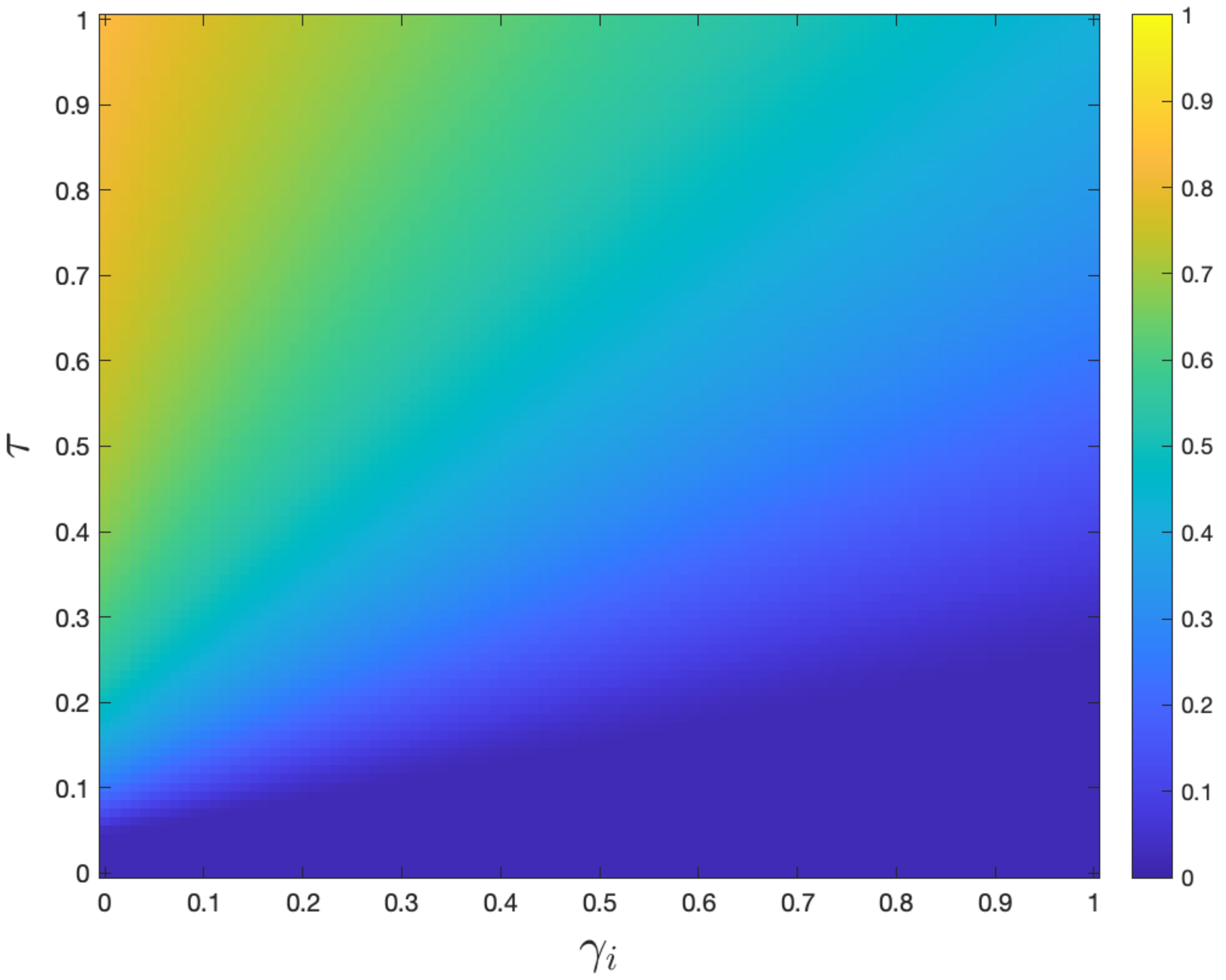
# Network Connectivity and Approximation Error

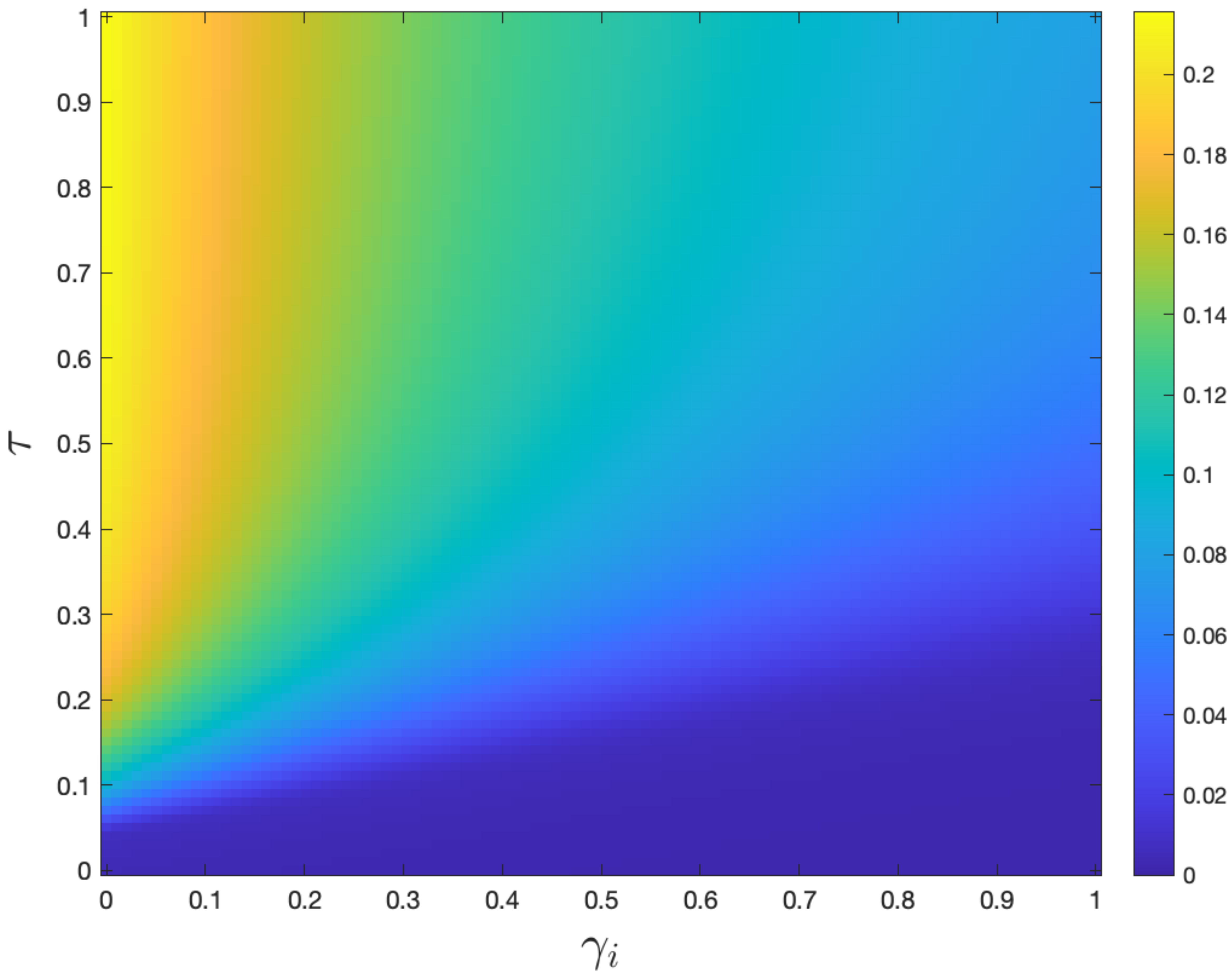

$N = 1000$

# Example for Parameter Sweep

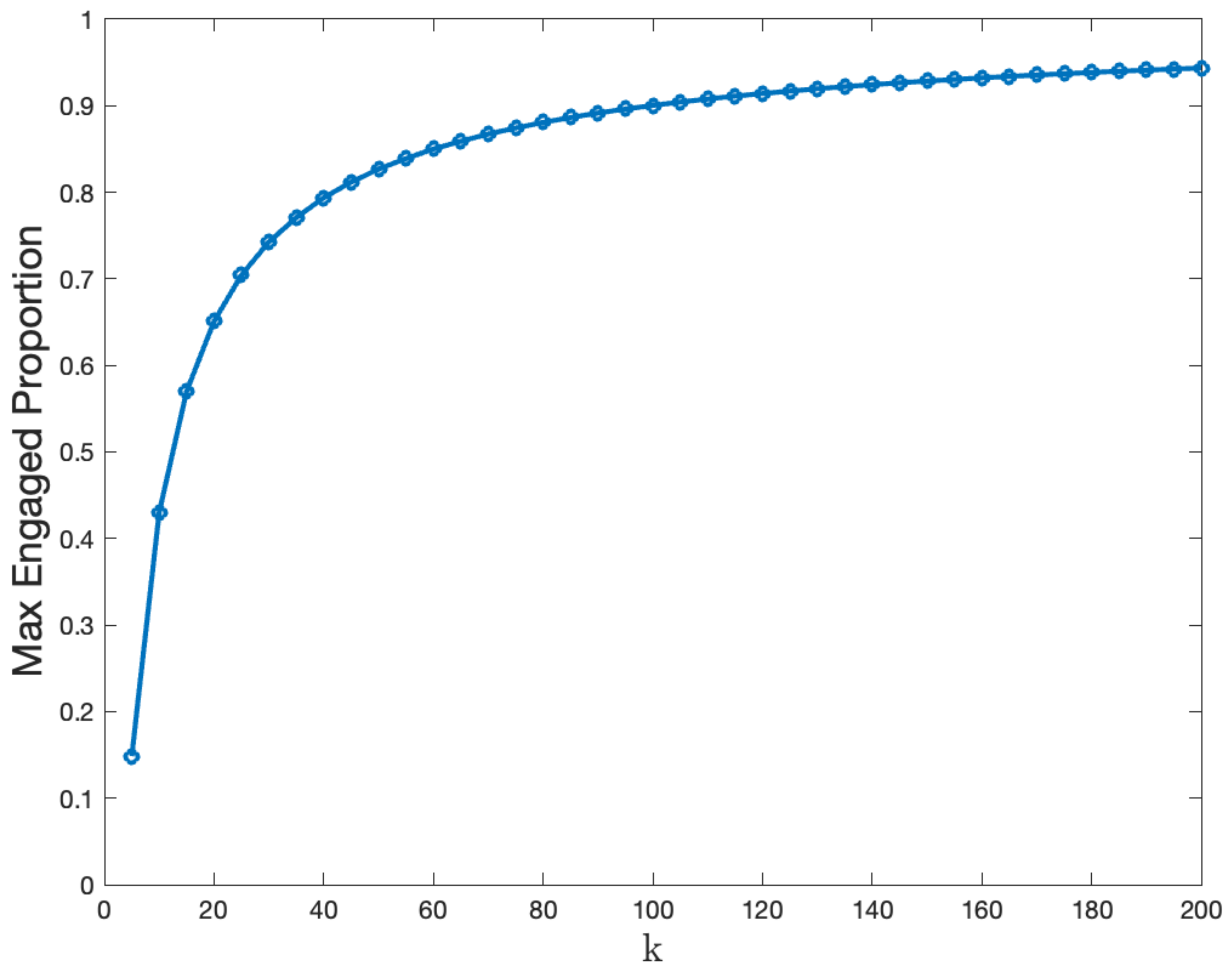
## Max Online Engagement Ratios

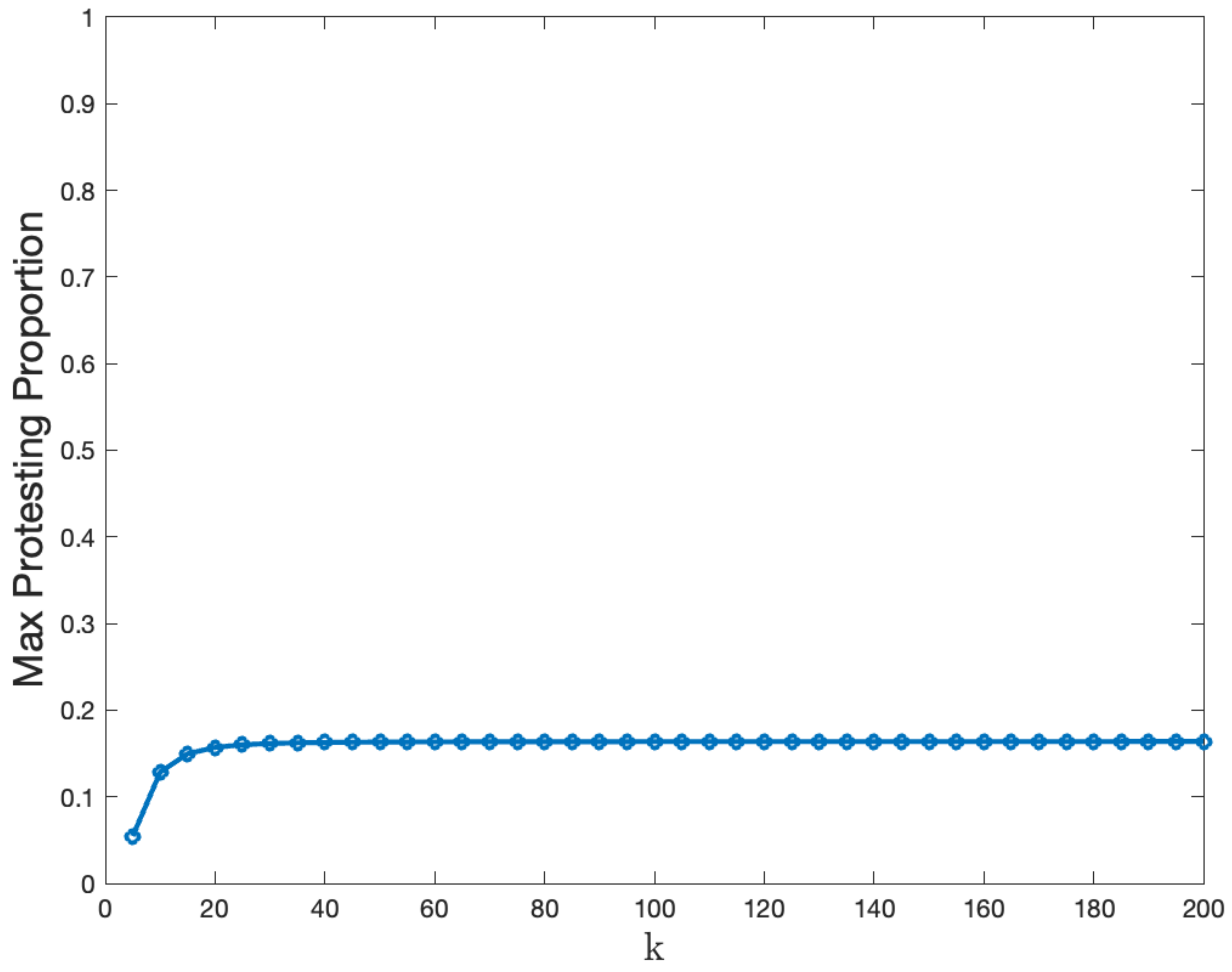



## Max Offline Protesting Ratios



# Network Connectivity and Outburst Size



# Summary

- **Localized Patterns on Ring Lattices**
  - We proved the existence of snaking bifurcation curves for sparse coupling and closed curve for all-to-all coupling.
  - We verified that the snaking curve bifurcate from homogeneous solutions and showed that their criticalities can be determined analytically with any $N, m$.
- **Modeling Coupled Online & Offline Dynamics on Networks**
  - We developed mean-field approximations for coupled continuum models on networks.
    - Coarse-grained predictions at varying levels of granularity offer a balance between alignment with the full original model and significant computational efficiency.
    - This enables further analysis and paves the way for future integration with data.

# References

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