Mentoring Undergraduate Research: a personal journey

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Patterns, Dynamics, and Data in Complex Systems (Brown U., 2025)

Go Bills!

Go Bills!

Go Bills!

My undergraduate student collaborators:

- **Brown U. (ICIAM):** Julia Bujalski, Jordan Collignon, Grace Dwyer, Melissa Morrissey, Harjasleen Malvai, Jordan Rosenthal-Kay
- **Calvin University:** Shamuel Auyeung, Jake Christiansen, Rebekah Coggin, Ashlyn DeGroot, Nate De Jong, Kaitlyn Eekhof, Elizabeth Hibma, Hwa-Pyeong Kim, Kris Miedema, Kaitlyn (Plaisier) Leisman, Ben Lewis, Emma Schmidt, Ike Timkovich, Duncan Waanders, Eric Yu

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Papers authored by students:

1. *Wave dynamics in the extended forced KdV*, co-authored with N. De Jong and K. Plaisier, SIAM J. Applied Math. Vol. 71, No. 3, 811-828 (2011)
2. *Tracking the movement of eigenvalues via a corresponding Evans function*, B. Lewis, SIAM Undergraduate Research Online Vol. 5 (2012)
3. *Finding eigenvalues for matrices acting on subspaces*, J. Christiansen, SIAM Undergraduate Research Online Vol. 5 (2012)
4. *Instability indices for matrix polynomials*, co-authored with E. Hibma, H.-P. Kim, and J. Timkovich, Vol. 439, No. 11, Lin. Alg. Appl. 3412-3434 (2013)
5. *The Krein matrix and an Interlacing Theorem*, S. Auyeung and E. Yu, SIAM Undergraduate Research Online Vol. 7 (2014)

6. *Numerically computing zeros of the Evans function*, R. Coggin, SIAM Undergraduate Research Online Vol. 8 (2015)
7. *Propagation of lead in the human body*, M. Morrissey and J. Collignon, SIAM Undergraduate Research Online Vol. 10 (2017)
8. *Consensus and clustering in opinion formation on networks*, co-authored with J. Bujalski, G. Dwyer, Q.-N. Le, H. Malvai, J. Rosenthal-Kay, and J. Ruiter, Phil. Trans. Royal Society A, (2017)
9. *Opinion formation dynamics in a modified contrarian model*, K. Eekhof, SIAM Undergraduate Research Online Vol. 12 (2019)
10. *Opinion dynamics with slowly evolving zealot populations*, A. DeGroot and E. Schmidt, SIAM Undergraduate Research Online Vol. 16 (2023)

Things to be aware of before starting:

- finding an appropriate project (peel off something from your own research?)
- only have 8-10 weeks to go from start (knowing nothing) to finish (hopefully, a paper)
- student enthusiasm is high
- if progress is going to be made, mentor must typically meet daily with the students

Things to be aware of before starting (cont.):

- student experience in doing research is low
- compartmentalization - students generally have little experience in synthesizing material from several different classes to work on one problem
- students probably have not been exposed to everything they need to know to work on the project (knowledge gap)
- students may not be familiar with needed software (R, MATLAB, LaTeX, etc.)

Two problems to talk about today:

1. counting unstable eigenvalues for $\star$-even polynomial eigenvalue problems
    - professional interest
2. opinion (belief) propagation in society
    - student interest

$\star$-even eigenvalue problems

Goal is to count number of (potentially) unstable eigenvalues for polynomials with matrix-valued coefficients

Consider $\star$-even matrix polynomials of the form

$$\boldsymbol{P}_d(\lambda) = \boldsymbol{A}_0 + \lambda \boldsymbol{A}_1 + \lambda^2 \boldsymbol{A}_2 + \cdots + \lambda^d \boldsymbol{A}_d,$$

where

- $\boldsymbol{A}_j \in \mathcal{M}_n(\mathbb{C})$
- $\boldsymbol{A}_0, \boldsymbol{A}_d$ are invertible
- $\boldsymbol{A}_{2j}$ are Hermitian, while $\boldsymbol{A}_{2j-1}$ are skew-Hermitian.

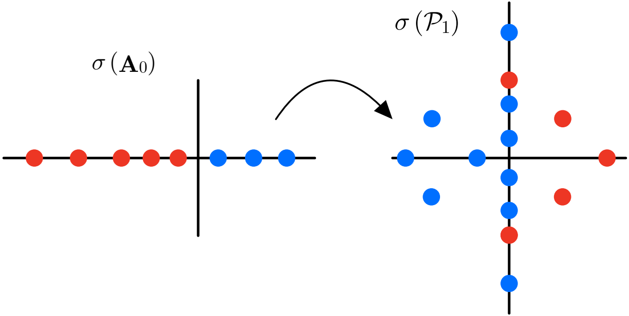
Fact: If $\boldsymbol{A}_j \in \mathcal{M}_n(\mathbb{R})$ the polynomial eigenvalues have the four-fold symmetry, $\{\pm\lambda, \pm\overline{\lambda}\}$.

The Hamiltonian-Krein index is the sum of three (potential) instability indices,

$$K_{\mathrm{Ham}} = k_{\mathrm{r}} + k_{\mathrm{c}} + k_{\mathrm{i}}^{-}.$$

Fact (K/co-authors, Pelinovsky/co-authors): Regarding the spectrum of $\mathcal{P}_1(\lambda) = \boldsymbol{A}_0 + \lambda \boldsymbol{A}_1$,

$$K_{\mathrm{Ham}} = \mathrm{n}(\boldsymbol{A}_0).$$



What if $d = 2$? Linearizing $(\boldsymbol{A}_0 + \lambda \boldsymbol{A}_1 + \lambda^2 \boldsymbol{A}_2)\boldsymbol{u} = \boldsymbol{0}$ via,

$$\boldsymbol{x}_1 = \boldsymbol{u}, \quad \boldsymbol{x}_2 = \lambda \boldsymbol{A}_2 \boldsymbol{u},$$

gives the first-order problem $(\boldsymbol{S}_2 + \lambda \boldsymbol{J}_2)\boldsymbol{x} = \boldsymbol{0}$, with

$$\boldsymbol{S}_2 = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{A}_2^{-1} \end{pmatrix}, \quad \boldsymbol{J}_2 = \begin{pmatrix} \boldsymbol{0}_n & -\boldsymbol{I}_n \\ \boldsymbol{I}_n & \boldsymbol{A}_1 \end{pmatrix}.$$

Fact (Bronski/Johnson/K): For $\mathcal{P}_2(\lambda) = \boldsymbol{A}_0 + \lambda \boldsymbol{A}_1 + \lambda^2 \boldsymbol{A}_2$,

$$K_{\mathrm{Ham}} = \mathrm{n}(\boldsymbol{S}_2) = \mathrm{n}(\boldsymbol{A}_0) + \mathrm{n}(\boldsymbol{A}_2).$$

What if $d \geq 3$? One $\star$-even linearization was given by Mehrmann and Watkins (2002) in a numerical linear algebra context with,

$$\boldsymbol{S}_d = \left(\begin{array}{c|cccccc} \boldsymbol{A}_0 & \boldsymbol{0}_n & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n \\ \hline \boldsymbol{0}_n & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_4 & \cdots & \boldsymbol{A}_d \\ \boldsymbol{0}_n & -\boldsymbol{A}_3 & -\boldsymbol{A}_4 & & & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{A}_4 & & & & \boldsymbol{0}_n \\ \vdots & \vdots & & & & \vdots \\ \boldsymbol{0}_n & \mp\boldsymbol{A}_d & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n \end{array} \right) \in \mathcal{M}_{dn}(\mathbb{C}),$$

where $(-1)^d \boldsymbol{A}_d \mapsto \mp \boldsymbol{A}_d$ (the exact form of the skew-Hermitian $\boldsymbol{J}_d$ not important here).

If $d = 3$,

$$\boldsymbol{S}_3 = \left(\begin{array}{c|cc} \boldsymbol{A}_0 & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \hline \boldsymbol{0}_n & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ \boldsymbol{0}_n & -\boldsymbol{A}_3 & \boldsymbol{0}_n \end{array}\right) \quad \rightsquigarrow \quad \mathrm{n}(\boldsymbol{S}_3) = \mathrm{n}(\boldsymbol{A}_0) + n.$$

Use,

$$\mathrm{n}\begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \boldsymbol{M}_2^{\mathrm{H}} & \boldsymbol{0}_N \end{pmatrix} = N,$$

where $\boldsymbol{M}_2 \in \mathcal{M}_N(\mathbb{C})$ is invertible, and $\boldsymbol{M}_1$ is Hermitian. The result generalizes for any odd d,

$$K_{\mathrm{Ham}} = \mathrm{n}(\boldsymbol{S}_{2\ell-1}) = \mathrm{n}(\boldsymbol{A}_0) + (\ell - 1)n.$$

The proof for d odd is straightforward.

The idea (of the students!) for $d = 2\ell$ is to embed $\boldsymbol{S}_{2\ell}$ into the larger odd-case matrix

$$\boldsymbol{S}_{\mathrm{new}} = \begin{pmatrix} \boldsymbol{S}_{2\ell} & \boldsymbol{0} \\ \boldsymbol{0} & (-1)^{\ell}\boldsymbol{A}_{2\ell} \end{pmatrix},$$

for which

$$\mathrm{n}(\boldsymbol{S}_{2\ell}) = \mathrm{n}(\boldsymbol{S}_{\mathrm{new}}) - \mathrm{n}((-1)^{\ell}\boldsymbol{A}_{2\ell}) = \mathrm{n}(\boldsymbol{S}_{\mathrm{new}}) + \mathrm{n}((-1)^{\ell-1}\boldsymbol{A}_{2\ell}) - n.$$

Regarding $\boldsymbol{S}_{\mathrm{new}}$ there is an invertible matrix $\boldsymbol{T}$ such that,

$$\boldsymbol{T}\boldsymbol{S}_{\mathrm{new}}\boldsymbol{T}^{\mathrm{T}} = \left(\begin{array}{c|c|c} \boldsymbol{A}_0 & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \hline \boldsymbol{0} & \boldsymbol{M}_2^{\mathrm{T}} & \boldsymbol{0}_{\ell n} \end{array}\right),$$

so

$$\mathrm{n}(\boldsymbol{S}_{\mathrm{new}}) = \mathrm{n}(\boldsymbol{T}\boldsymbol{S}_{\mathrm{new}}\boldsymbol{T}^{\mathrm{T}}) = \mathrm{n}(\boldsymbol{A}_0) + \ell n.$$

Theorem (K/Hibma/Kim/Timkovich)

The Hamiltonian-Krein index for the $\star$-even matrix pencil

$$\boldsymbol{P}_d(\lambda) = \boldsymbol{A}_0 + \lambda \boldsymbol{A}_1 + \lambda^2 \boldsymbol{A}_2 + \cdots + \lambda^d \boldsymbol{A}_d$$

is

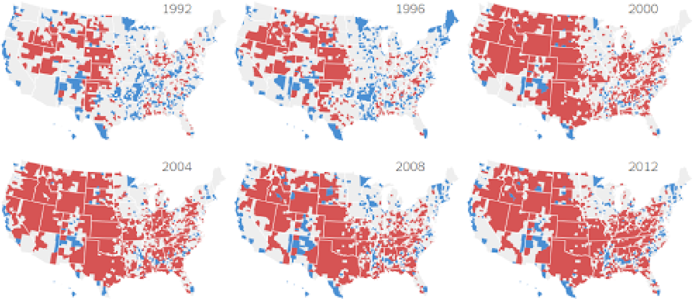
$$K_{\mathrm{Ham}} = \begin{cases} \mathrm{n}(\boldsymbol{A}_0) + (\ell - 1)n, & d = 2\ell - 1 \\ \mathrm{n}(\boldsymbol{A}_0) + (\ell - 1)n + \mathrm{n}((-1)^{\ell-1}\boldsymbol{A}_{2\ell}), & d = 2\ell. \end{cases}$$

Opinion dynamics

Goal is to understand “opinion" propagation and polarizartion.

Counties that voted for the Republican or Democratic presidential candidate by 20 percentage points or more

In 1992, 38% of voters lived in landslide counties.

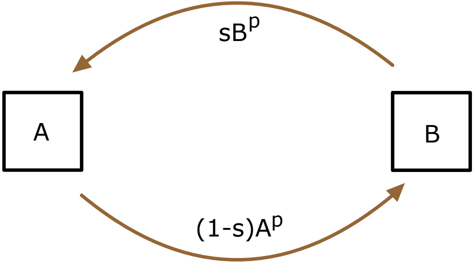


In 2012, 50% of voters lived in landslide counties.

We start with a population having two different:

1. languages (no bilingualism)
2. opinions (no middle ground)
3. religious affiliations

The basic compartment model by Abrams/Strogatz (Nature, 2003):

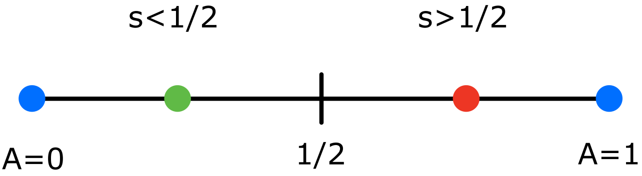


Experimentally, $p \sim 1.31 \pm 0.25$.

The governing ODE (using $A + B = 1$):

$$\dot{A} = sA^p(1 - A) - (1 - s)(1 - A)^p A, \quad 0 < s < 1.$$

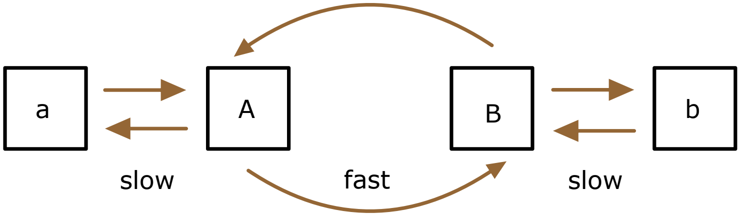
For $p > 1$ (low volatility) the prestige parameter, s, provides:



- $0 < s < 1/2$ means A is more prestigious
- $1/2 < s < 1$ means B is more prestigious

WLOG assume $p = 2$.

Look at a 4-compartment model to understand effect of evolving zealots.



The extended system is,

$$
\begin{aligned}
\dot{A} &= s(a+A)^2(1-a-b-A)-(1-s)(1-a-A)^2A+\epsilon g(A,a,b)\\
\dot{a} &= \epsilon h(A,a,b)\\
\dot{b} &= \epsilon m(A,a,b).
\end{aligned}
$$

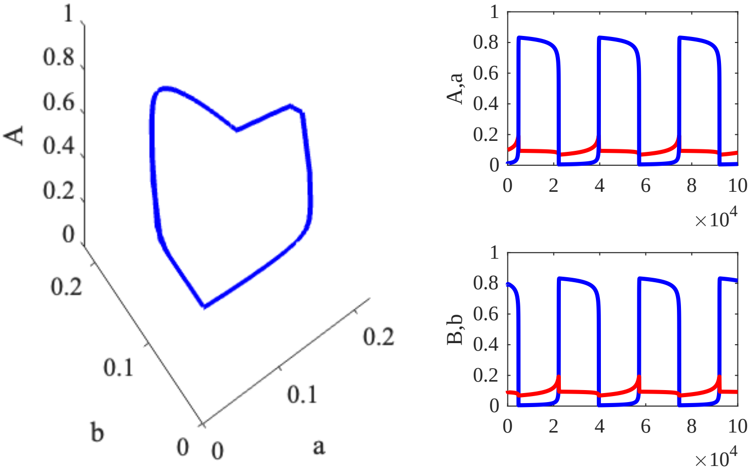
Effects to be modeled include:

- **indoctrination:** zealots convert moderates to zealots ($0 < w < 1$)
- **deradicalization:** moderates convince zealots to become moderates ($0 < 1 - w < 1$)
- **spontaneous radicalization:** moderates become zealots for no readily apparent reason ($0 < d_- < d < d_+ < 1$)
- **repulsive radicalization:** moderates become zealots in reaction to opposing zealots ($0 < r \ll 1$ - not so important)

Interpretation:

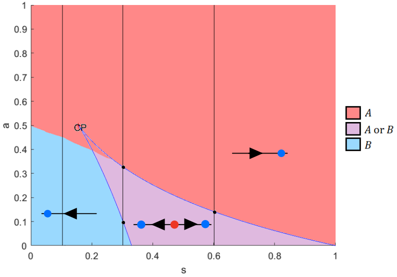
- $0 < w < 1/2$ means deradicalize faster than indoctrinate
- $1/2 < w < 1$ means indoctrinate faster than deradicalize

The most interesting solution is the periodic orbit:

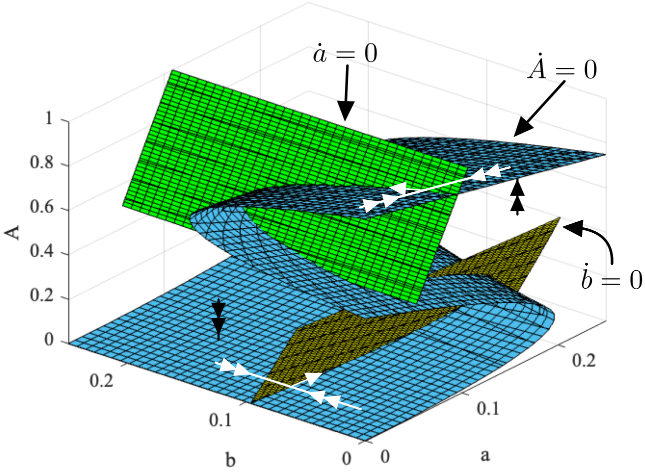


Central ideas behind constructing singular solutions:

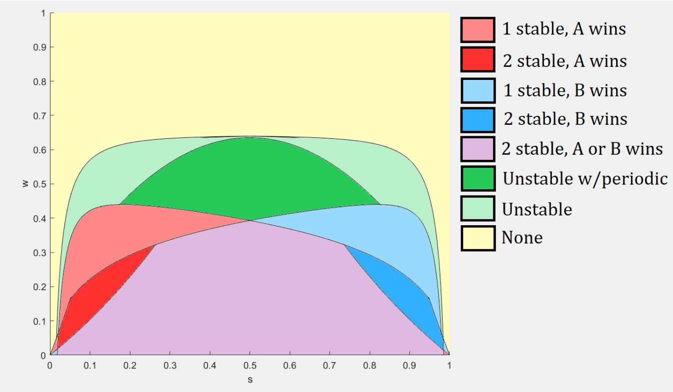
- construct attracting invariant surfaces, A_{B} ($A \sim 0$) and A_{T} ($A \sim 1$)
- construct $\dot{a} = 0$ and $\dot{b} = 0$ surfaces
- show these surfaces are attracting on A_{B} and A_{T}
- construct two-parameter bifurcation diagrams for frozen a and b (Matcont)



The final result in (A, a, b)-space:



Possible dynamics as a function of the parameters:



Possible directions to go:

- add contrarians to the mix
- allow prestige and/or level of indoctrination to be slowly varying instead of constant
- look at networks (extend previous work with P. Kevrekidis)