TOPOLOGICAL ORIGAMI: BEYOND THE PAPER DONUT

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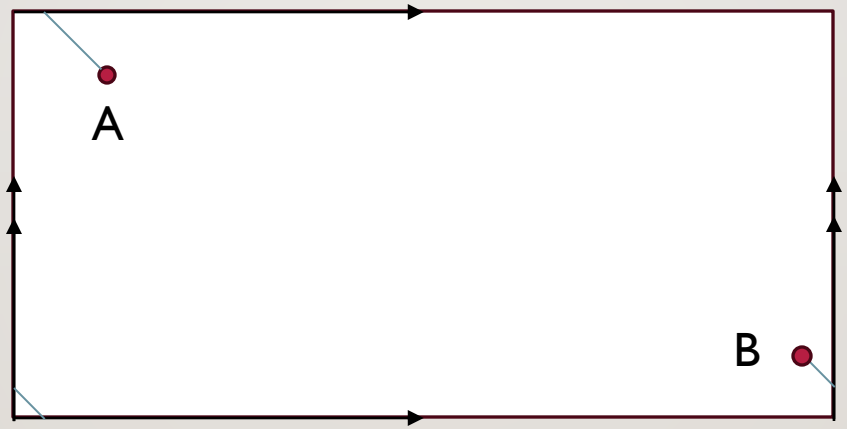
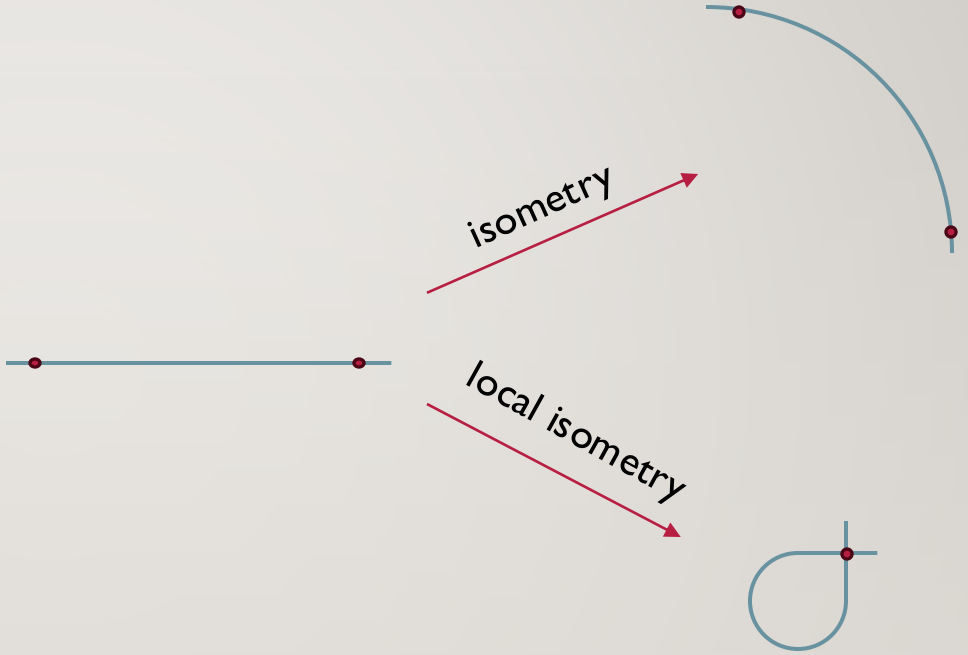
THE FLAT TORUS VIA EDGE IDENTIFICATIONS



ISOMETRIC MAPS

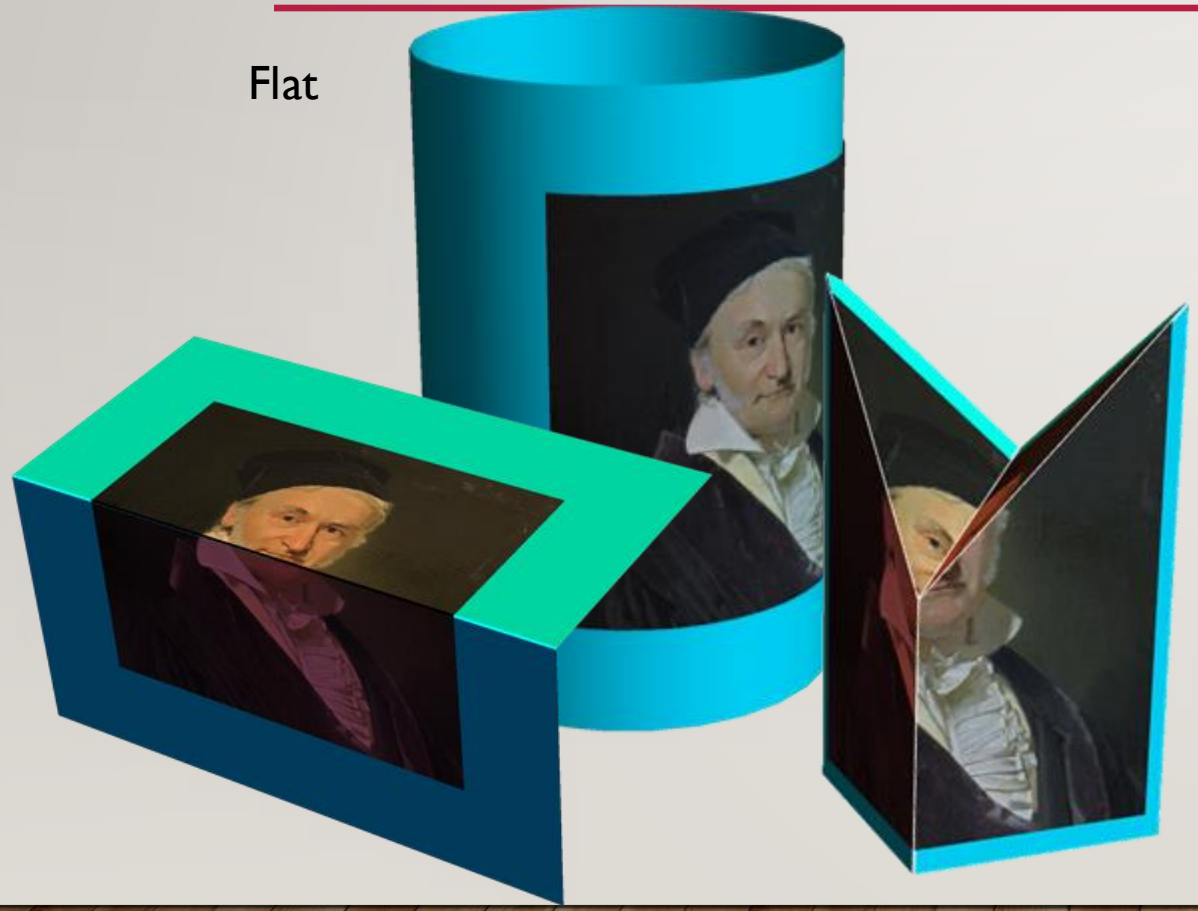
A map between metric spaces is called an **isometry** if it preserves distances between points.

A map is called a **local isometry** if every point has a neighborhood to which the restriction of the map is an isometry.



FLATNESS
LOCALLY ISOMETRIC TO THE EUCLIDEAN PLANE

Flat

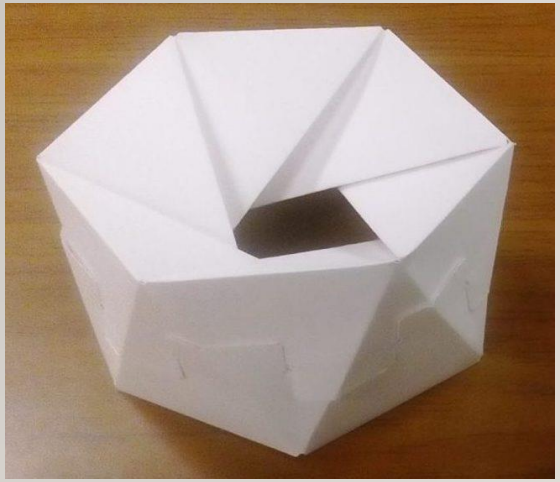


Not Flat

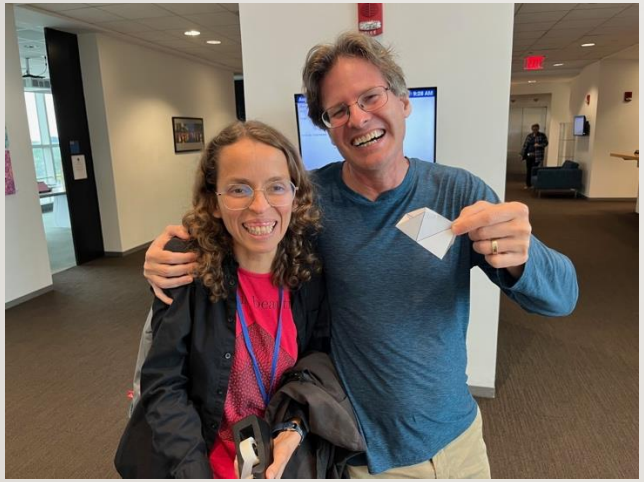
FLAT TORUS USING ORIGAMI TOPOLOGY

Need extra structure to keep track of multiply-covered surfaces (see Demaine & O'Rourke, 2007)

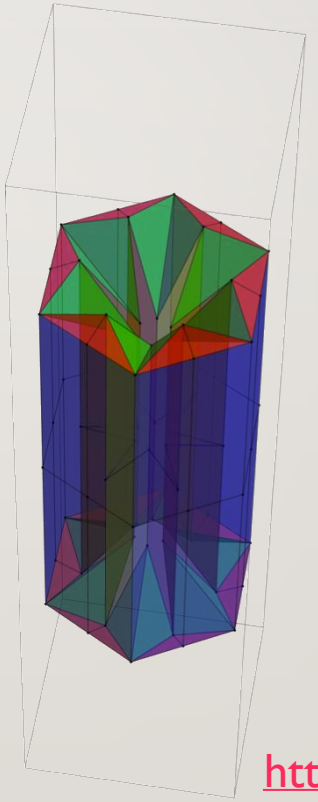
"PAPER DONUTS" (FLAT TORI)

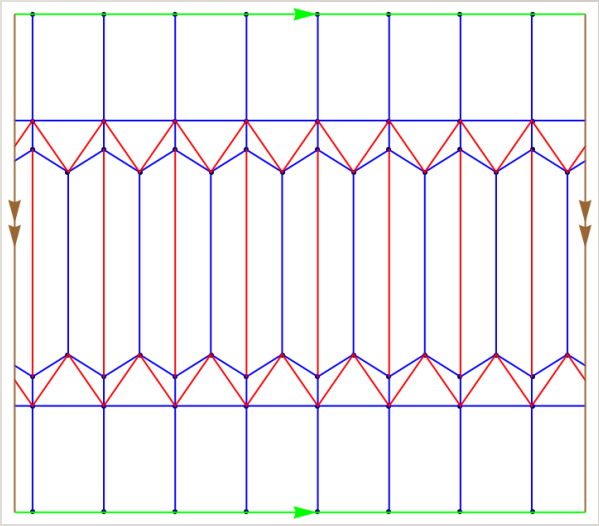


(*Diplotorus*, Málaga & Lelièvre, 2020)



(Schwartz & Málaga, 30 min ago)

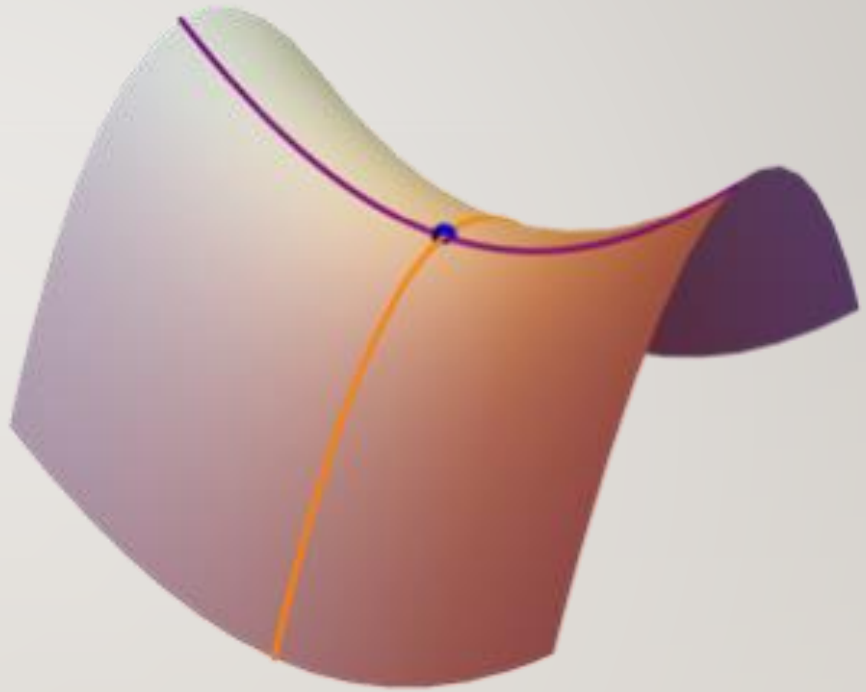




https://www.desmos.com/3d/de131ca2ef

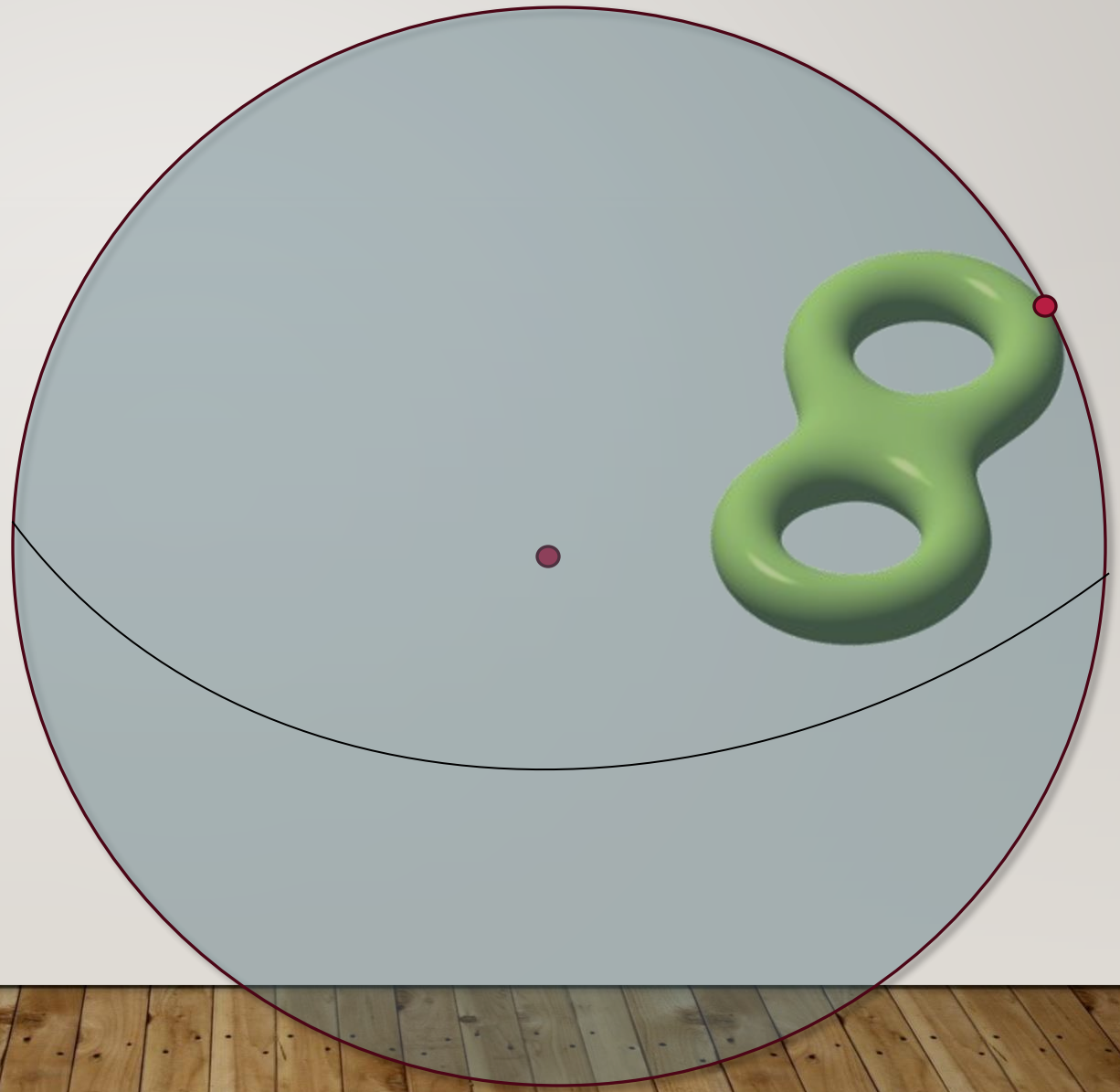
GAUSSIAN CURVATURE

- Normal curvatures
- The **Gaussian curvature** at a smooth point on a surface in $\mathbb{R}^3$ is the product of the maximum and minimum signed normal curvatures at that point.
- Gauss's *Theorema Egregium* states that Gaussian curvature is preserved under smooth local isometry.



EVERY COMPACT SMOOTH SURFACE HAS A POSITIVE CURVATURE POINT

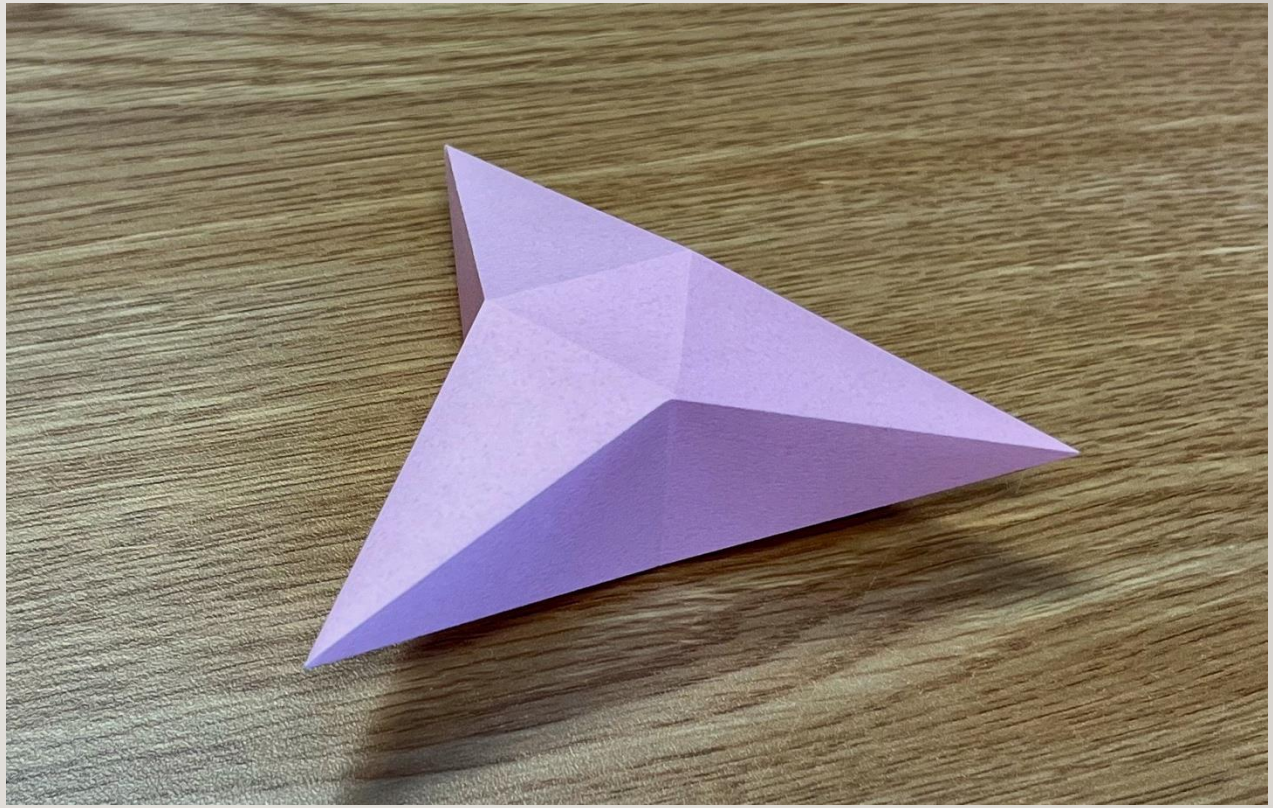
At the farthest point from a fixed point, the surface has curvature at least as large as that of the sphere.



BEYOND: WHY STOP WITH THE TORUS?

A PAPER SPHERE?

An embedded sphere must have at least 3 non-flat points.

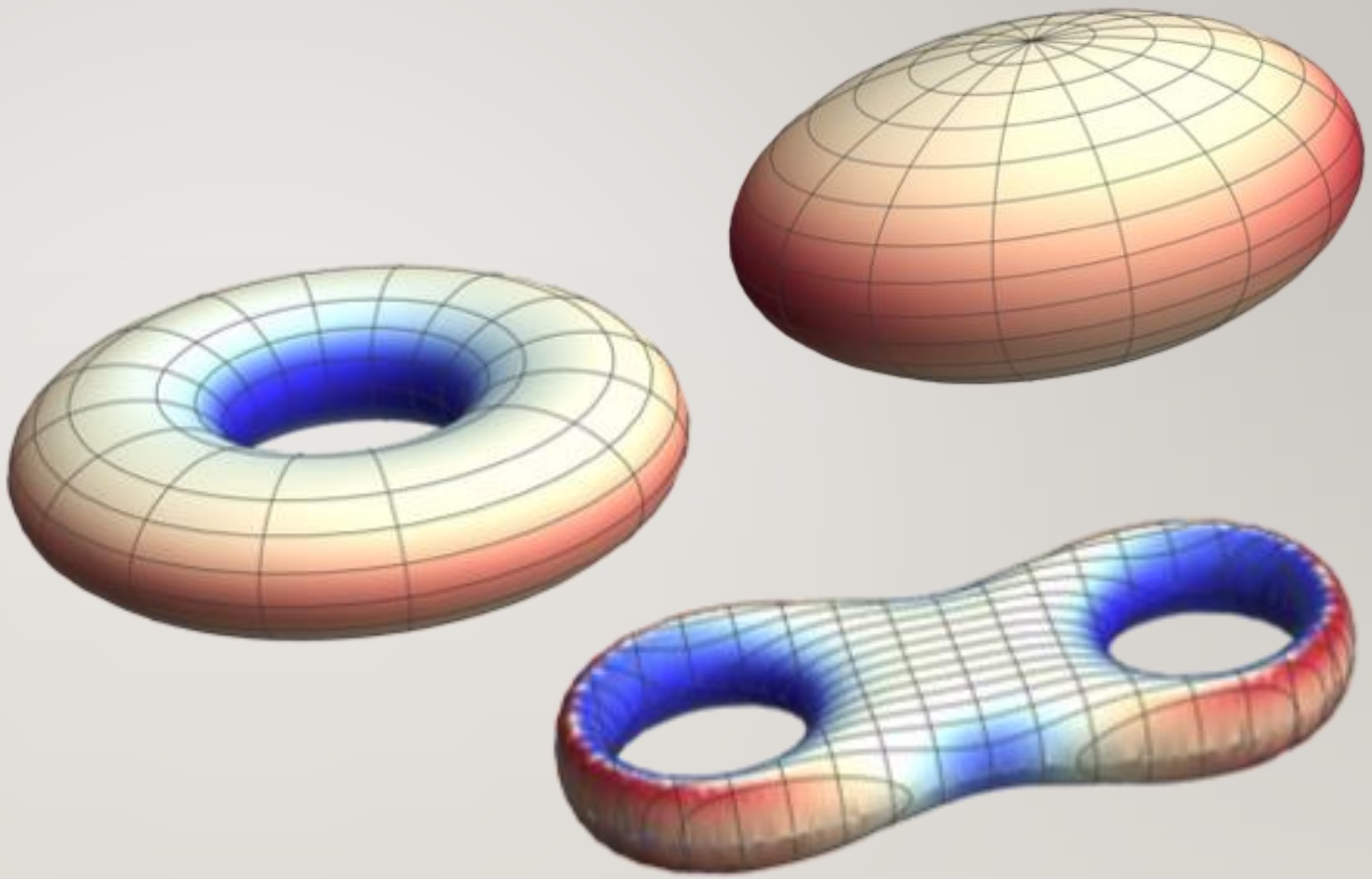


"Three Points of Defect", *Polyplane* (Paul, 2022)

GAUSS-BONNET THEOREM

The average Gaussian curvature of a compact surface without boundary equals $\frac{2\pi\chi}{A}$, where χ is the Euler characteristic and A is the area.

$\Longrightarrow$ Only the torus and Klein Bottle admit flat (zero-curvature) metrics.

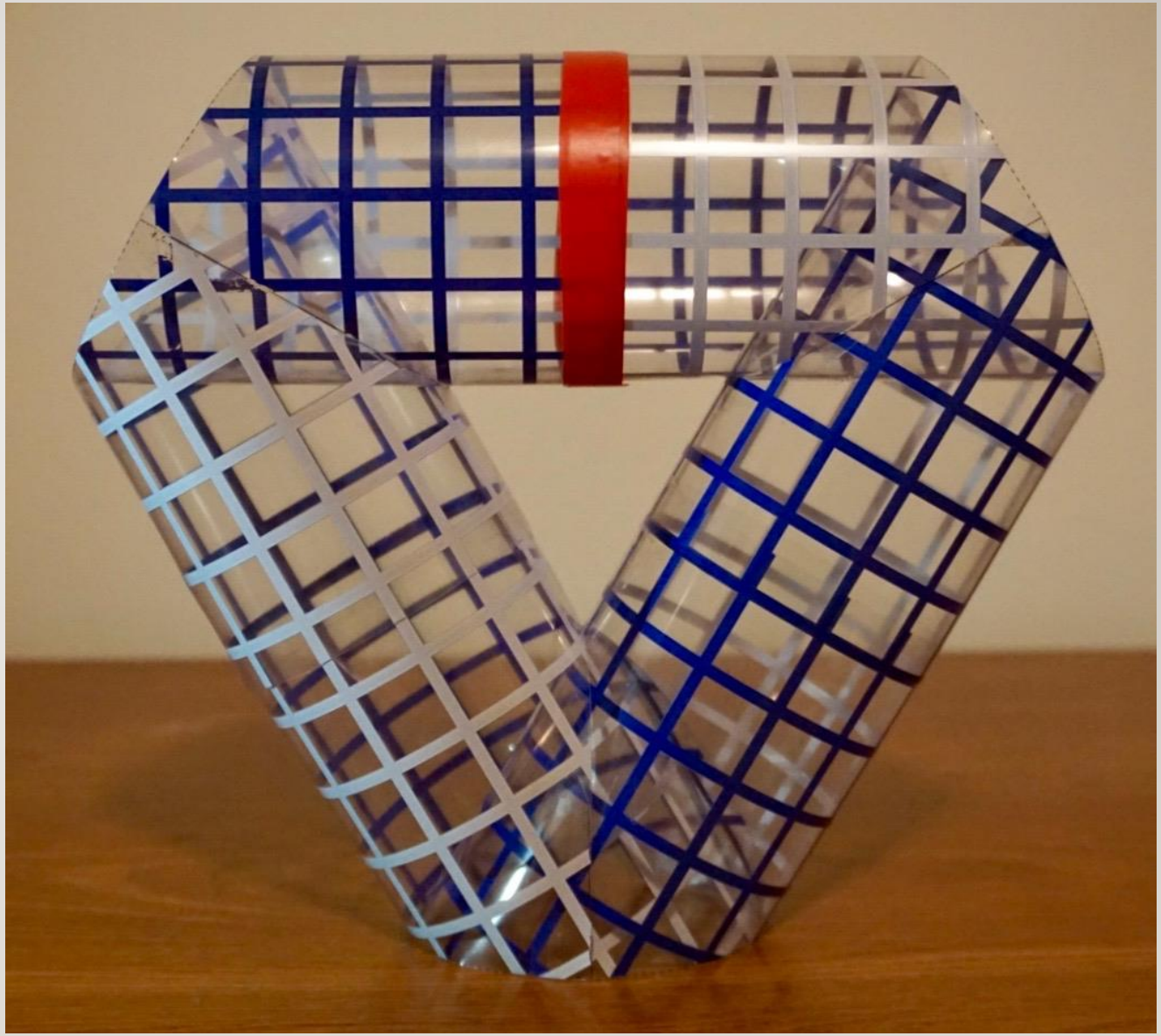


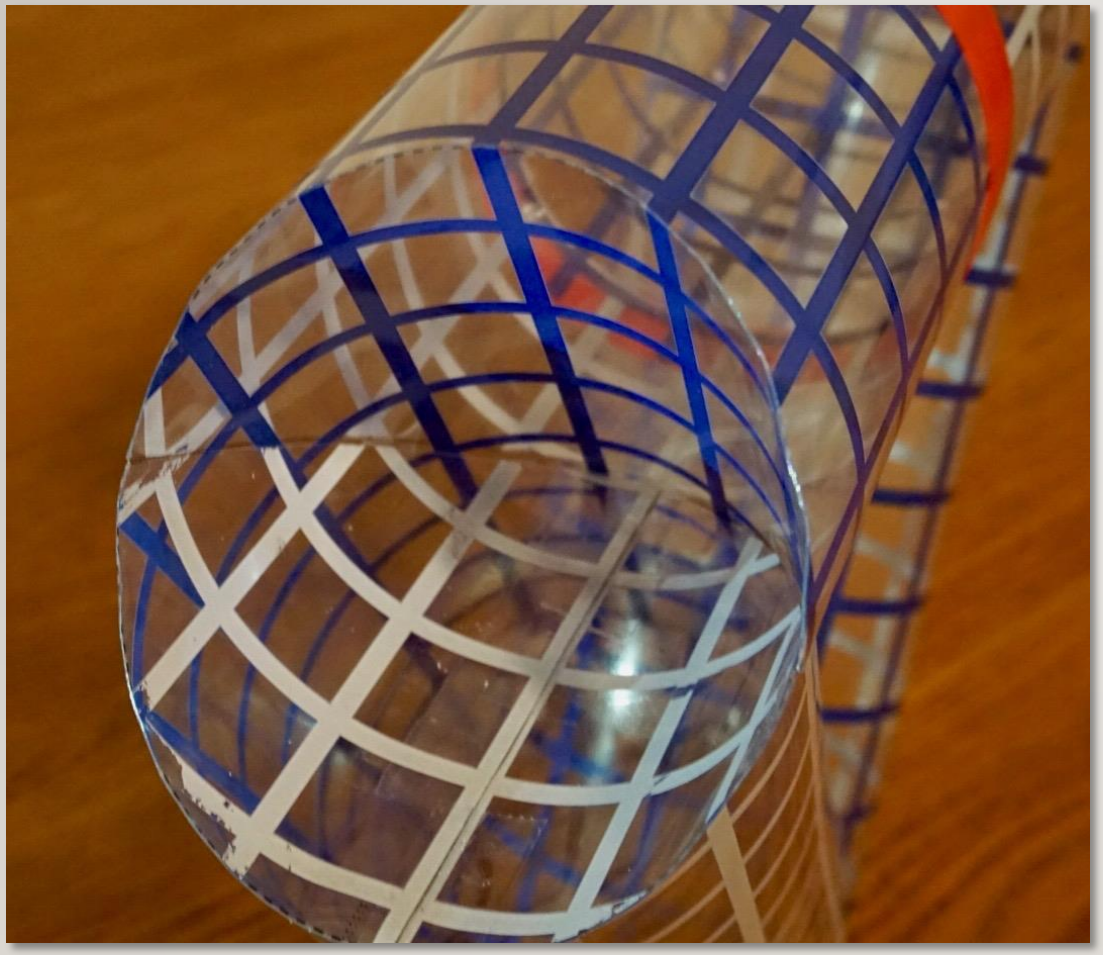
FLAT KLEIN BOTTLE WITH CURVED CREASE ORIGAMI

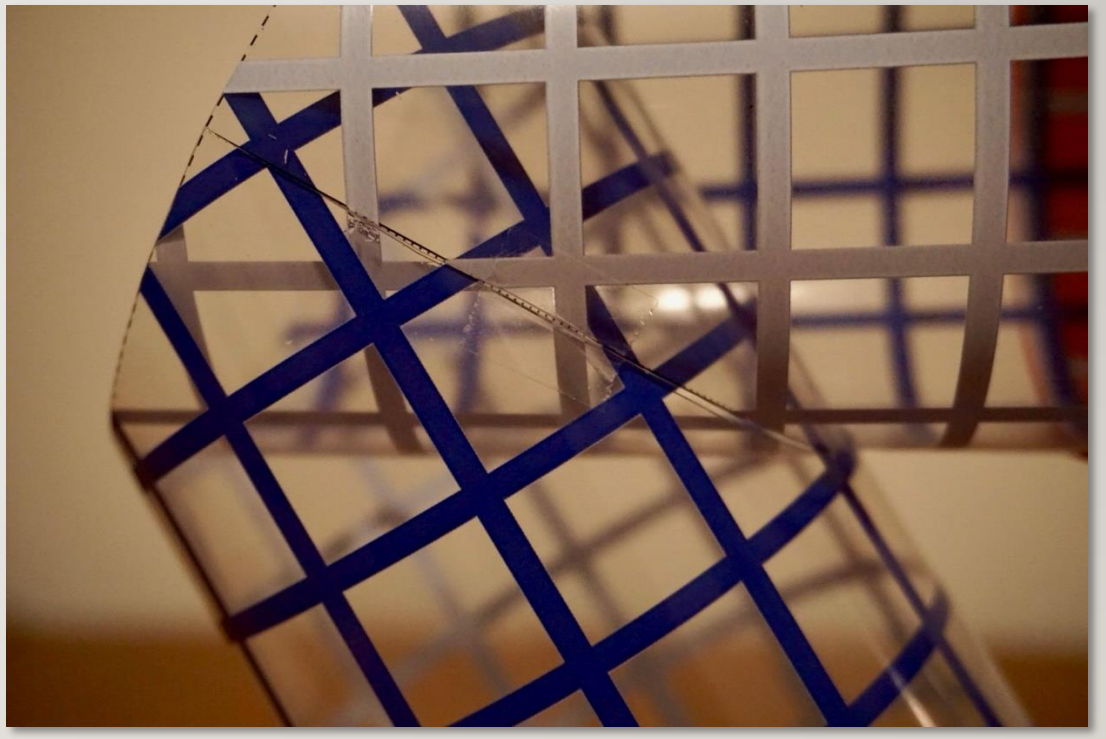
The image of a piecewise smooth arc length preserving map from a flat Klein bottle into Euclidean three space.

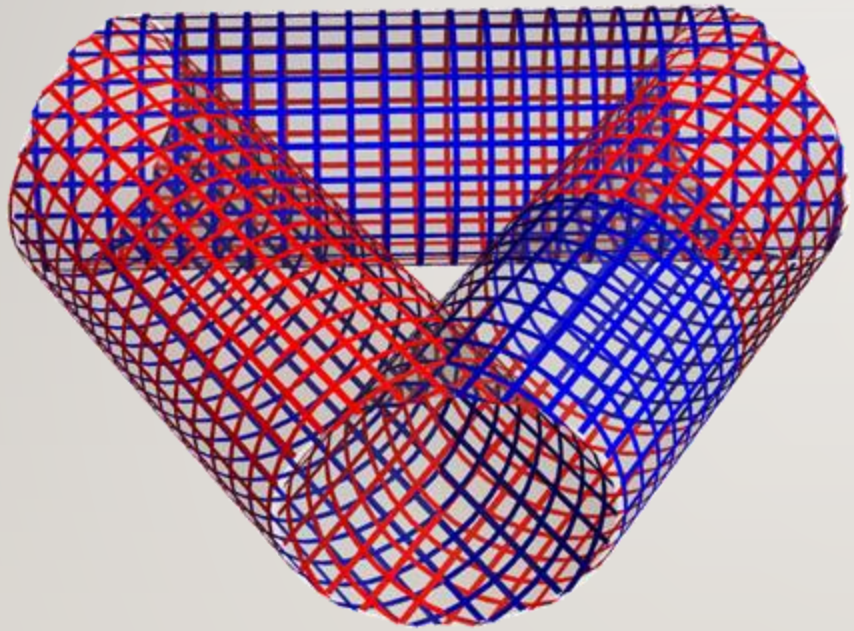
Almost a local isometry!

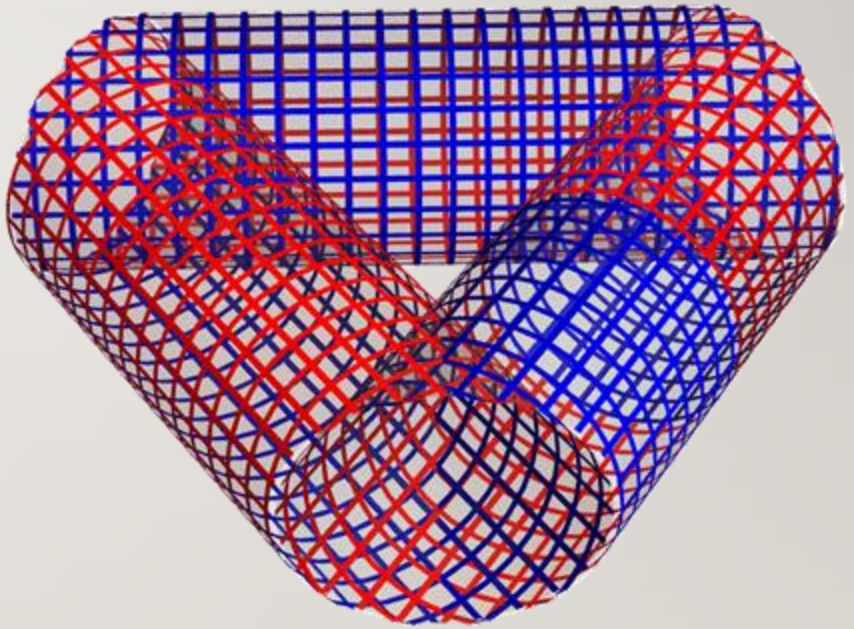
(Paul, 2021)



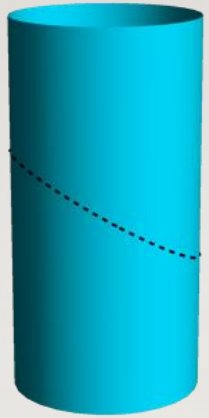


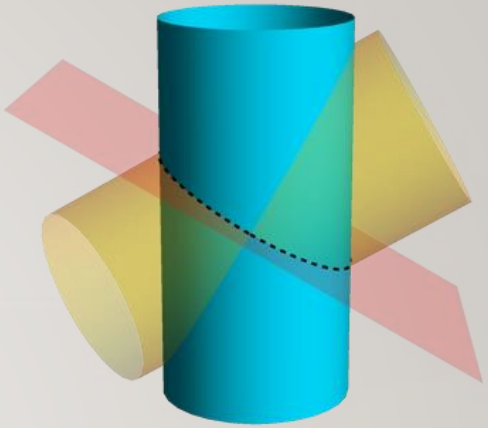


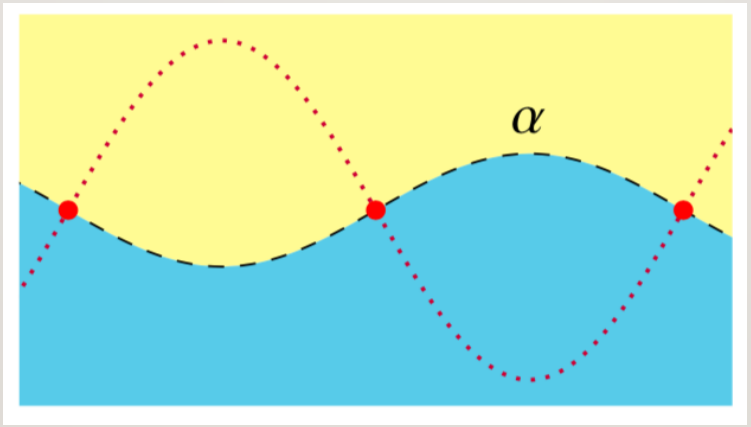


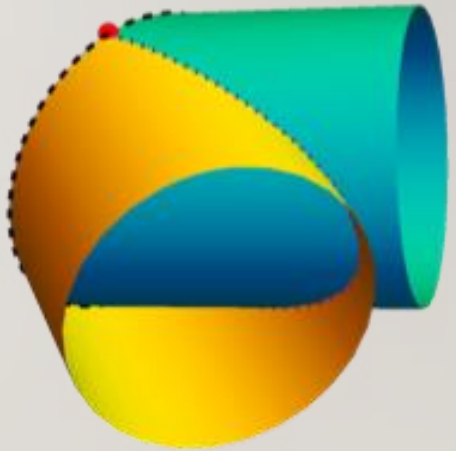


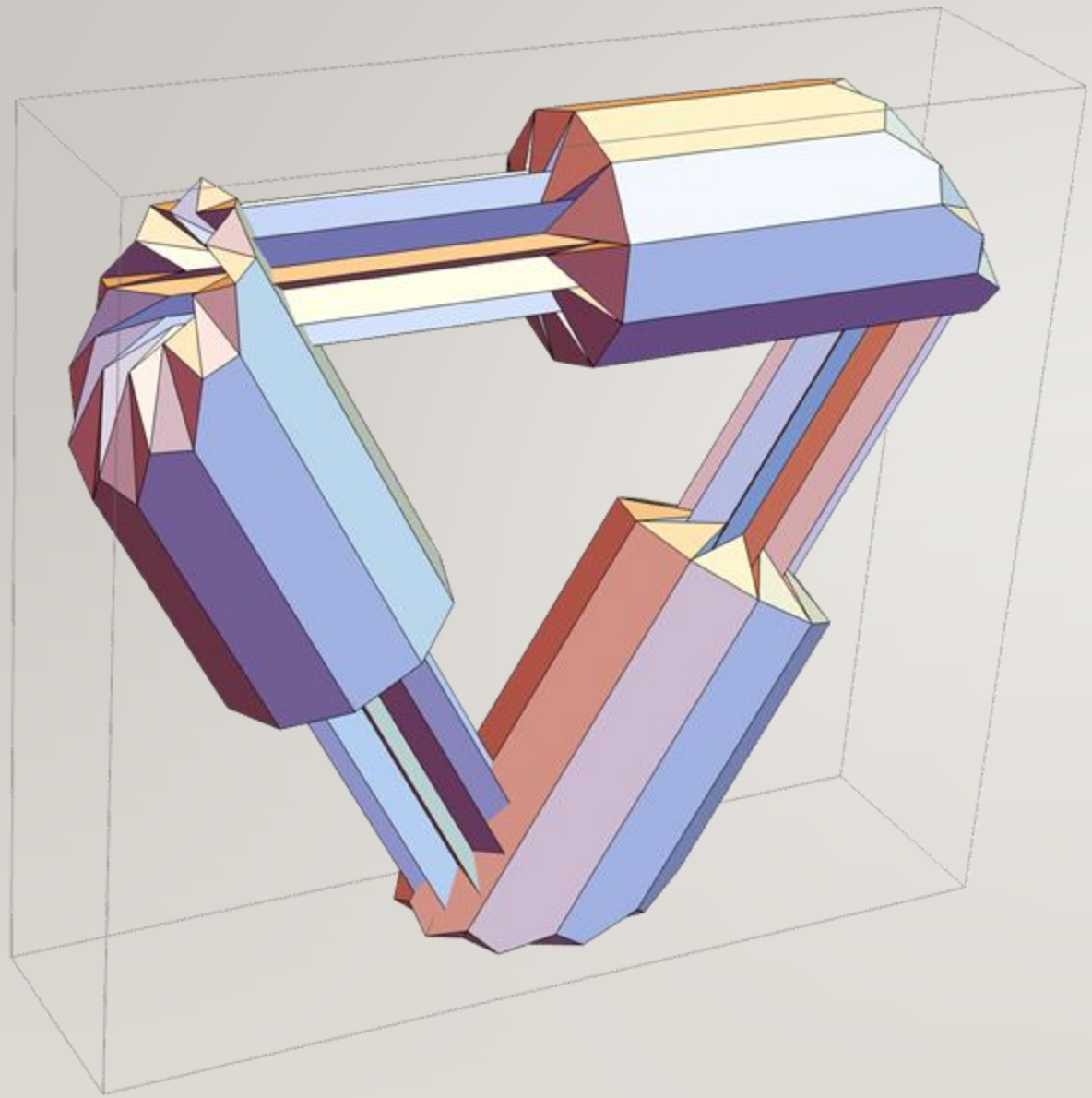
CURVED CREASE ORIGAMI





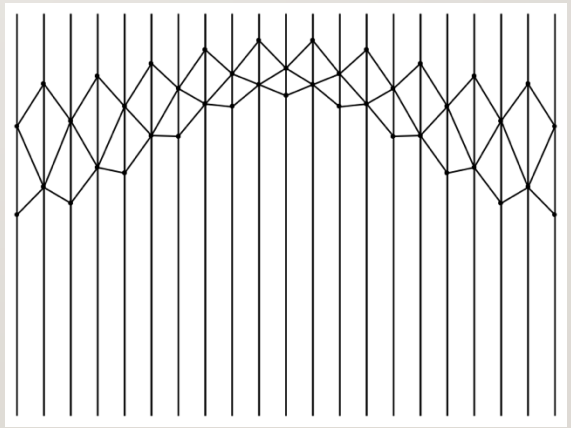


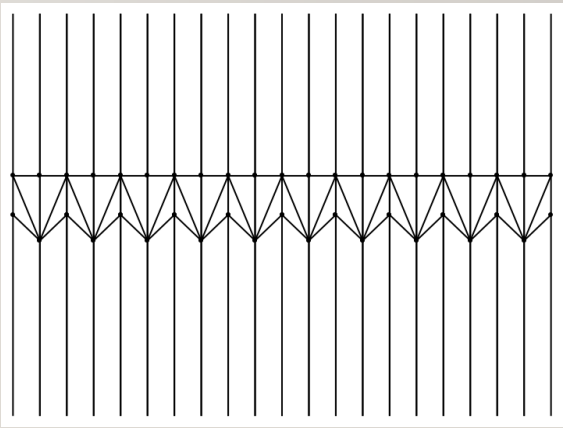




A Piecewise Smooth Isometric Immersion of a flat Klein Bottle

(Paul, 2025)





Fold patterns

Since no other compact surfaces have genus 0, we can't hope to make them with flat paper.

THE END???

- Can we make positive Euler characteristic surfaces (sphere, projective plane) with "spherical paper"?
- Can we make negative Euler characteristic surfaces (double torus…) with "hyperbolic paper"?

NON-EUCLIDEAN "PAPER"

Hyperbolic Paper (with Aaron Abrams)

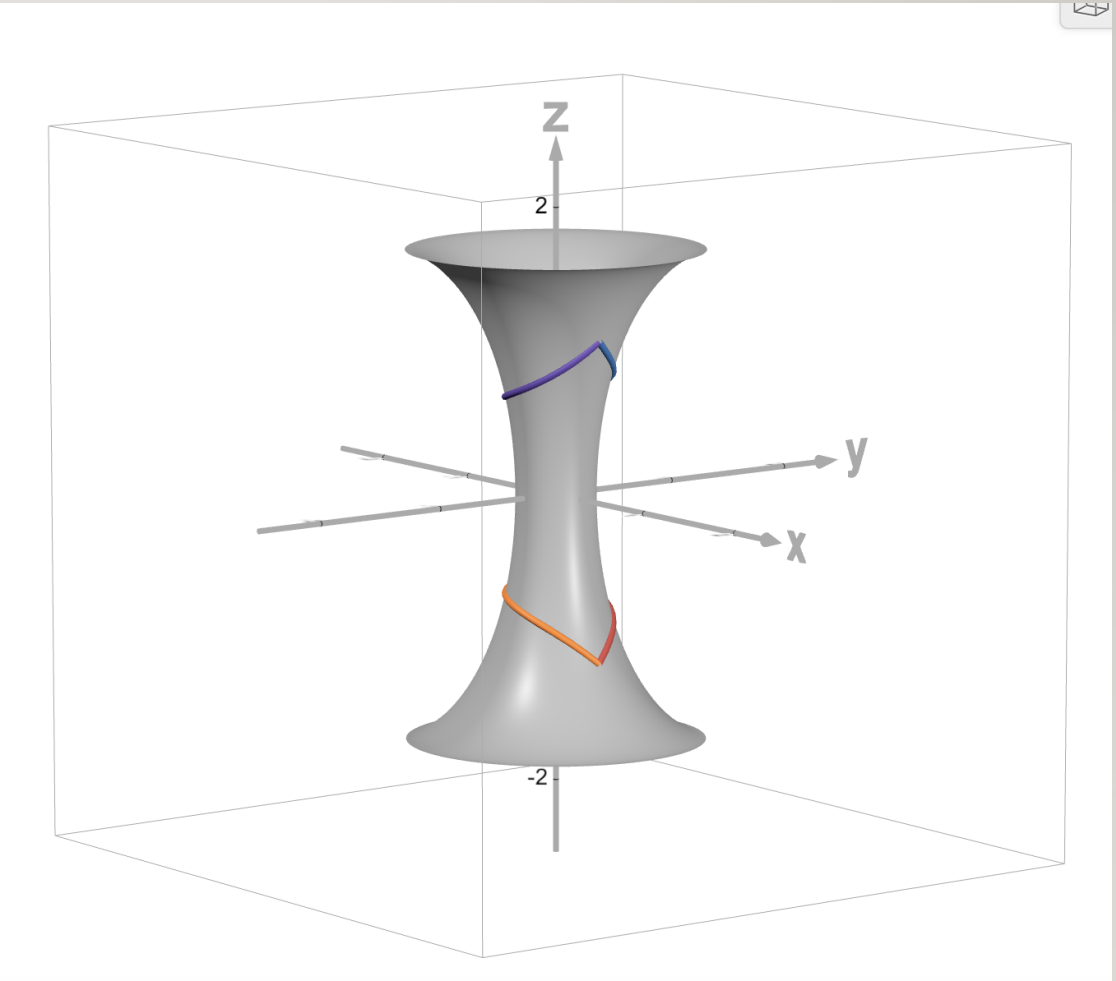
Images from *Warped Realities* exhibit at MoMath

Topological Origami

How faithfully can we represent the geometry and topology of constant curvature surfaces in Euclidean three space?

CONSTANT CURVATURE RIGHT-ANGLE HEXAGON

https://www.desmos.com/3d/0pnoky45en



Scan for origami fold patterns, references, other info

THANK YOU!

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