Singular amplitude equations and Eckhaus type stability criteria for convective Turing bifurcation with conservation laws

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I. Intro: Classical Turing bifurcation/amplitude equations

"The chemical baisis of morphogenesis," 1952.

Brilliant mechanism for pattern formation

Homogeneous state $u(x,t) \equiv u_0$ of physical time-evolution

$$\partial_t u = \mathcal{F}(u, \partial_x u, \dots, \partial_x^m u), \qquad \mathcal{F}(u_0, 0, \dots, 0) = 0, \tag{1}$$

for definiteness $m = 2$ (reaction-convection-diffusion).
Linearized equations

$$u_t = (A\partial_x^2 + B\partial_x + C)u, \tag{2}$$

Fourier transform

$$\hat{u}_t = (-Ak^2 + Bik + C)\hat{u}, \qquad \text{symbol } S(ik) = -Ak^2 + Bik + C. \tag{3}$$

IDEA: $(B = 0)$ Stable reaction (C) and diffusion (A) yield stability of S for small, large k. But noncommutation of matrices can yield instability at finite wave number k ∼ periodic patterns...

Graphical view via dispersion relation

Destabilization of constant state as bifurcation parameter is varied. Spectra given by dispersion relation, i.e., e-values of Fourier symbol, $\lambda(k) \in \sigma(-k^2A + ikB + C)$ (reaction convection diffusion).

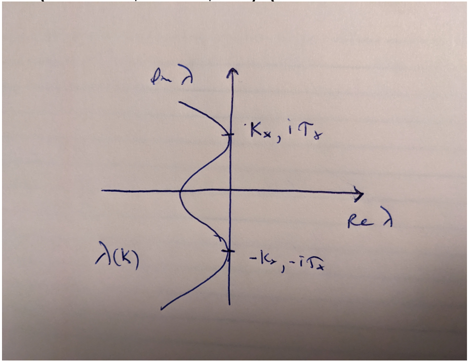

Figure: Dispersion rel. at bifurcation (critical curve). Transl. inv. $\Rightarrow$ conjugate symm.; $x \to x - (\tau_*/k_*)t \to$ double root at $\tau_* = 0$, SO(2) bif to periodic traveling wave $\bar{u}(kx + \tau t)$, $\tau = \tau(k)$.

Eckhaus' weakly unstable approximation

Let r, $\lambda(k_*)$ be critical e-vectors/values of $S(\mu) = -k^2 A(\mu) + ikB(\mu) + C(\mu)$ at bif. point $\mu = 0$, $\Rightarrow$ exact nondecaying spatially-periodic solution of the linearized equations

$$u(x,t) = e^{i(k_* x + \Im\lambda(k_*,0)t)} r + c.c. \tag{4}$$

remaining modes time-exponentially decaying at different rates.

For $\mu \sim \epsilon^2$ (weakly unstable), seek formal asymptotic solution

$$\begin{aligned} U^\varepsilon(x,t) &= \frac{1}{2}\varepsilon A(\hat{x},\hat{t}) e^{i\xi} r + \mathcal{O}(\varepsilon^2) + c.c., \\ \xi &= k_*\Big(x + \frac{\Im\lambda(k_*,0)}{k_*} t\Big). \end{aligned} \tag{5}$$

Note: complex vs. real, SO(2) vs. O(2) symmetry.

Amplitude equations and Eckhaus stability criterion

With scaling $\hat{x} = \varepsilon(x + \Im\partial_k\lambda(k_*, 0)t)$, $\hat{t} = \varepsilon^2 t$, automatic solution to order $O(\varepsilon^2)$. At $O(\varepsilon^3)$, get complex Ginzburg-Landau eqn.:

$$A_{\hat{t}} = -\frac{1}{2}\partial_k^2\lambda(k_*, 0)A_{\hat{x}\hat{x}} + \partial_\mu\lambda(k_*, 0)A + \gamma|A|^2 A. \qquad \text{(cGL)}$$

NOTE: 1. Explicit (periodic) exponential solutions

$$A = e^{i(\kappa\hat{x} + \omega\hat{t})}\alpha, \quad \alpha \equiv \text{constant}. \qquad (6)$$

2. Linearized e-value equations about periodics reduce to constant-coefficient, explicitly solvable! (2×2 disp. relation.)

Resulting conditions are Eckhaus "sideband" stability conditions

Rigorous verification ([Mielke-Schneider], early 90's)

Chris Jones: "If you understand the existence problem well enough then you understand the stability problem" (Here, pushed to limit!)

Existence: In simplest O(2) (stationary) setting, seek Hopf bifurcation

$$Lu = \mu M(\mu)u + N(\mu, u), \quad u \text{ periodic}$$

L const-coeff, dim(kernel)=2, $N = O(|u|^2)$. Lyapunov-Schmidt reduction: Letting v, w denote components in ker and range of L, solve for w in terms of v, $\to$ 2D O(2) (radial) bifurcation, $\mu \sim \epsilon^2$.

Stability: Bloch-Floquet spectrum solves $\lambda u = (L + O(\sigma) + O(\epsilon))u$, reducible (again) by L-S reduction to 2×2 matrix problem $\mathcal{M}(\lambda, \sigma)r = \lambda r$. By direct comparison, agrees to lowest order with dispersion relation for (cGL), $\Rightarrow$ (eventually) Eckhaus condition necessary and sufficient for spectral stability.

Summary, and challenge...

All three huge results. Together, definitive: used constantly in applications from engineering to quantum physics, basis for systematic exploration of pattern formation.

BUT: (our motivation: not unique...) does not apply to modern biomorphology models featuring conservation laws... e.g., Ambrosi-Gamba-Serini model for vasculogenesis (2004):

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = \beta \nabla c - \gamma \rho u,$$
$$\partial_t(c) - D\Delta c = \alpha\rho - \tau^{-1} c, \tag{7}$$

ρ and u density and velocity of endothelial cells, P pressure, γ drag coefficient against extracellular matrix, c chemo-attractant concentration, α and β release and cell response rates, τ half-life.

GOAL: Analogs of these important tools in cons. law case!

II. Prequel: the model of Matthews-Cox

Matthews and Cox (2000), for O(2) symmetric models (e.g., binary mixtures) with conservation laws, deduced by symmetry that amplitude equations should have form

$$\begin{aligned} A_{\hat{t}} &= aA_{\hat{x}\hat{x}} + bA + c|A|^2A + dAB, \\ B_{\hat{t}} &= eB_{\hat{x}\hat{x}} + f(|A|^2)_{\hat{x}\hat{x}}, \end{aligned} \tag{8}$$

and verified this formally for a specific model, later verified rigorously by Sukhtayev (2018). Here, $A \in \mathbb{C}$ is amplitude and $B \in \mathbb{R}^m$ a vector of "mean modes" $\sim$ conserved variables.

Note, exponential solutions with $|A|$, B constant. m new param's!

The model of Häcker-Schneider-Zimmermann

With only SO(2) symmetry, the amplitude eqns become, rather

$$
\begin{aligned}
A_{\hat t} &= aA_{\hat x\hat x} + bA + c|A|^2A + dAB,\\
B_{\hat t} &= eB_{\hat x\hat x} + \epsilon^{-1}(fB + h|A|^2)_{\hat x} + \partial_{\hat x}\Re(gA\bar A_{\hat x}),
\end{aligned}
\tag{9}
$$

singular convection in mean modes! Decoupled case $d = 0$ derived for Bénard-Marangoni and thin film flow in [HSZ2011].

Key observation: Similar to low Mach number limit, rapid convection/averaging does not prevent existence up to $O(1)$ time. (Write B eqn. in Fourier space and use var. of constants.)

• B equation appears at ε^4 order, hence for $d = 0$ may be replaced simply by $B = -h|A|^2/f$, **"Darcy's law,"** with A satisfying (cGL), $\Rightarrow$ rigorous validation to $O(\varepsilon^2)$ [HSZ], for bounded time localized "Darcy" ($=$ CGL) solutions.

Again, (nonlocalized) exponential solutions $|A|, B$ constant. Our goal is to determine time-asymptotic stability of these, for $d \neq 0$.

Converse: "If you understand linearized/nonlinear stability well enough..."

In 2003/2004, Oh-Zumbrun and Denis Serre investigated exact Floquet spectral perturbation of critical modes for periodic waves of systems with conservation laws, not necessarily small. (Noncons. case treated in [Schneider95].*)

Serre made the important observation that the finite dimensional critical eigenspace had a nontrivial 2×2 Jordan block precisely when wave speed is nonconstant among nearby waves, a possibility that occurs only in the SO(2) and not the O(2) case.

This suggested singularity of the associated projector, hence failure of linearized stability. However, in this case the "disease was the cure," as shown in [Johnson-Z10] (see also [JRNZ14]*). Namely, the same conservation principles leading to the Jordan block show that the lower left corner of the block vanishes to first order in the Bloch wave expansion in terms of the Floquet number σ.

“Balancing” /unfolding of Jordan block

That is, expressed in standard basis elements, the Jordan block expands as

$$M(\sigma) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \sigma \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} + O(\sigma^2).$$

A “balancing transformation” $M \to \tilde{M} := SMS^{-1}$, with $S = \mathrm{diag}\{\sigma, 1\}$ converts this to

$$\tilde{M}(\sigma) = \sigma\Big(\begin{pmatrix} * & * \\ * & * \end{pmatrix} + O(\sigma)\Big),$$

which generically splits into analytic spectra $\sigma\lambda_j$, $\lambda_1(\sigma) \neq \lambda_2(\sigma)$.

THUS, NO OBSTRUCTION TO STABILITY

III. Current: issues and main results

FORMAL DERIVATION OF AMPLITUDE EQNS: m new critical (zero) modes for m cons. laws.

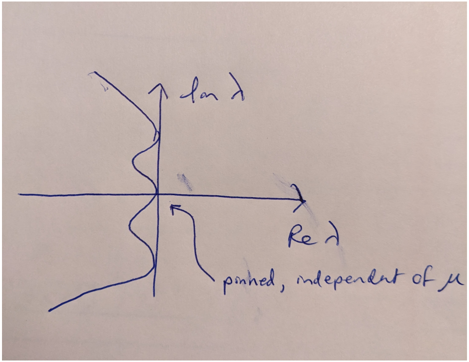

Figure: Dispersion relation at bifurcation (critical curve), conservation law case, zero mode independent of bif. parameter μ.

Not much changed- but now gives singular term in SO(2) case...

Preliminary observations

• In singular case, a Darcy reduction $B = -h|A|^2/f + B_0$ similar as in [HSZ10] may be shown well-posed to time $O(1)$ even with coupling $d \neq 0$ and data near periodic (exp.) solution. But now modifies (cGL) part through $dAB = -dh|A|^2 A + B_0 A$ term, yielding modified existence/Eckhaus stability criteria. ("A" macroscopic behavior, $\sim$ nonnormally hyperbolic slow manifold)

Paradigm shift: Different from low-Mach, plasma dynamics, etc., consider singular amplitude equation, not Darcy, as primary object.

• The "usual miracle": for all cases, linearized amplitude equations reduce to constant coefficient, in principle soluble by linear algebra/disp. relation. For singular system, a two-parameter spectral perturbation problem...

Main results

O(2) case. For general RD systems, extend Matthews-Cox, Sukhtayev results for specal system: formal amplitude eqns., necessary and sufficient for ("diffusive") spectral stability.
• No Darcy reduction, stability involves both A and B equation.
• In principle soluble by const. coeff., case $B \in \mathbb{R}$ explored in [MC].

SO(2) case.* For general conv. reaction diff. systems: HSZ-type amplitude eqns, now fully coupled, nec. and suff for stability.
• Stability again involves both A and B equation. Darcy necessary.
• Stability computation simplifies, yielding $m+1$ Eckhaus-type criteria as nec. and suff. conditions. Reminiscient of relaxation, Chapman-Enskog and Kawashima conditions. Compare numerics [Barker-Jung-Z,2018] (incredibly stiff), analysis [BJNRZ13].

Focus on singular case, $m = 1$: linearized stability system

SPECTRAL PERTURBATION PROBLEM, $\lambda \in \mathrm{spec}(M(\sigma, \epsilon)$:

$$M(\varepsilon, \sigma) = \begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 0 & A_0\Im(d) \\ 0 & 0 & 0 \end{pmatrix} + (\sigma/\epsilon)M_1 + \sigma^2 M_2. \quad (10)$$

(Coordinatization by $(\Re A, \Im A, B)$, $\sigma =$ Bloch-Floquet number, $\sim$ sideband stability.)

OBS. 1. As ϵ^{-1} enters with σ, existence, co-periodic stability are nonsingular, go as usual (same for exact spectra, PDE).

2. M_0 rank one- generically, 2×2 Jordan block! -inherited from nonlinear theory, Serre, JZ10... And, SAME CURE!

Balanced, block-diagonalized system

Projecting out critical modes, then balancing to remove Jordan block, get $O(\epsilon,\sigma)=\sigma O_1+\sigma^2 O_2+O(\sigma^3)$, with

$$O_1 = i\begin{pmatrix} -2\kappa(\Re(a)q+\Im(a)) & O(1)\\ 4i\kappa\epsilon^{-1}A_0 h\Re(a) & \epsilon^{-1}(2A_0hp+f)\end{pmatrix},$$
$$O_2 = -m^{-1}\begin{pmatrix} O(1) & -pq\epsilon^{-1}(2A_0hp+f)\\ O(\epsilon^{-2}) & \epsilon^{-2}2A_0ph(2A_0hp+f)\end{pmatrix}. \tag{11}$$

Relaxation analogy: Separation of scales allows further reduction, "effective diffusions"

$$\mu_t=\ell_t O_2 r_t = O(\epsilon^{-1}) =: \epsilon^{-1}\mu_t^0 + h.o.t.\ (\sim \mathit{Darcy}!),$$
$$\mu_c=\ell_c O_2 r_c = \epsilon^{-2}\Re(d)hf\Re(\hat{c})/\Re(c)^2 2A_0^2 + O(\epsilon^{-1}) =: \epsilon^{-2}\mu_c^0 + h.o.t. \tag{12}$$

giving 2nd-order coefficients. Explicit Eckhaus conditions $\Re\mu_j<0$.

Further details

• Radius of convergence only $O(\epsilon)$, not $O(1)$ as nonsingular case. Requires separate consideration of several parameter regimes; however, each has property that stability properties are constant through each regime, hence inherited from boundaries. Reminiscent of studies of JNRZ for KdV limit of KdV-KS, again relaxation structure.

• Rigorous validation. Lyapunov-Schmidt reduction gives approximate matrix perturbation problem

$$0 = \det\Big(M(\lambda,\sigma,\epsilon) - \lambda \mathrm{Id}\Big) = \det\Big(M(\sigma,\epsilon) - \lambda(\mathrm{Id} + \mathcal{E})\Big).$$

Previously analyzed by quadratic formula [M,S] or Weierstrass preparation in char. poly. We instead invert $(\mathrm{Id} + \mathcal{E})$ by Neumann expansion to obtain smaller error term $\sim \mathcal{E}M$ in matrix problem. Then apply the detailed matrix pert. computations already done to conclude that resulting "convergence error" small as well.

IV. Discussion and open problems

• Opens applications to biomorphology, systematic study!

• More important than usual (Turing/cGL) case, since numerically stiff. Real need for analysis...

• (bounded-time) Darcy and (time-asymptotic) Eckhaus slow modes agree, i.e., Darcy behaves like approximate slow manifold in both finite-time approximation and spectral sense- reminiscent of classical case, Mielke-Schneider analysis.

• Vectorial case $m > 1$: requires an additional "consistent transfer" condition for $|\sigma| \in [\epsilon/C, \epsilon/C]$, $\Leftrightarrow$ absence of real roots of an explicitly computable polynomial. (Hidden subtlety.)

OPEN PROBLEMS:

• Systematic exploration for bio-models! With existing nonlinear theory, complete toolkit ($m = 1$: Darcy cond's. $+ \Re c < \Re \hat{c}$).

• "Emergent dynamics:" modulation approx'n for large patterns?

• Multi-dimensions, secondary bifurcations.

Classical Turing vs. patterns with conservation law (2D)

(a) (b)

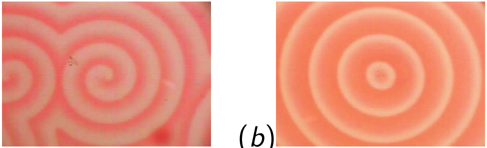

Figure: Classical Turing (Belousov-Zhabatinski): a) spiral. b) target.

(a) (b)

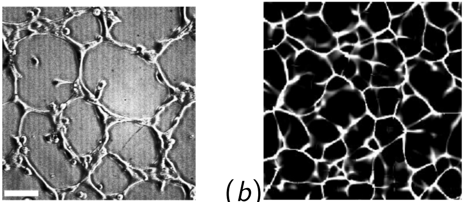

Figure: Vasculogenesis (AGS), $m = 1$): a) in vitro. b) numerical.

THANKS FOR YOUR ATTENTION