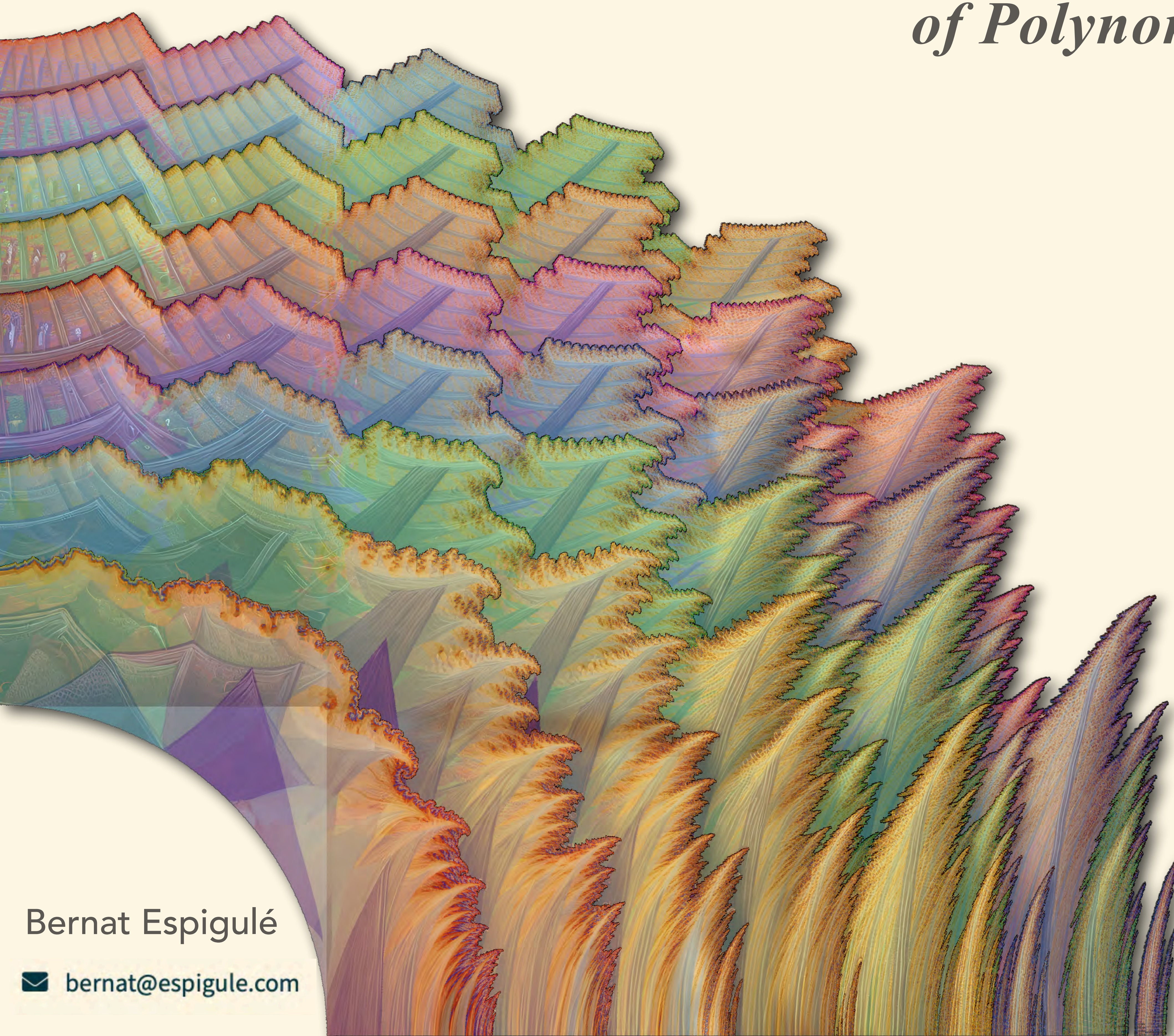
Collinear Fractals - Unveiling the Hidden Geometry of Polynomial Roots



Joint work with my PhD supervisors, Joan Saldaña, and David Juher

Universitat de Girona

IFUdG 2022–2024 Santander

Low dimensional dynamical systems: topology, geometry, combinatorics and bifurcations

PID2023-146424NB-I00

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

AGENCIA ESTATAL DE INVESTIGACIÓN

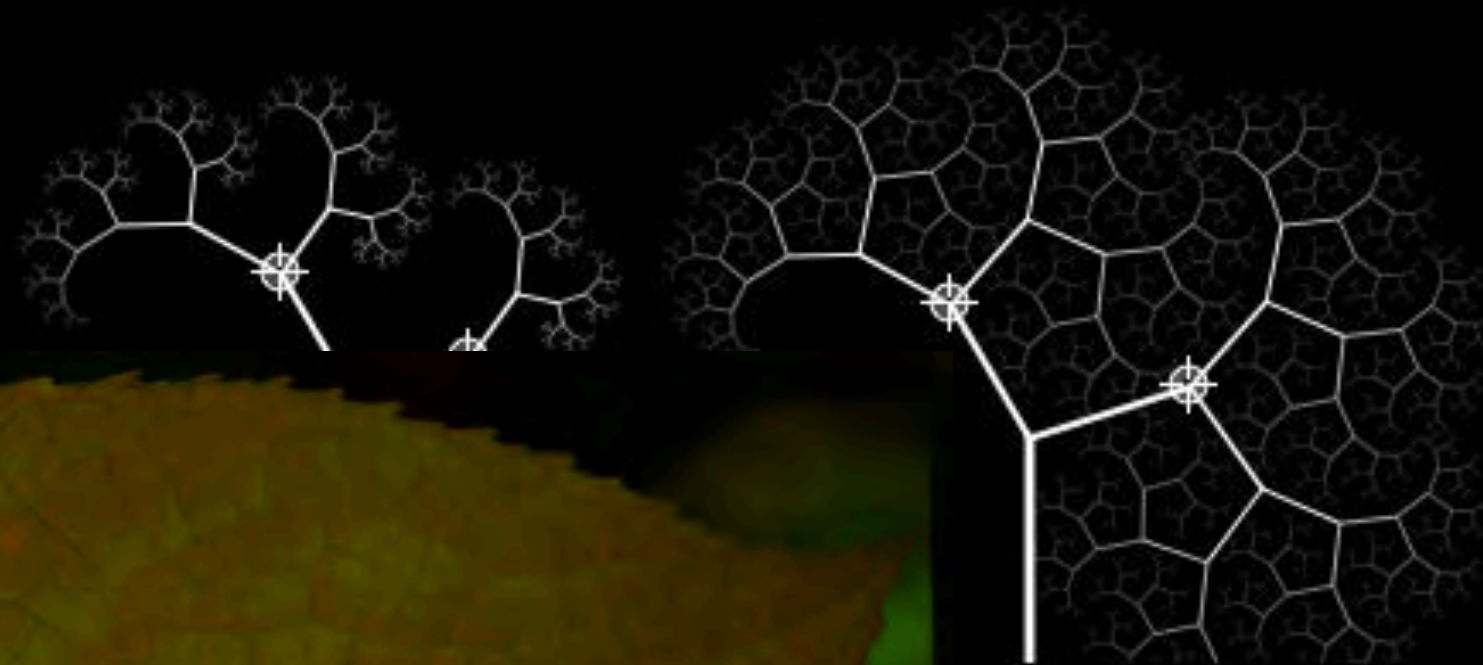
Bernat Espigulé

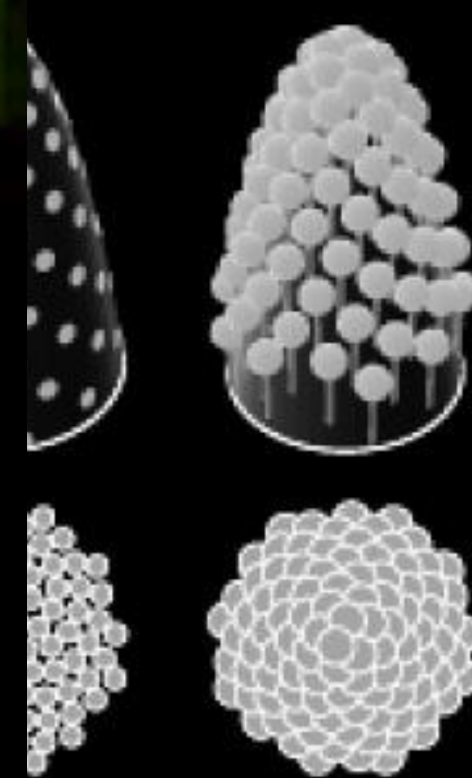
bernat@espigule.com

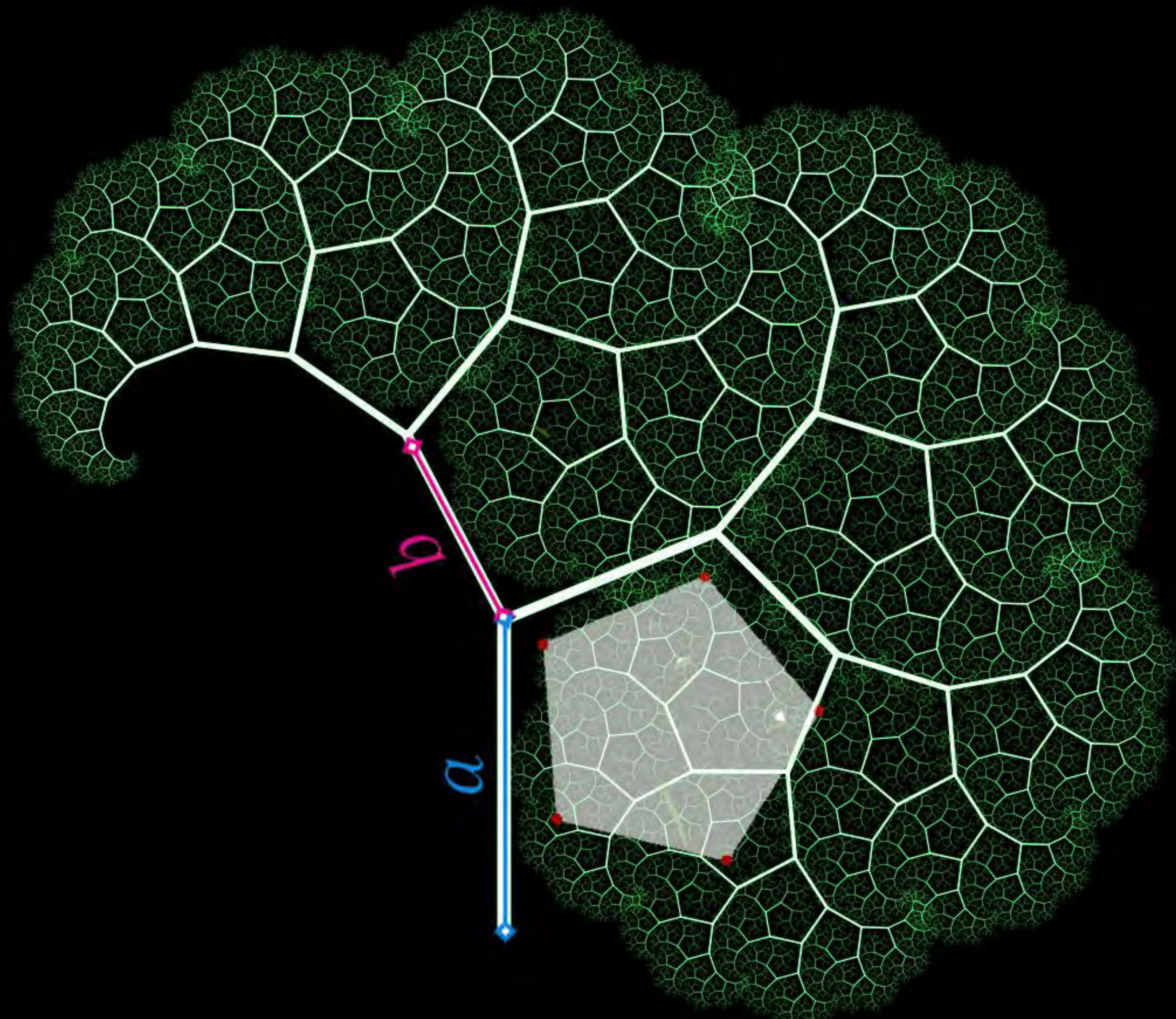

1989 Figueres

2011 University of California, Santa Barbara

The wisest and noblest teacher is nature itself.
—Leonardo da Vinci



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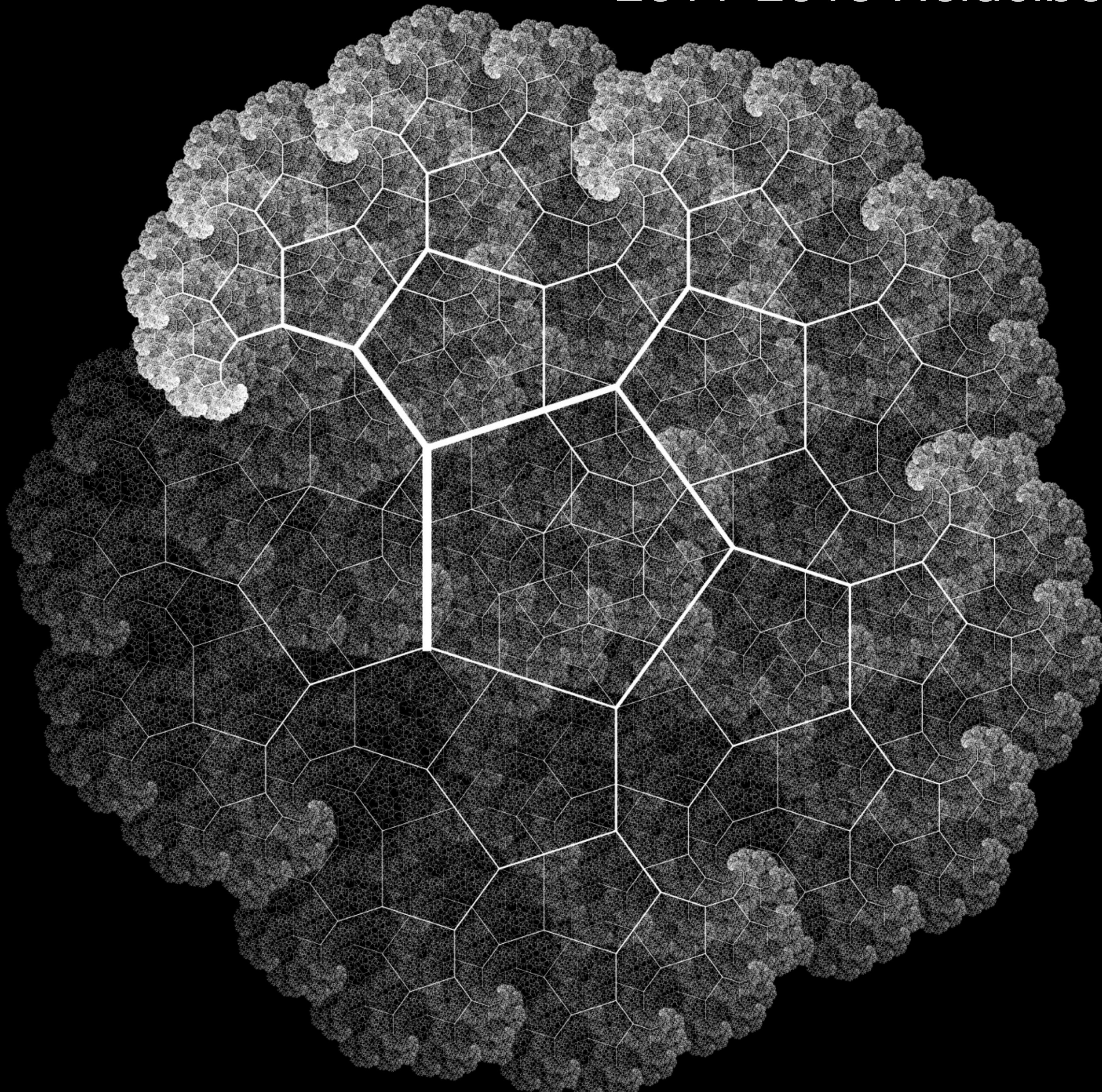
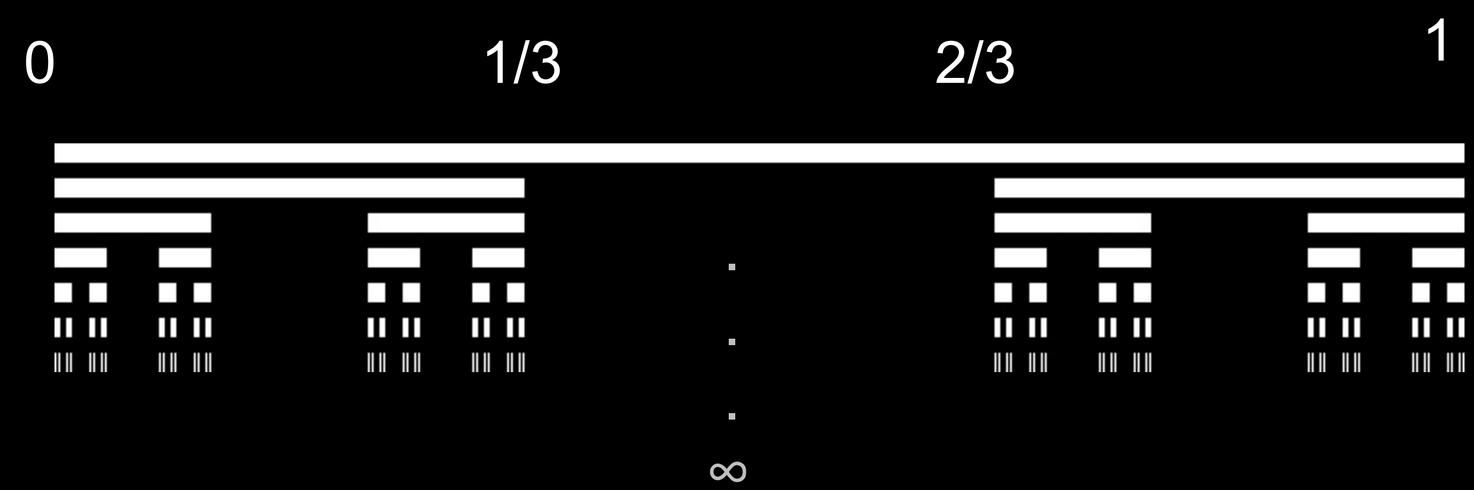
2011-2013 Heidelberg Universität



Georg Cantor
(1845 –1918)

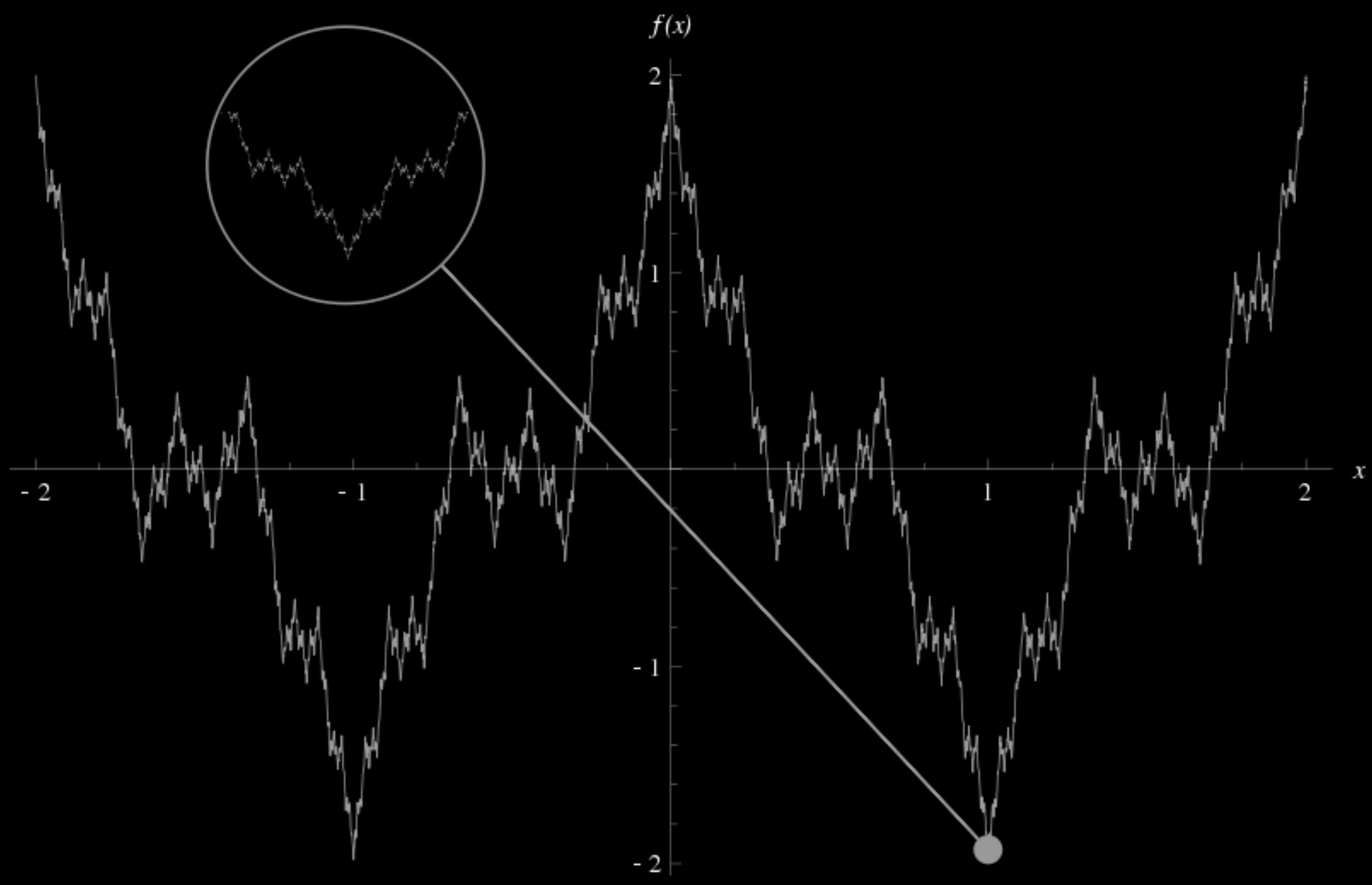


$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right).$$

"Über unendliche, lineare Punktmannigfaltigkeiten V"
[On infinite, linear point-manifolds (sets)],
Mathematische Annalen, vol. 21, pages 545–591. (1883)

Helge von Koch
(1870 –1924)

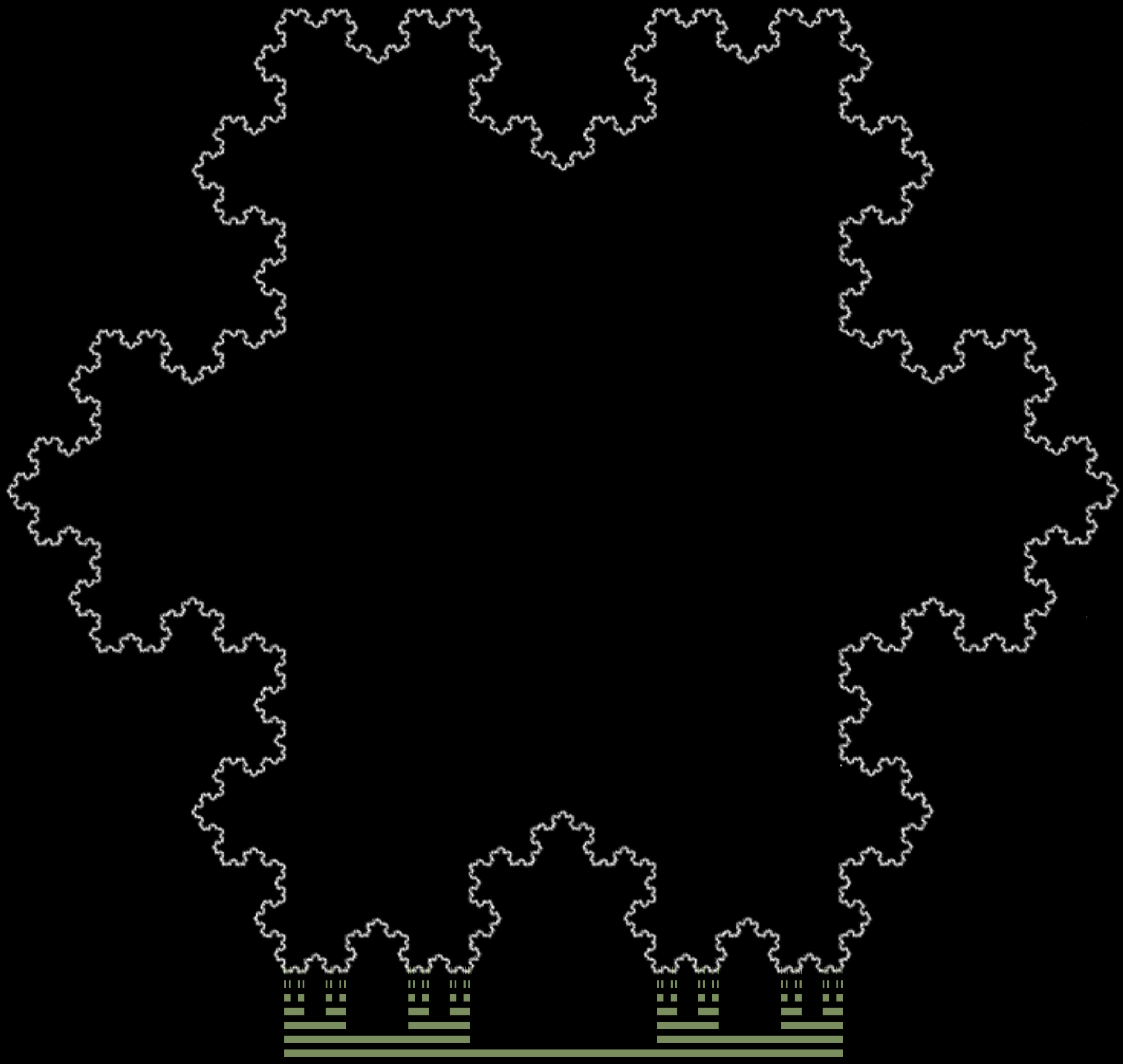
Weierstrass function, 1872



"Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire."

Archiv för Matemat., Astron. och Fys. 1, 681-702, 1904.

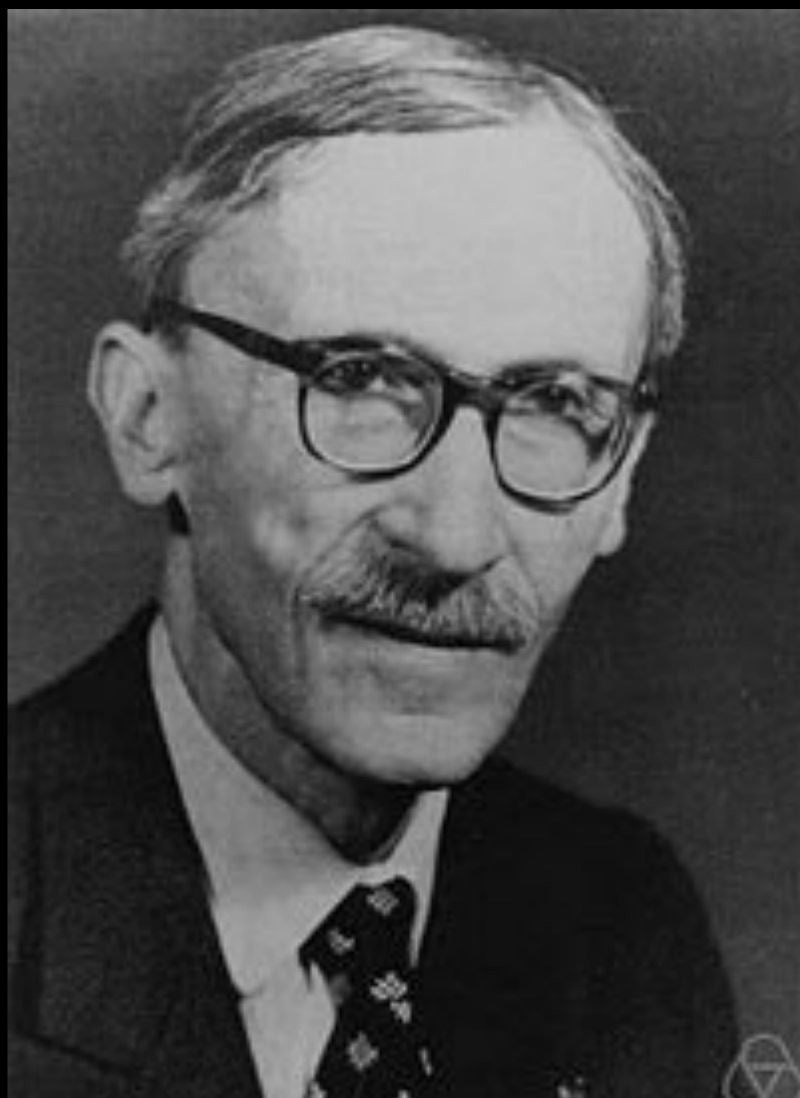
Helge von Koch
(1870 –1924)

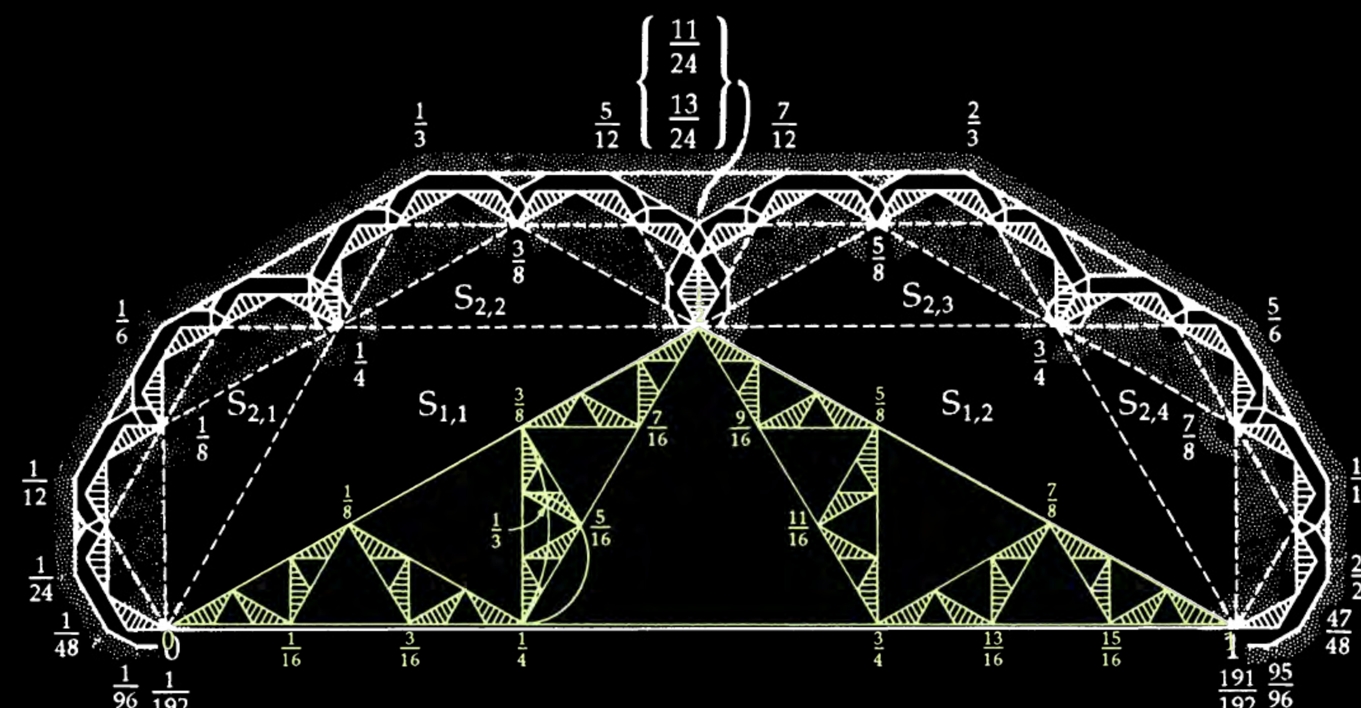


"Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire."

Archiv för Matemat., Astron. och Fys. 1, 681-702, 1904.

Paul Lévy
(1886 –1971)

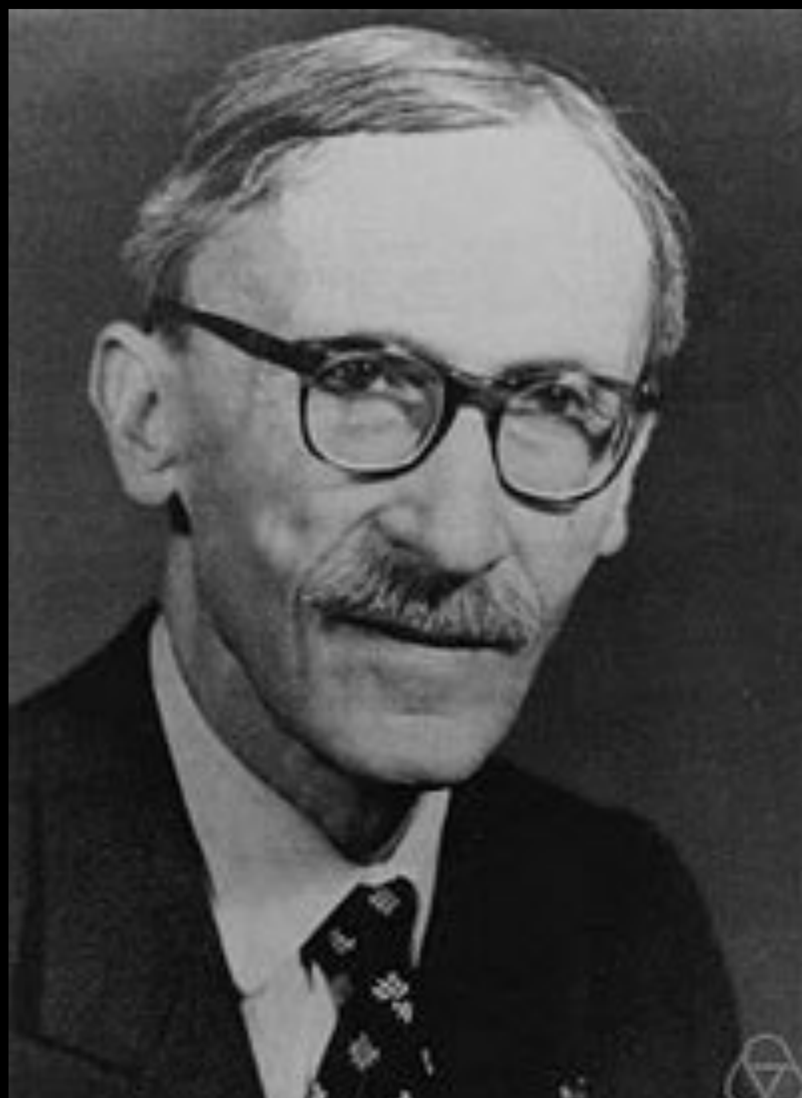


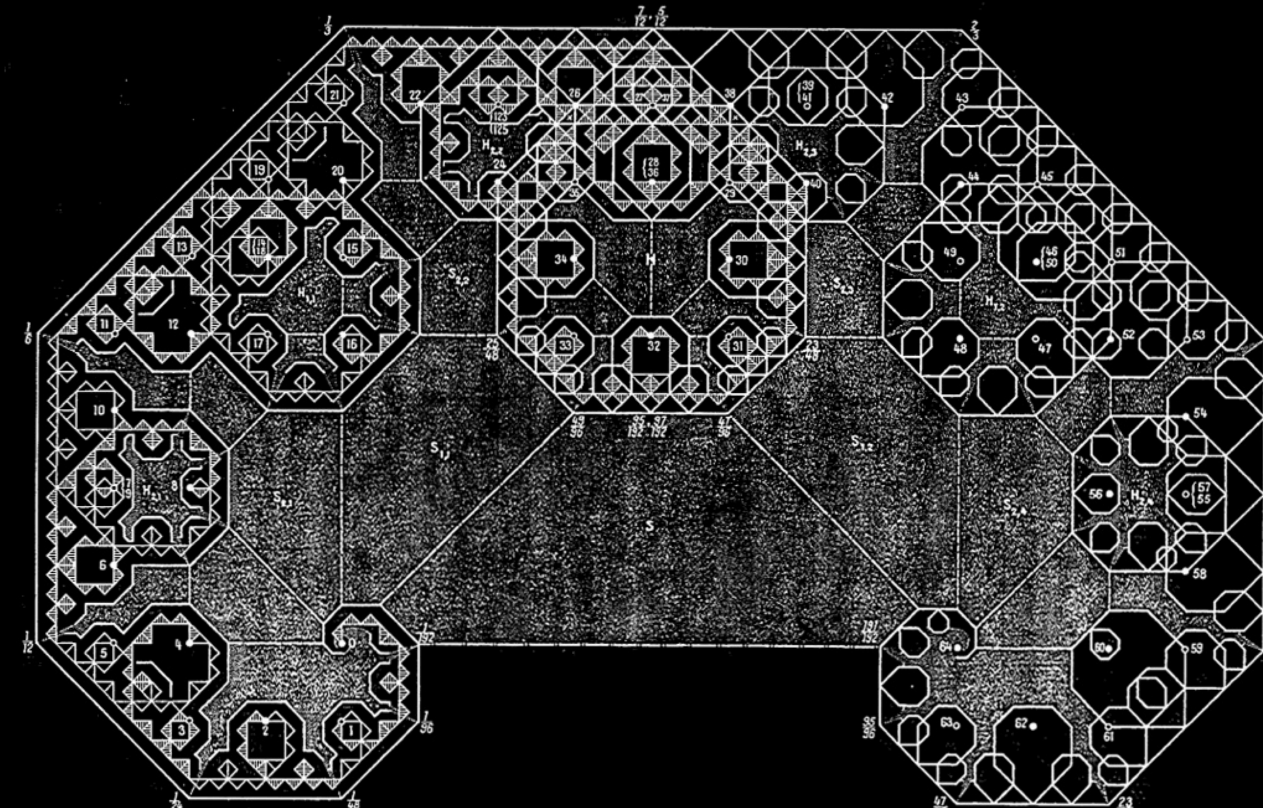


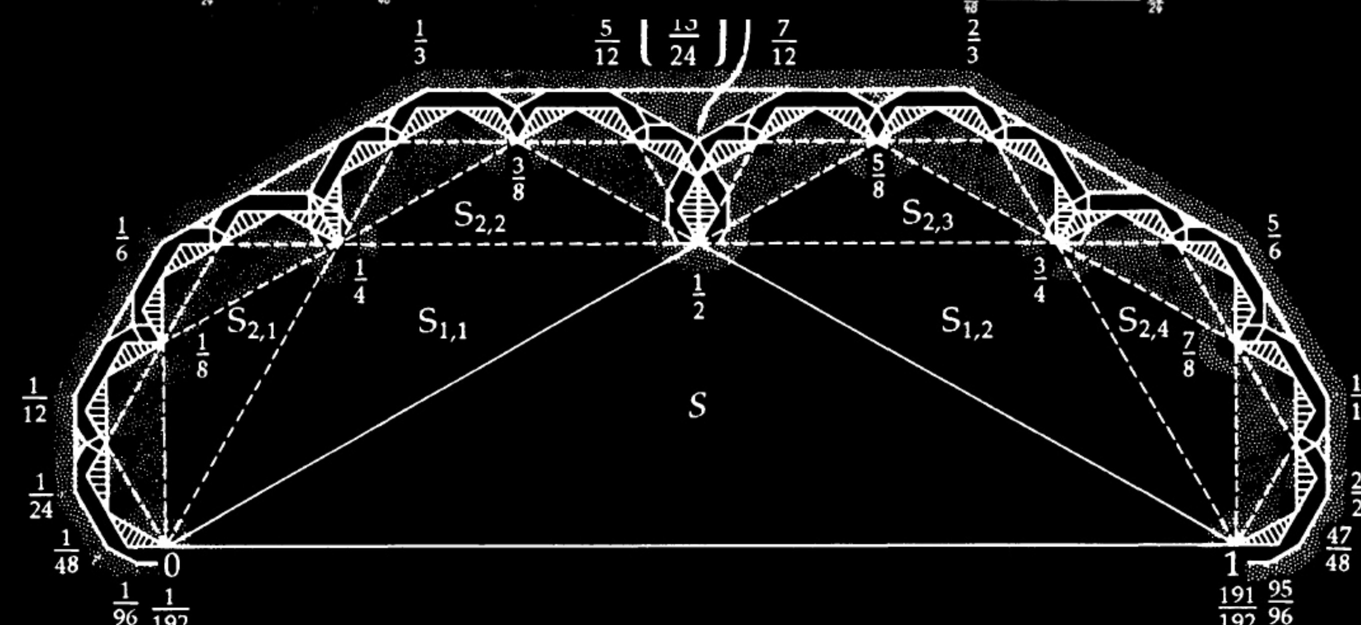
"Les courbes planes ou gauches et les surfaces composées de parties semblales au tout. »

J. l'École Polytech., 227-247 and 249-291, 1939.

Paul Lévy
(1886 –1971)





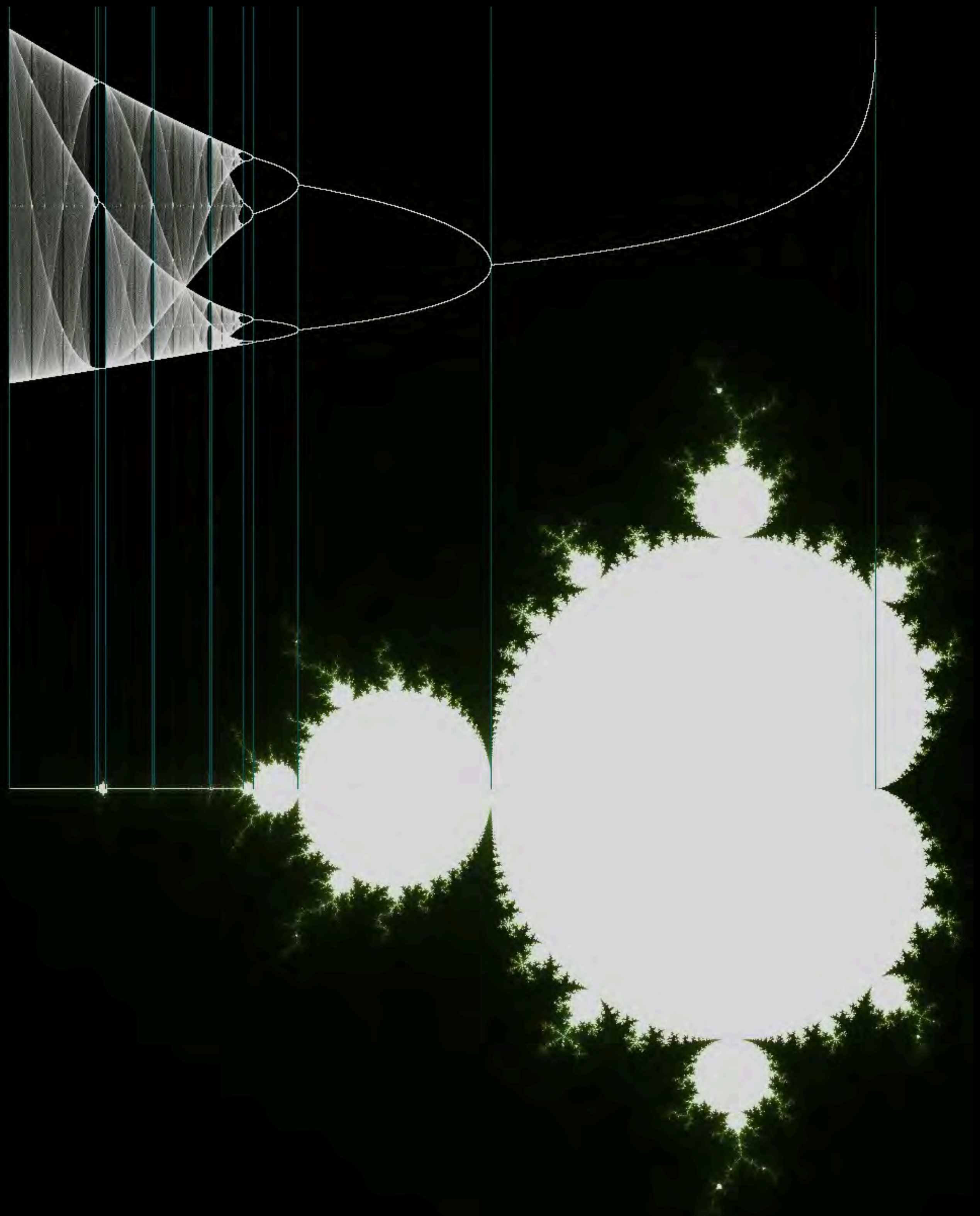


"Les courbes planes ou gauches et les surfaces composées de parties semblales au tout. »

J. l'École Polytech., 227-247 and 249-291, 1939.

Benoît Mandelbrot
(1924 –2010)

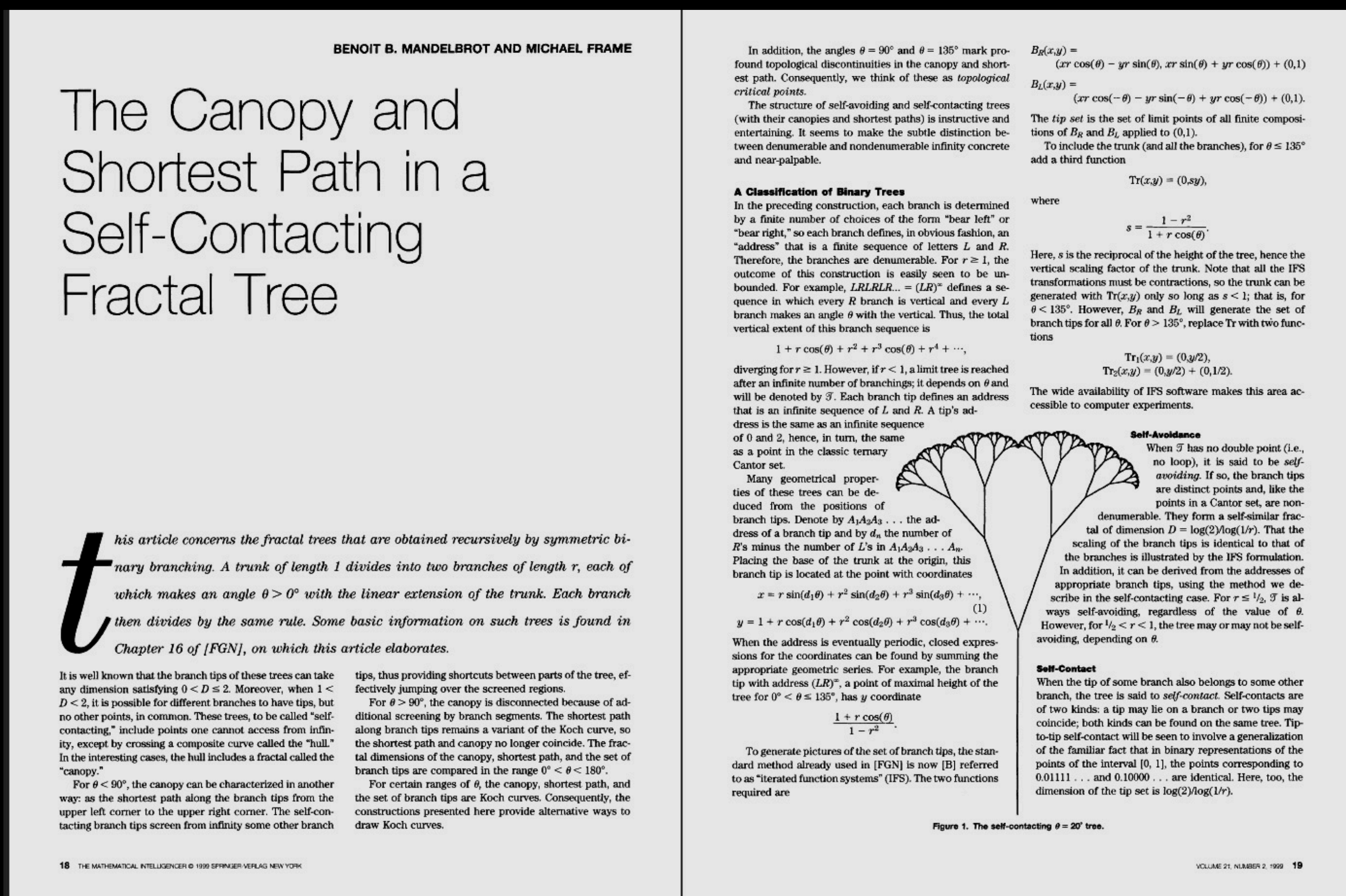
« Les objectes fractals.
Forme, hasard et dimension »
Paris: Flammarion. 1975



Self-Contacting Fractal Trees

Michael Frame and Benôit B. Mandelbrot

The Mathematical Intelligencer, 1999 Springer-Verlag


BENOIT B. MANDELBROT AND MICHAEL FRAME

The Canopy and Shortest Path in a Self-Contacting Fractal Tree

This article concerns the fractal trees that are obtained recursively by symmetric binary branching. A trunk of length 1 divides into two branches of length r, each of which makes an angle $\theta > 0°$ with the linear extension of the trunk. Each branch then divides by the same rule. Some basic information on such trees is found in Chapter 16 of [FGN], on which this article elaborates.

It is well known that the branch tips of these trees can take any dimension satisfying $0 < D \leq 2$. Moreover, when $1 < D < 2$, it is possible for different branches to have tips, but no other points, in common. These trees, to be called "self-contacting," include points one cannot access from infinity, except by crossing a composite curve called the "hull." In the interesting cases, the hull includes a fractal called the "canopy."

For $\theta < 90°$, the canopy can be characterized in another way: as the shortest path along the branch tips from the upper left corner to the upper right corner. The self-contacting branch tips screen from infinity some other branch tips, thus providing shortcuts between parts of the tree, effectively jumping over the screened regions.

For $\theta > 90°$, the canopy is disconnected because of additional screening by branch segments. The shortest path along branch tips remains a variant of the Koch curve, so the shortest path and canopy no longer coincide. The fractal dimensions of the canopy, shortest path, and the set of branch tips are compared in the range $0° < \theta < 180°$.

For certain ranges of θ, the canopy, shortest path, and the set of branch tips are Koch curves. Consequently, the constructions presented here provide alternative ways to draw Koch curves.

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In addition, the angles $\theta = 90°$ and $\theta = 135°$ mark profound topological discontinuities in the canopy and shortest path. Consequently, we think of these as *topological critical points.*

The structure of self-avoiding and self-contacting trees (with their canopies and shortest paths) is instructive and entertaining. It seems to make the subtle distinction between denumerable and nondenumerable infinity concrete and near-palpable.

A Classification of Binary Trees

In the preceding construction, each branch is determined by a finite number of choices of the form "bear left" or "bear right," so each branch defines, in obvious fashion, an "address" that is a finite sequence of letters L and R. Therefore, the branches are denumerable. For $r \geq 1$, the outcome of this construction is easily seen to be unbounded. For example, $LRLRLR... = (LR)^\infty$ defines a sequence in which every R branch is vertical and every L branch makes an angle θ with the vertical. Thus, the total vertical extent of this branch sequence is

$$1 + r\cos(\theta) + r^2 + r^3\cos(\theta) + r^4 + \cdots,$$

diverging for $r \geq 1$. However, if $r < 1$, a limit tree is reached after an infinite number of branchings; it depends on θ and will be denoted by $\mathcal{T}$. Each branch tip defines an address that is an infinite sequence of L and R. A tip's address is the same as an infinite sequence of 0 and 2, hence, in turn, the same as a point in the classic ternary Cantor set.

Many geometrical properties of these trees can be deduced from the positions of branch tips. Denote by $A_1A_2A_3 \ldots$ the address of a branch tip and by d_n the number of R's minus the number of L's in $A_1A_2A_3 \ldots A_n$. Placing the base of the trunk at the origin, this branch tip is located at the point with coordinates

$$x = r\sin(d_1\theta) + r^2\sin(d_2\theta) + r^3\sin(d_3\theta) + \cdots,$$
$$y = 1 + r\cos(d_1\theta) + r^2\cos(d_2\theta) + r^3\cos(d_3\theta) + \cdots. \quad (1)$$

When the address is eventually periodic, closed expressions for the coordinates can be found by summing the appropriate geometric series. For example, the branch tip with address $(LR)^\infty$, a point of maximal height of the tree for $0° < \theta \leq 135°$, has y coordinate

$$\frac{1 + r\cos(\theta)}{1 - r^2}.$$

To generate pictures of the set of branch tips, the standard method already used in [FGN] is now [B] referred to as "iterated function systems" (IFS). The two functions required are

$$B_R(x,y) = (xr\cos(\theta) - yr\sin(\theta), xr\sin(\theta) + yr\cos(\theta)) + (0,1)$$
$$B_L(x,y) = (xr\cos(-\theta) - yr\sin(-\theta) + yr\cos(-\theta)) + (0,1).$$

The *tip set* is the set of limit points of all finite compositions of B_R and B_L applied to (0,1).

To include the trunk (and all the branches), for $\theta \leq 135°$ add a third function

$$\mathrm{Tr}(x,y) = (0,sy),$$

where

$$s = \frac{1 - r^2}{1 + r\cos(\theta)}.$$

Here, s is the reciprocal of the height of the tree, hence the vertical scaling factor of the trunk. Note that all the IFS transformations must be contractions, so the trunk can be generated with $\mathrm{Tr}(x,y)$ only so long as $s < 1$; that is, for $\theta < 135°$. However, B_R and B_L will generate the set of branch tips for all θ. For $\theta > 135°$, replace Tr with two functions

$$\mathrm{Tr}_1(x,y) = (0,y/2),$$
$$\mathrm{Tr}_2(x,y) = (0,y/2) + (0,1/2).$$

The wide availability of IFS software makes this area accessible to computer experiments.

Self-Avoidance

When $\mathcal{T}$ has no double point (i.e., no loop), it is said to be *self-avoiding.* If so, the branch tips are distinct points and, like the points in a Cantor set, are nondenumerable. They form a self-similar fractal of dimension $D = \log(2)/\log(1/r)$. That the scaling of the branch tips is identical to that of the branches is illustrated by the IFS formulation. In addition, it can be derived from the addresses of appropriate branch tips, using the method we describe in the self-contacting case. For $r \leq 1/2$, $\mathcal{T}$ is always self-avoiding, regardless of the value of θ. However, for $1/2 < r < 1$, the tree may or may not be self-avoiding, depending on θ.

Self-Contact

When the tip of some branch also belongs to some other branch, the tree is said to *self-contact.* Self-contacts are of two kinds: a tip may lie on a branch or two tips may coincide; both kinds can be found on the same tree. Tip-to-tip self-contact will be seen to involve a generalization of the familiar fact that in binary representations of the points of the interval [0, 1], the points corresponding to 0.01111 . . . and 0.10000 . . . are identical. Here, too, the dimension of the tip set is $\log(2)/\log(1/r)$.

Figure 1. The self-contacting $\theta = 20°$ tree.

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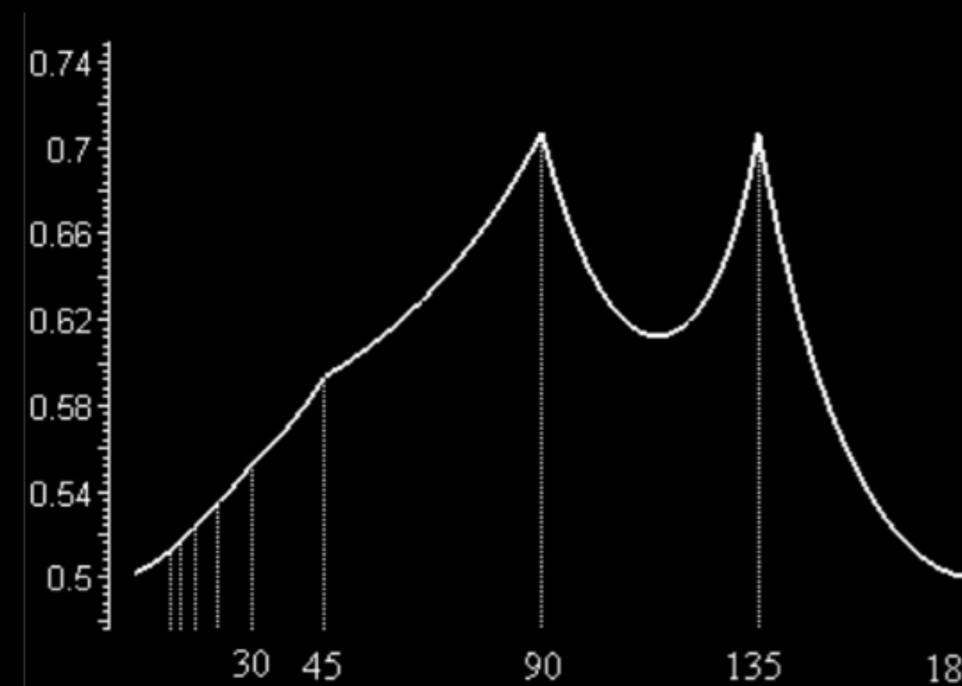
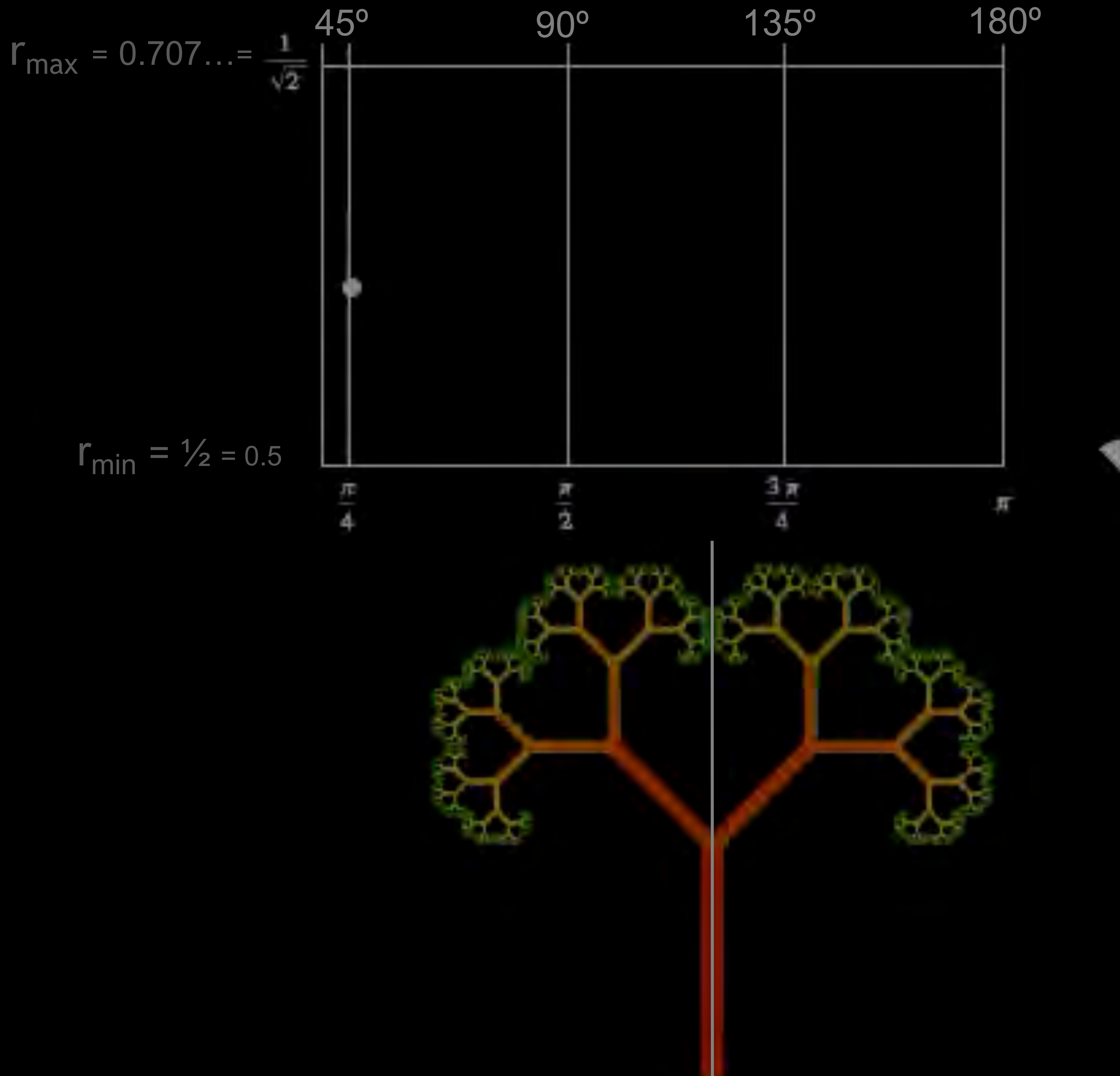
Mathematical Details for the article of Frame & Mandelbrot
Self-Contacting Fractal Trees
by *Prof. Donald C. West* © (1999)

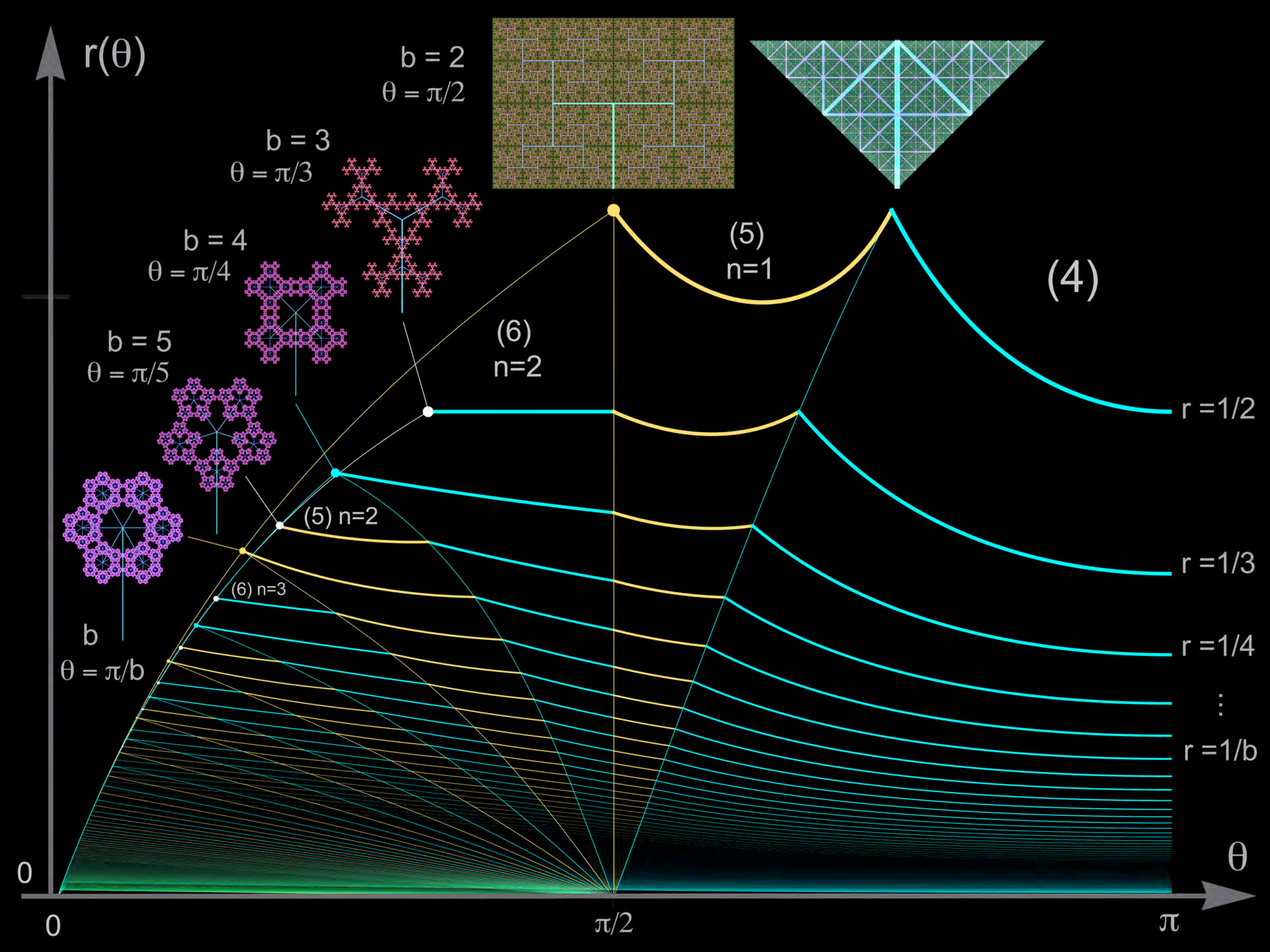


http://faculty.plattsburgh.edu/don.west/trees/index.htm

Tara D. Taylor
"Computational Topology and Fractal Trees",
PhD Thesis, Dalhousie University, Canada (2005)

Tara D. Taylor "TOPOLOGICAL BAR-CODES OF FRACTALS: A NEW CHARACTERIZATION OF SYMMETRIC BINARY FRACTAL TREES" CONVEX AND FRACTAL GEOMETRY, VOLUME 84, (2009)

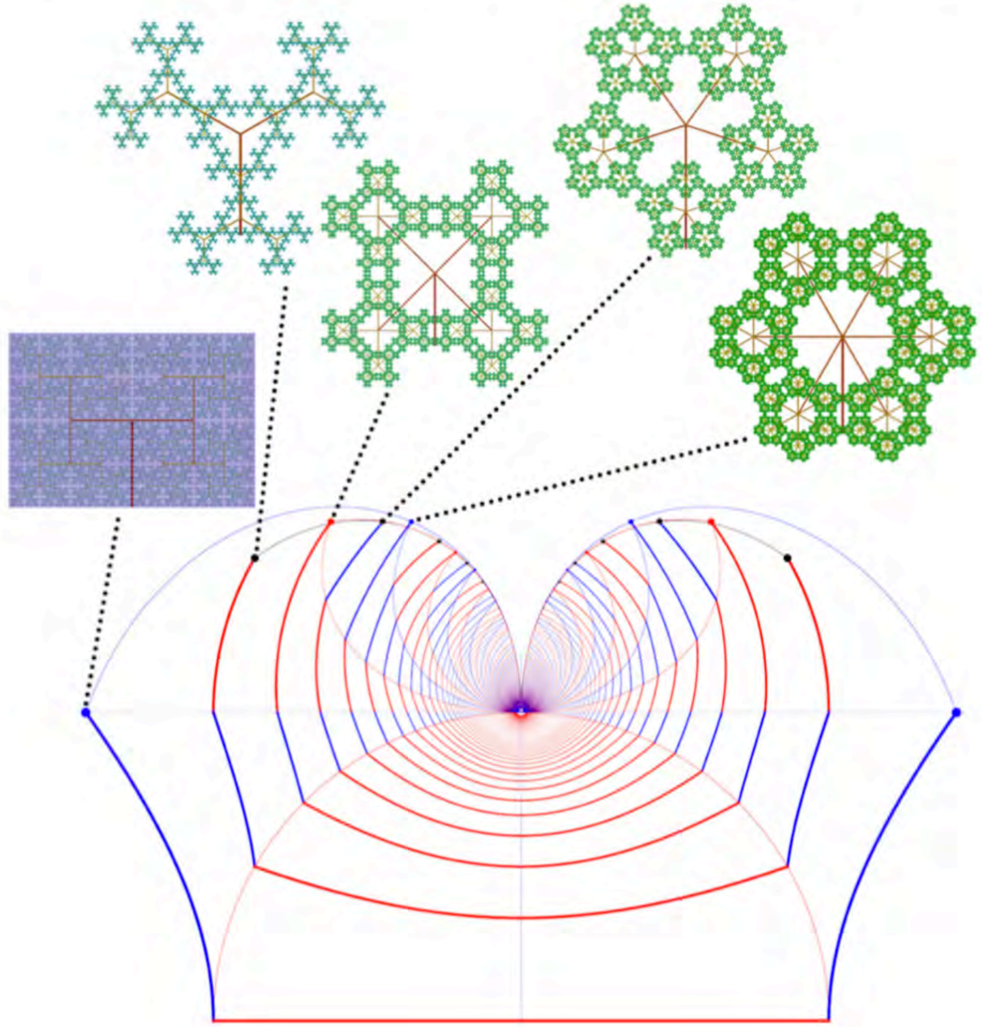

45º
90º
135º
180º
r max = 0.707…= 1/√2
r min = ½ = 0.5
π/4
π/2
3π/4
π



r(θ)
b = 2
θ = π/2
b = 3
θ = π/3
b = 4
θ = π/4
b = 5
θ = π/5
b
θ = π/b
(5)
n=1
(4)
(6)
n=2
(5) n=2
(6) n=3
r =1/2
r =1/3
r =1/4
r =1/b
0
0
π/2
π
θ

Publications 2013

Generalized self-contacting symmetric fractal trees. Journal Symmetry, 24 (1-4): 320–338.



Bridges 2013

Unfolding Symmetric Fractal Trees http://archive.bridgesmathart.org/2013/bridges2013-295.html

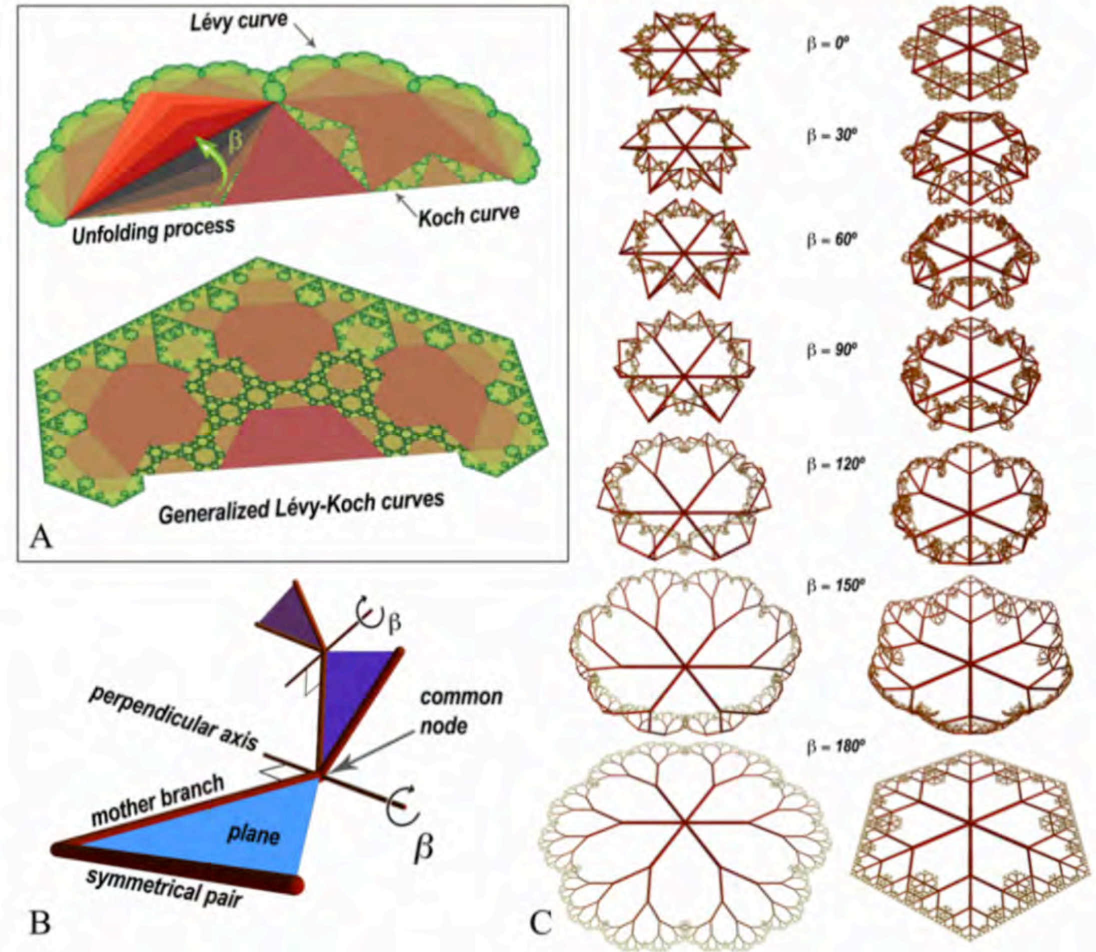
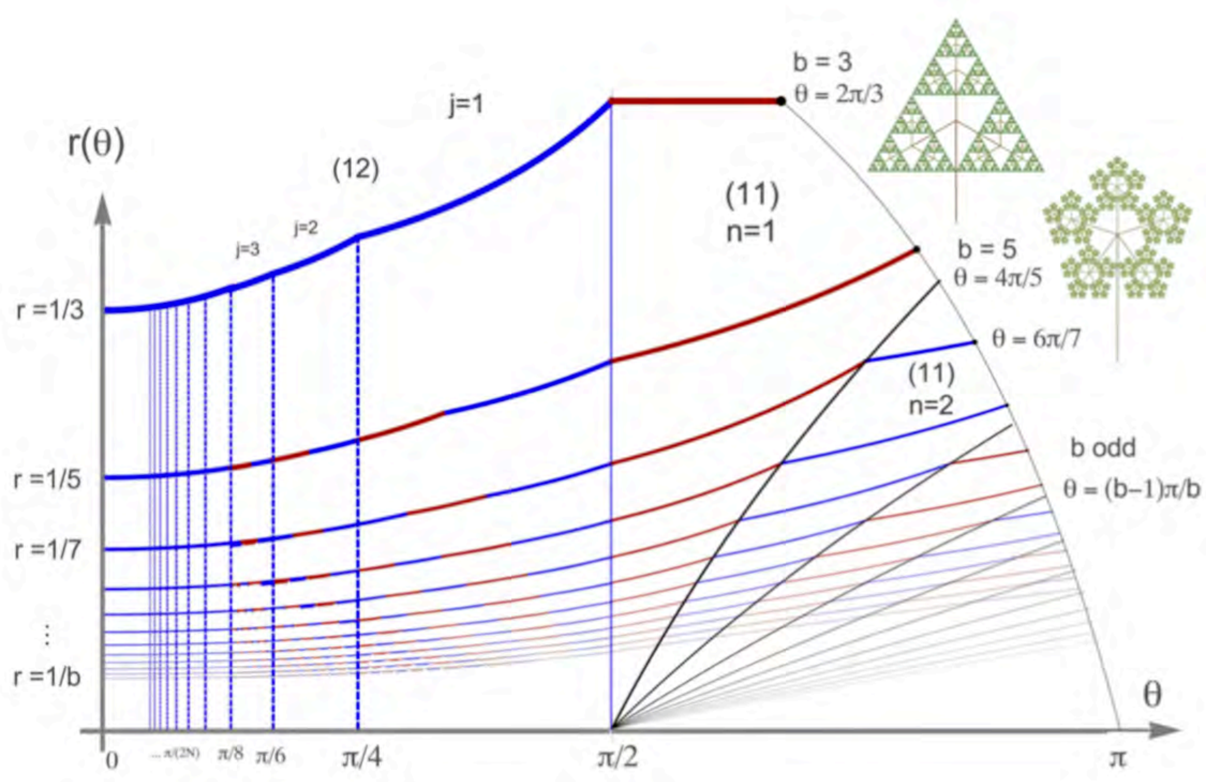
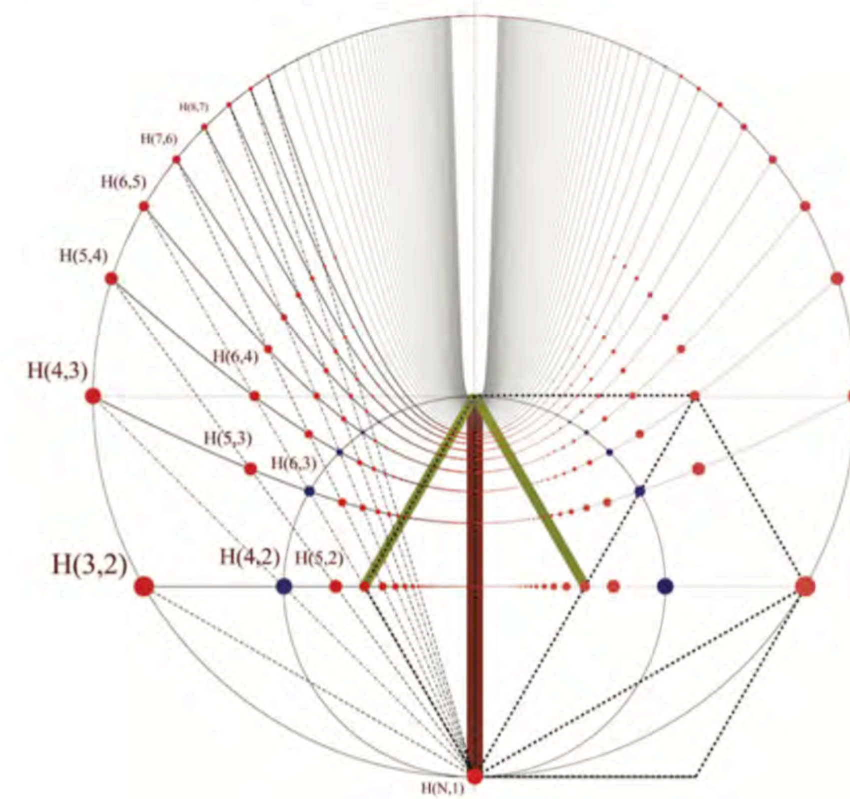
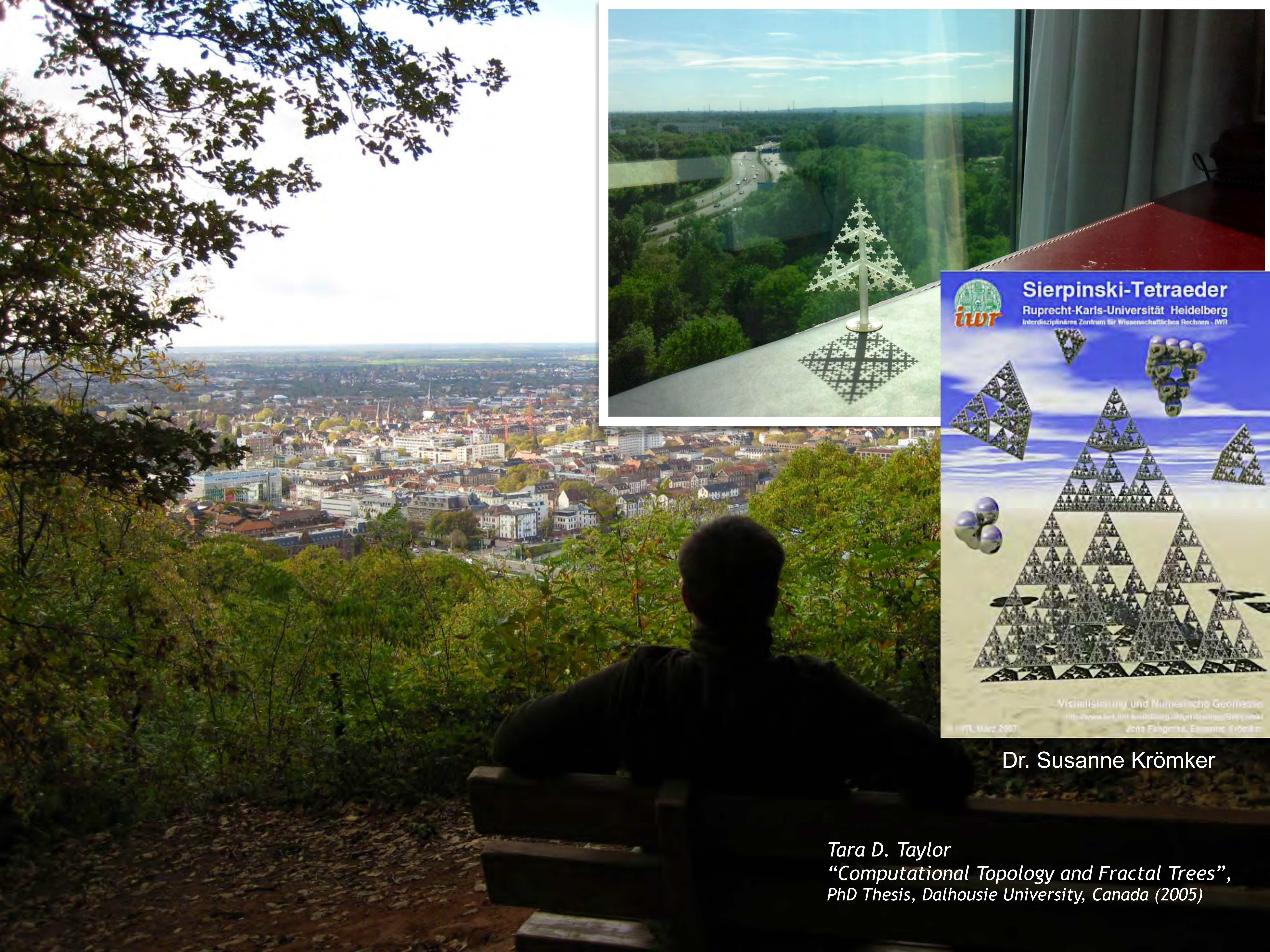


Figure 3: *A: Lévy curve constructed by unfolding a Koch curve, and the example of a new generalization associated to higher-order harmonic trees $H(N, b > 2)$. B: schematic diagram showing the branch planes and the axes of rotation. C: image sequences of the unfolding process applied to $H(6,2)$ (left) and $H(6,3)$ (right).*







Dr. Susanne Krömker

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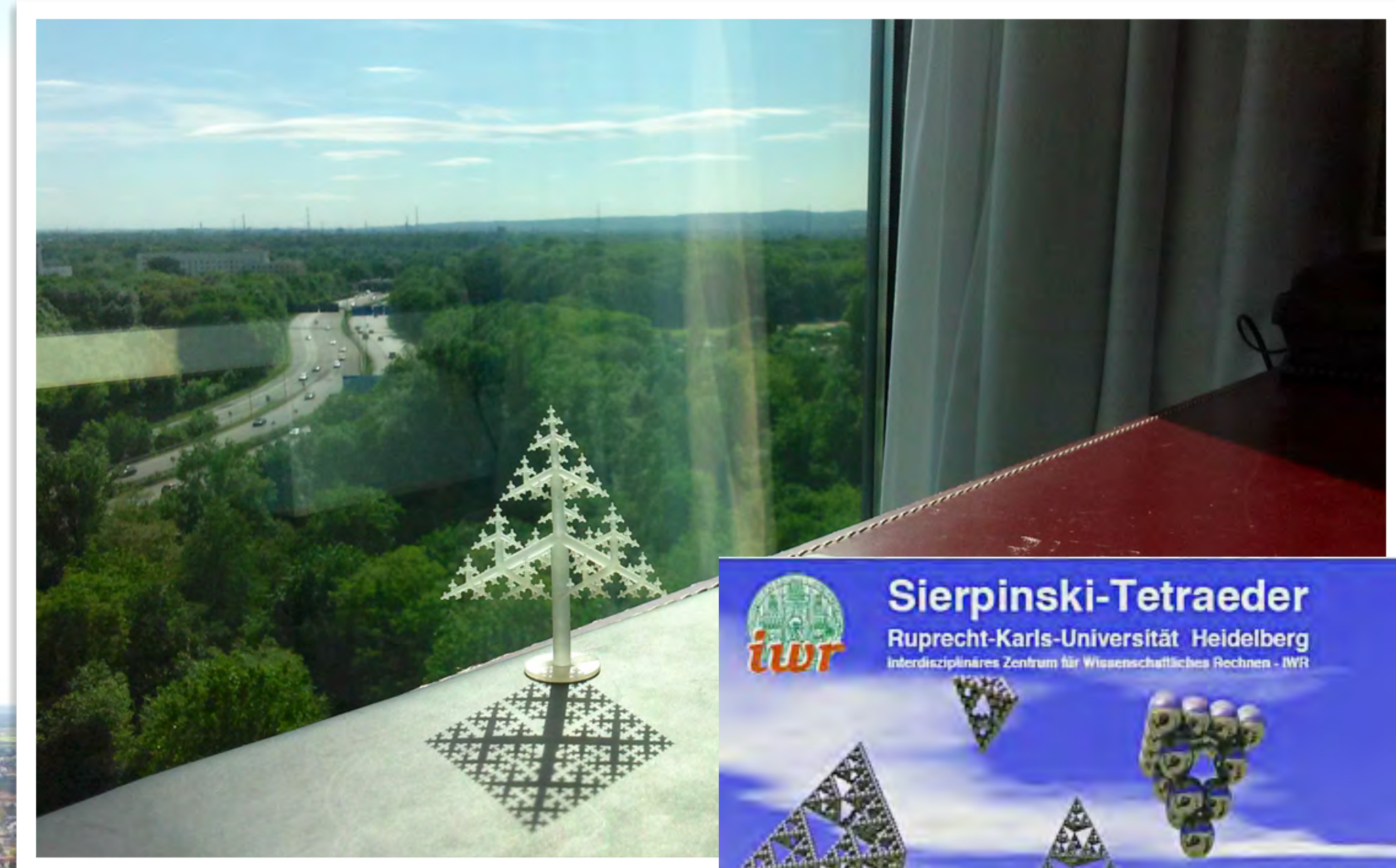
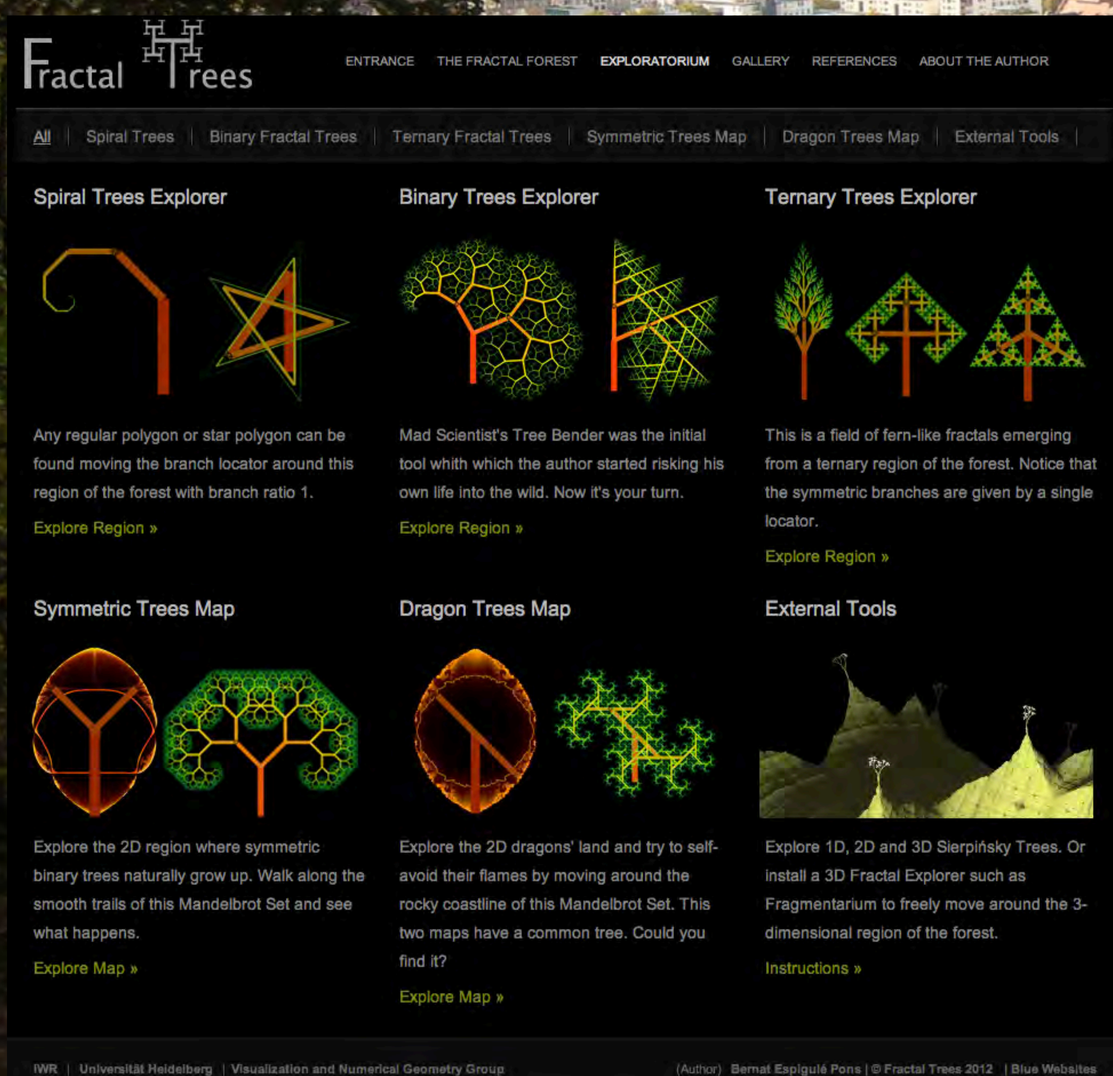
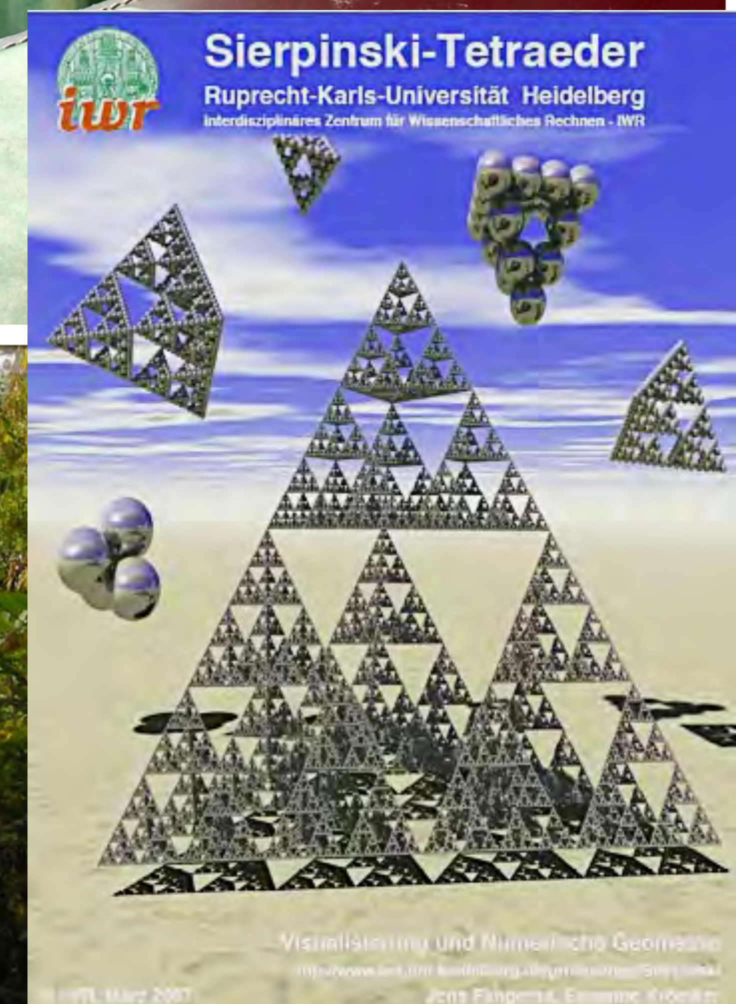
PhD Thesis, Dalhousie University, Canada (2005)







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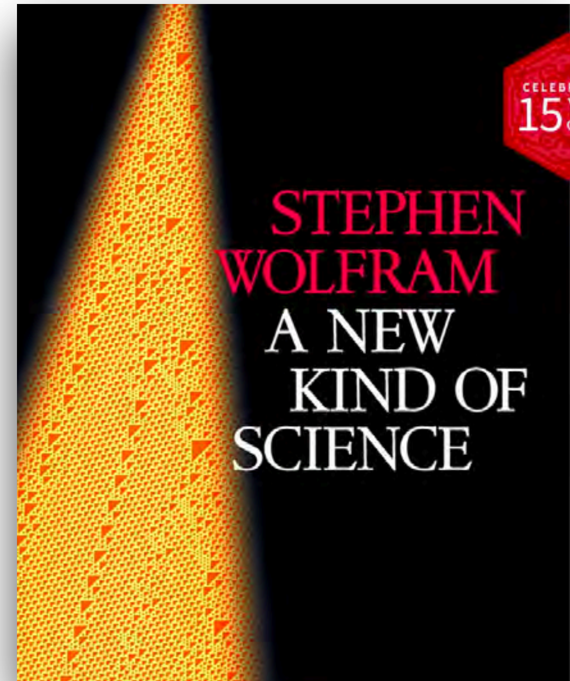


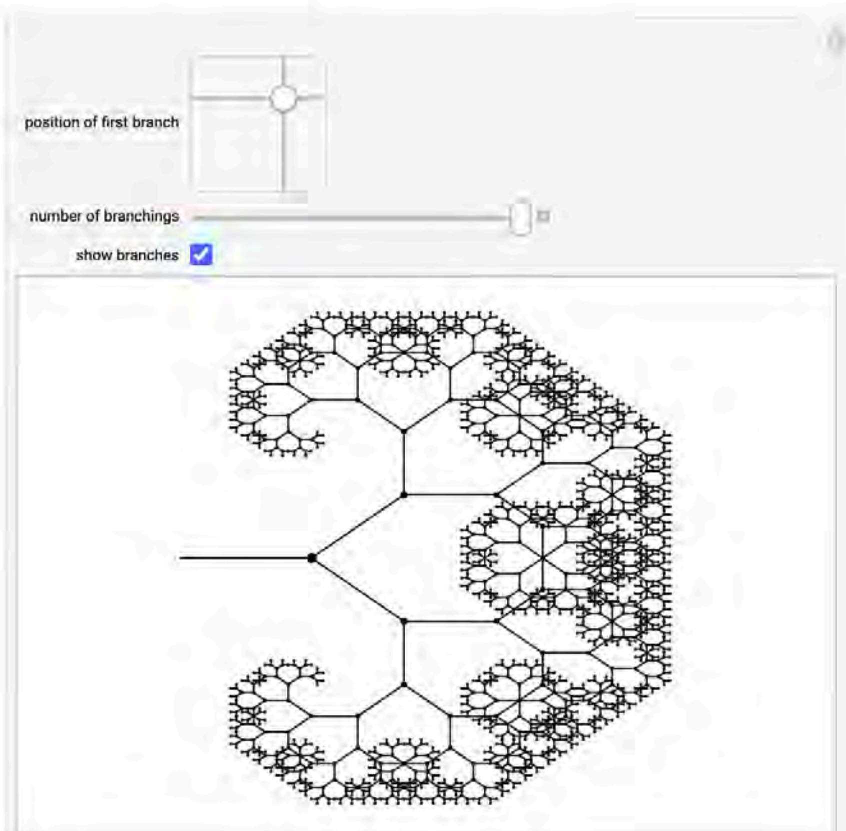
WOLFRAM Demonstrations Project

blog.wolfram.com/author/michael-trott/

Chris Carlson christophercarlson.com

Theodore Gray

Limits of Tree Branching



Stephen Wolfram (2011), "Limits of Tree Branching"
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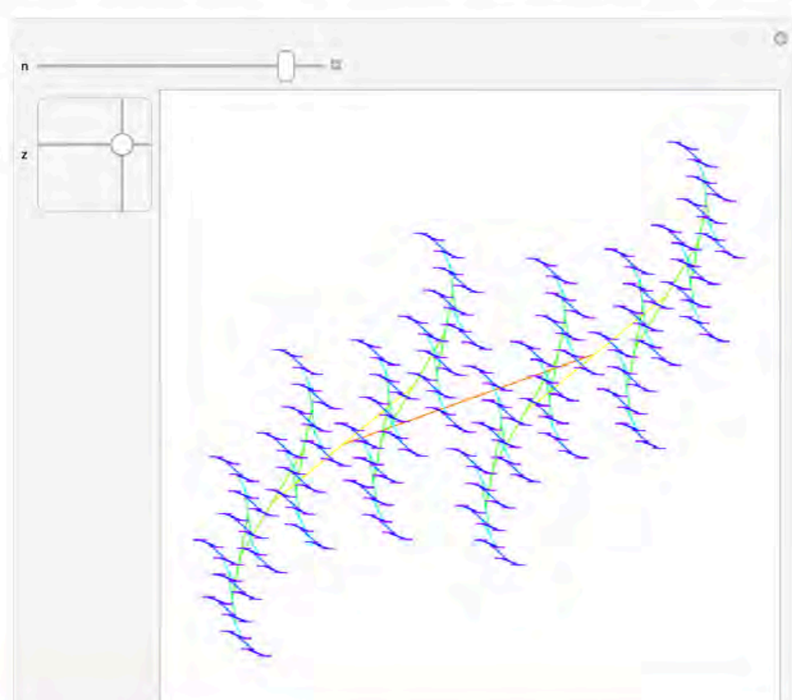
demonstrations.wolfram.com/LimitsOfTreeBranching/

Michael Trott (1959-2025)

All Possible Sums and Differences of Powers



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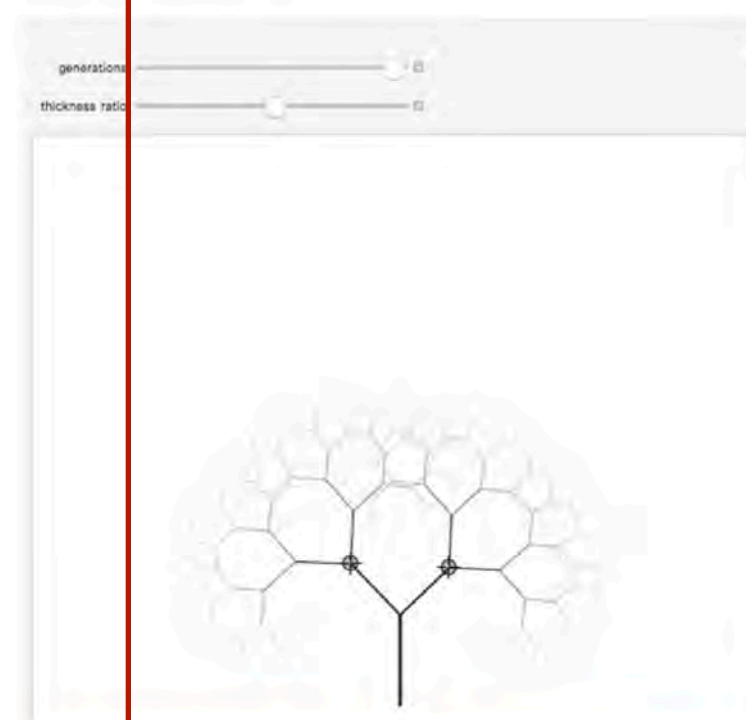
Tree Bender



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Tara D Taylor. *Computational topology and fractal trees.* PhD thesis, Dalhousie University, 2005.

Tara D Taylor. Homeomorphism classes of self-contacting symmetric binary fractal trees. *Fractals*, 15(01):9–25, 2007.

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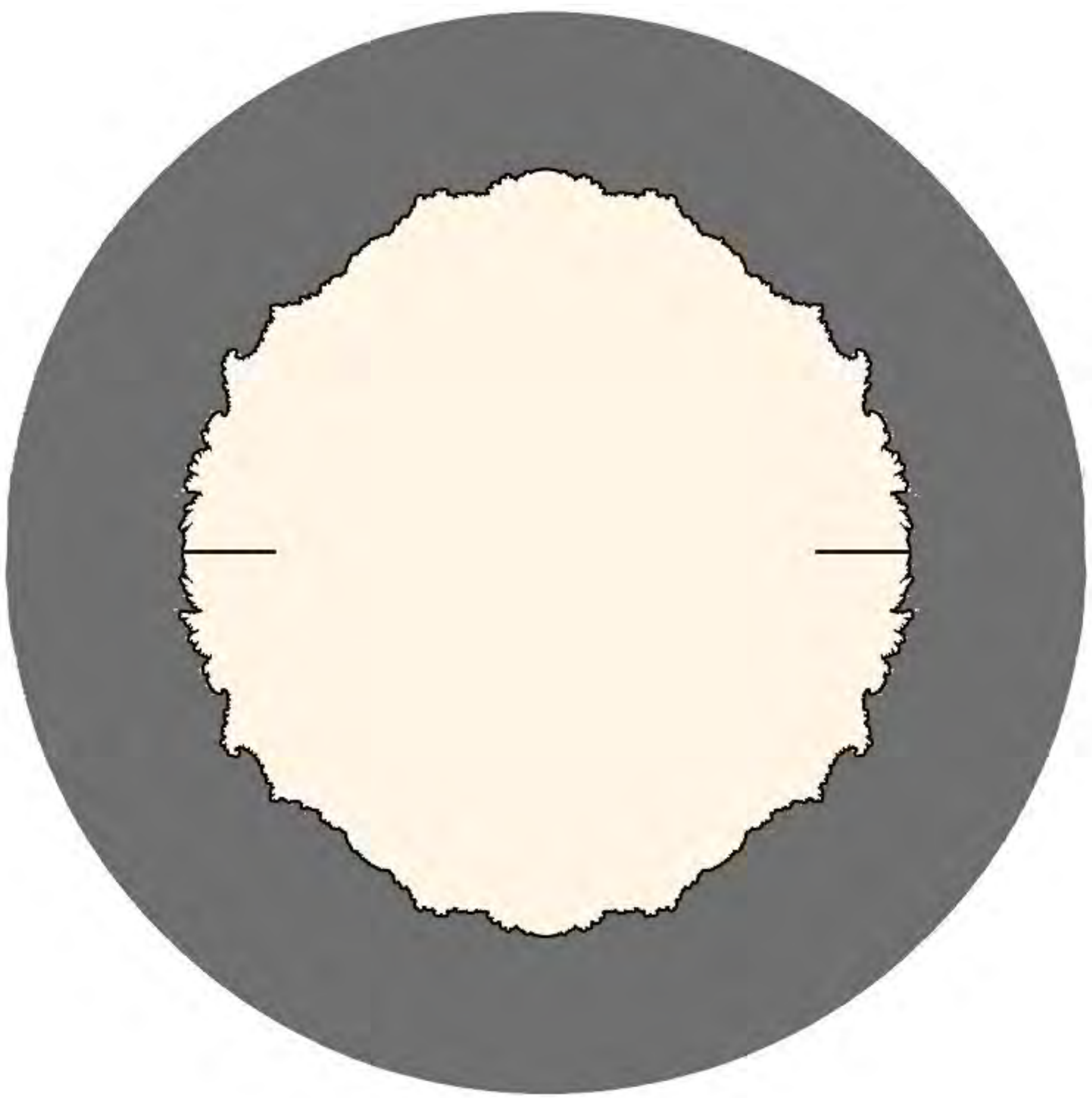

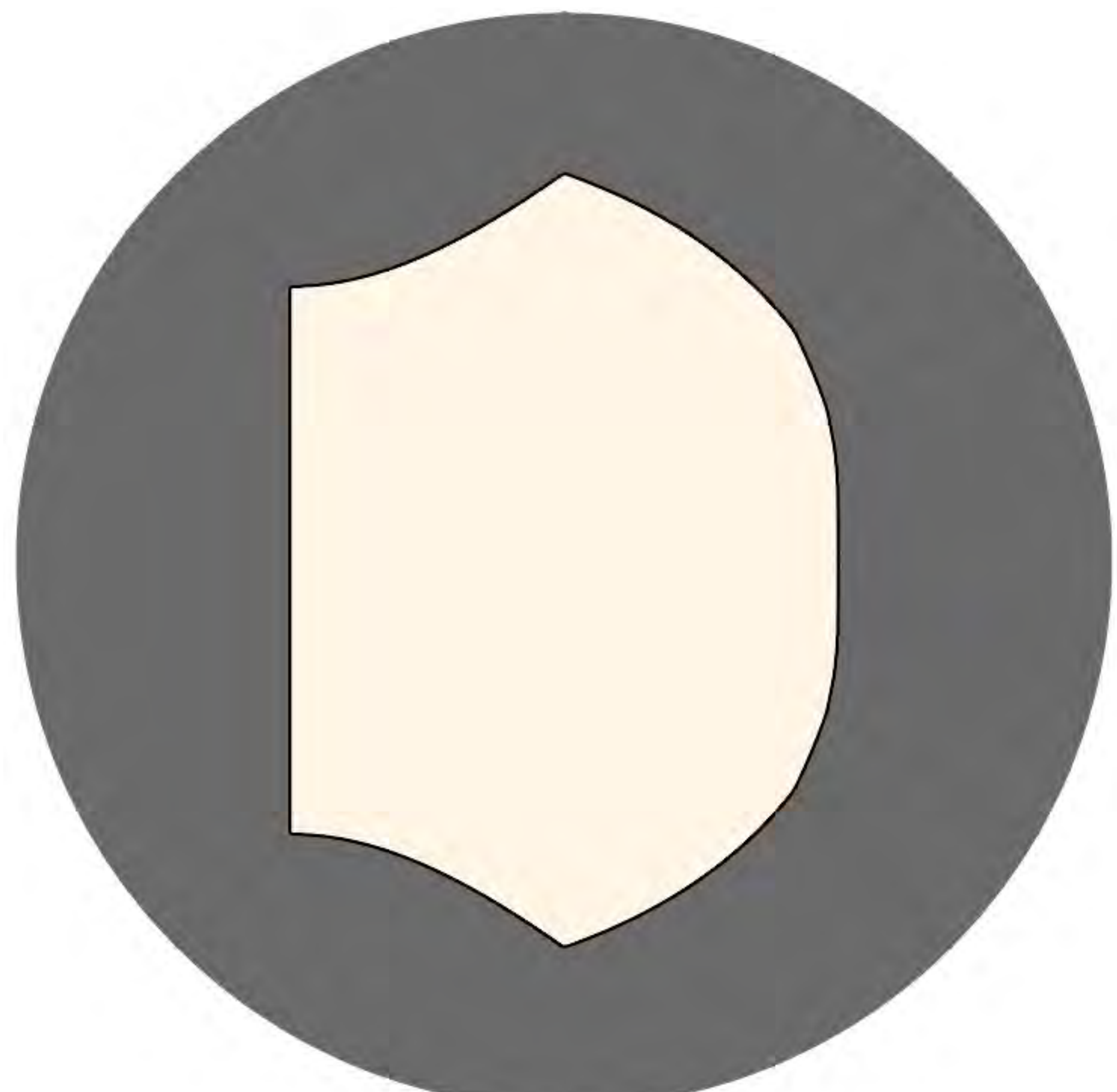

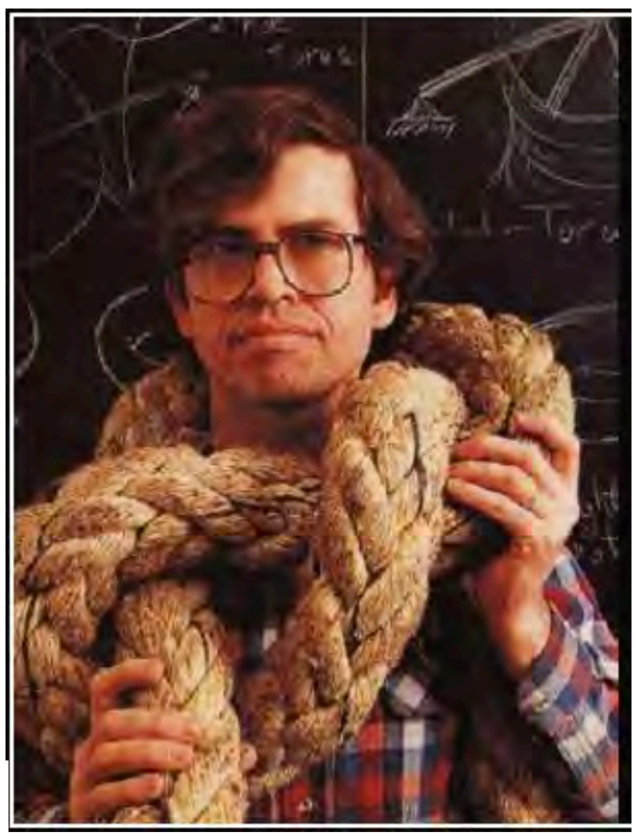

Bill Thurston

(1946 - 2012)

mathoverflow

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Perron number distribution

60

18

A *Perron number* is a real algebraic integer λ that is larger than the absolute value of any of its Galois conjugates. The Perron-Frobenius theorem says that any non-negative integer matrix M such that some power of M is strictly positive has a unique positive eigenvector whose eigenvalue is a Perron number. Doug Lind proved the converse: given a Perron number λ, there exists such a matrix, perhaps in dimension much higher than the degree of λ. Perron numbers come up frequently in many places, especially in dynamical systems.

My question:

> What is the limiting distribution of Galois conjugates of Perron numbers λ in some bounded interval, as the degree goes to infinity?

share cite improve this question

edited Jan 11 '11 at 4:04

asked Jan 11 '11 at 3:50
Bill Thurston
21.4k ● 11 ● 82 ● 113

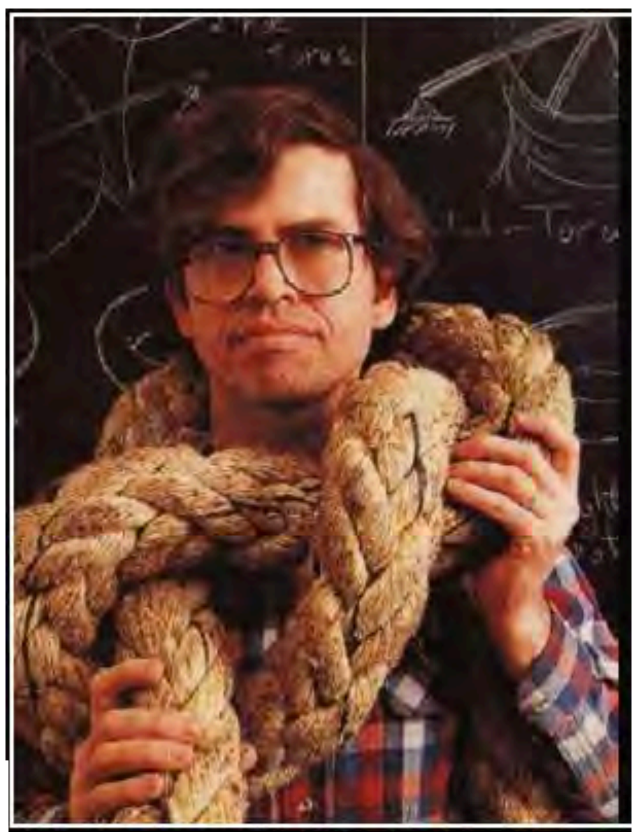
2 Should be related to the distribution at math.ucr.edu/home/baez/roots ; there are references at that link, I think. – Qiaochu Yuan Jan 11 '11 at 4:20

2 @Qiaochu Yuan: Thanks for bringing it up. I actually intended to check out and point to those references, until my question got too long. I was trying to take a slice of things in a way that eliminates the fractal distribution of roots of polynomials with bounded coefficients. My motivation for this question originated in trying understand topological entropy for postcritically finite iterated polynomials, where a Mandelbrot-like distribution comes up that is very related to those exhibited by Baez (and others). – Bill Thurston Jan 11 '11 at 4:41

Thurston’s last paper

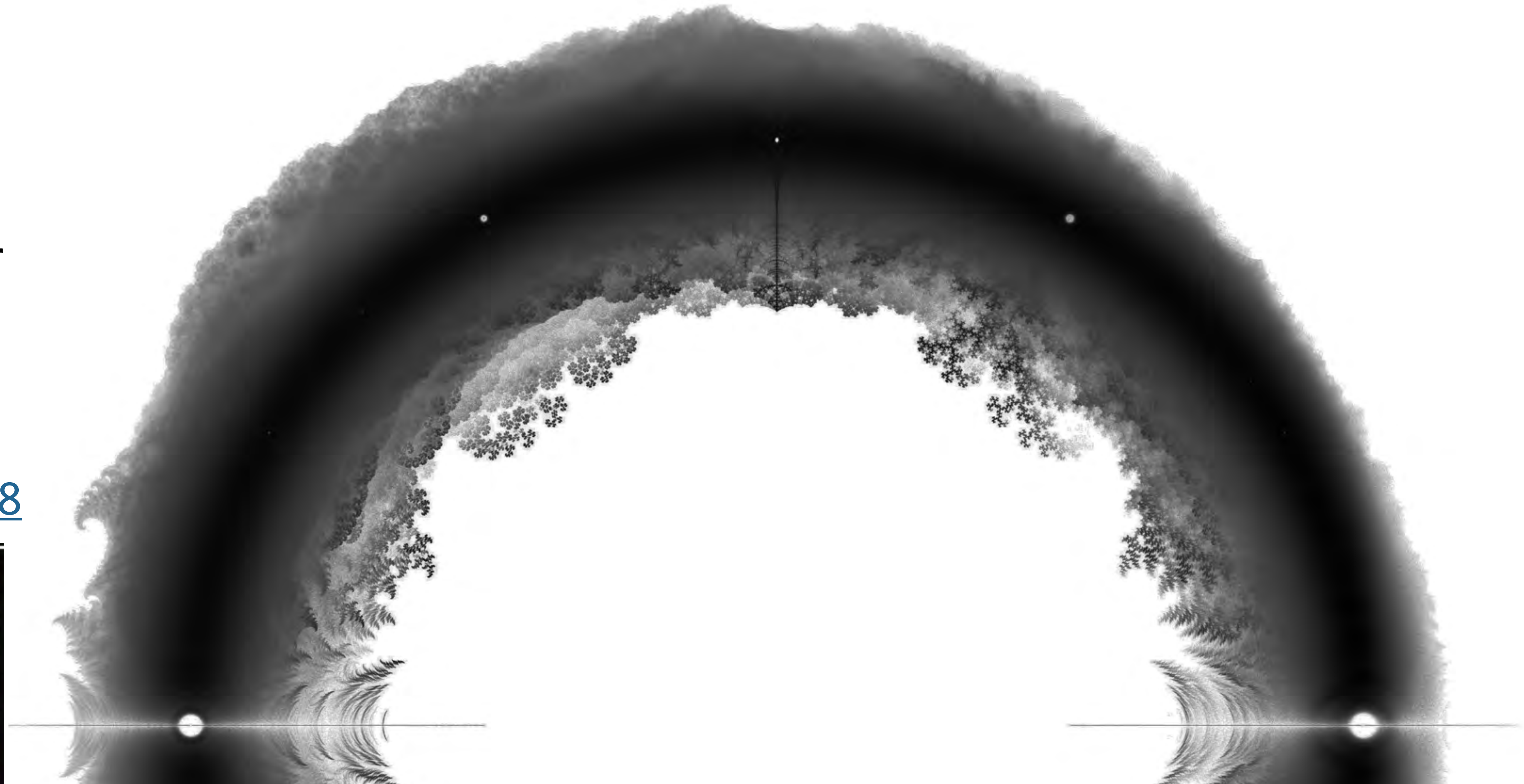
ENTROPY IN DIMENSION ONE

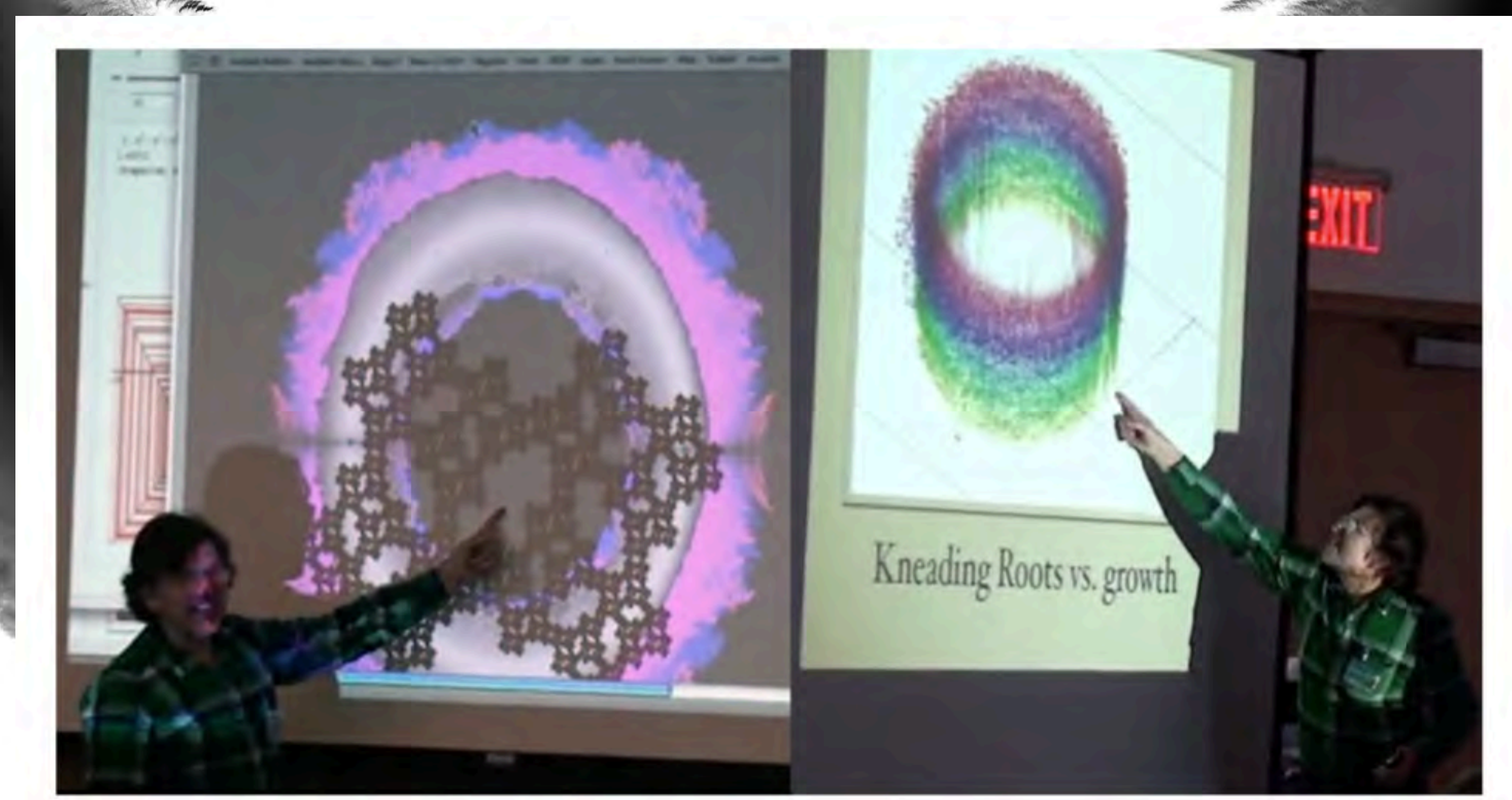
arxiv.org/abs/1402.2008



Bill Thurston

(1946 – 2012)





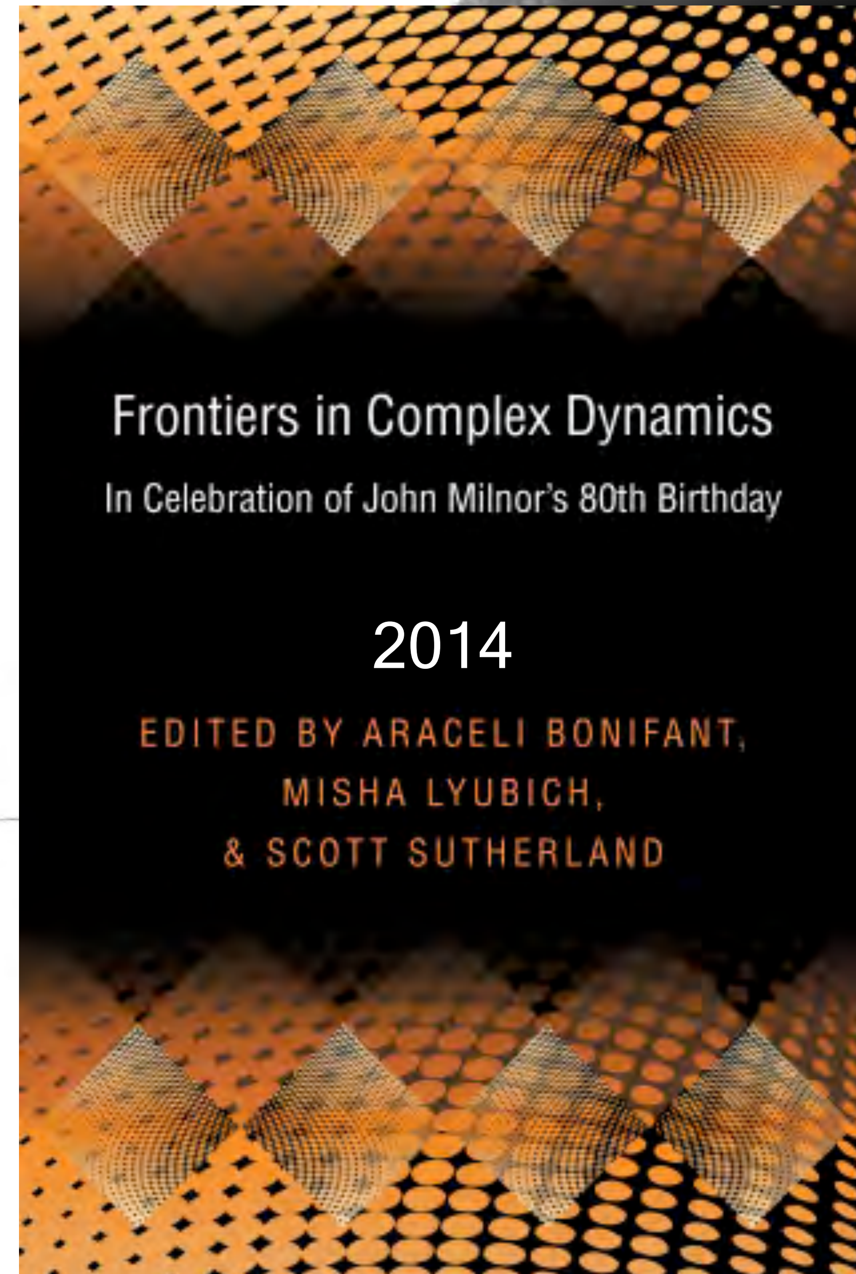
Bill Thurston presenting *M*, *M*_0, and the Thurston Set, Jackfest 2011

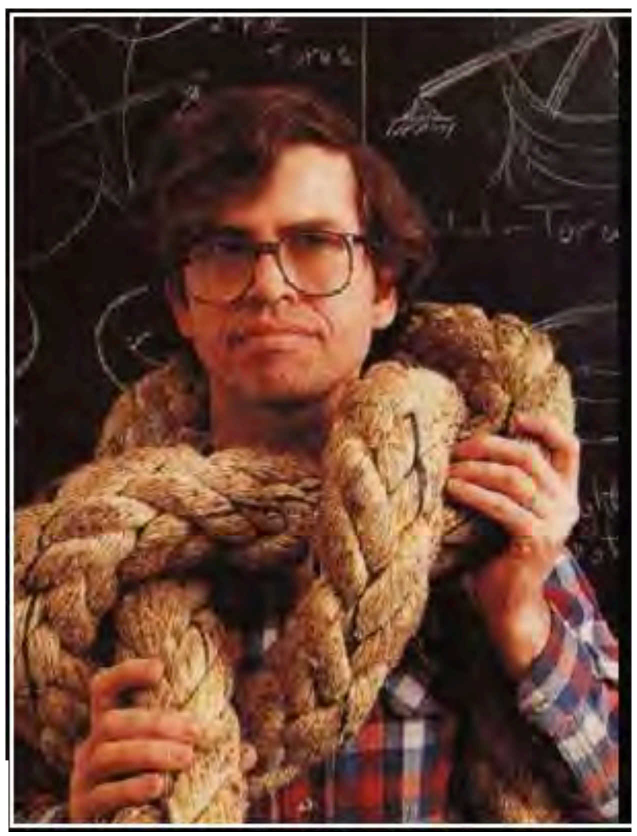
Structure of Entropy: The hidden dimensions

Thurston’s last paper

ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008





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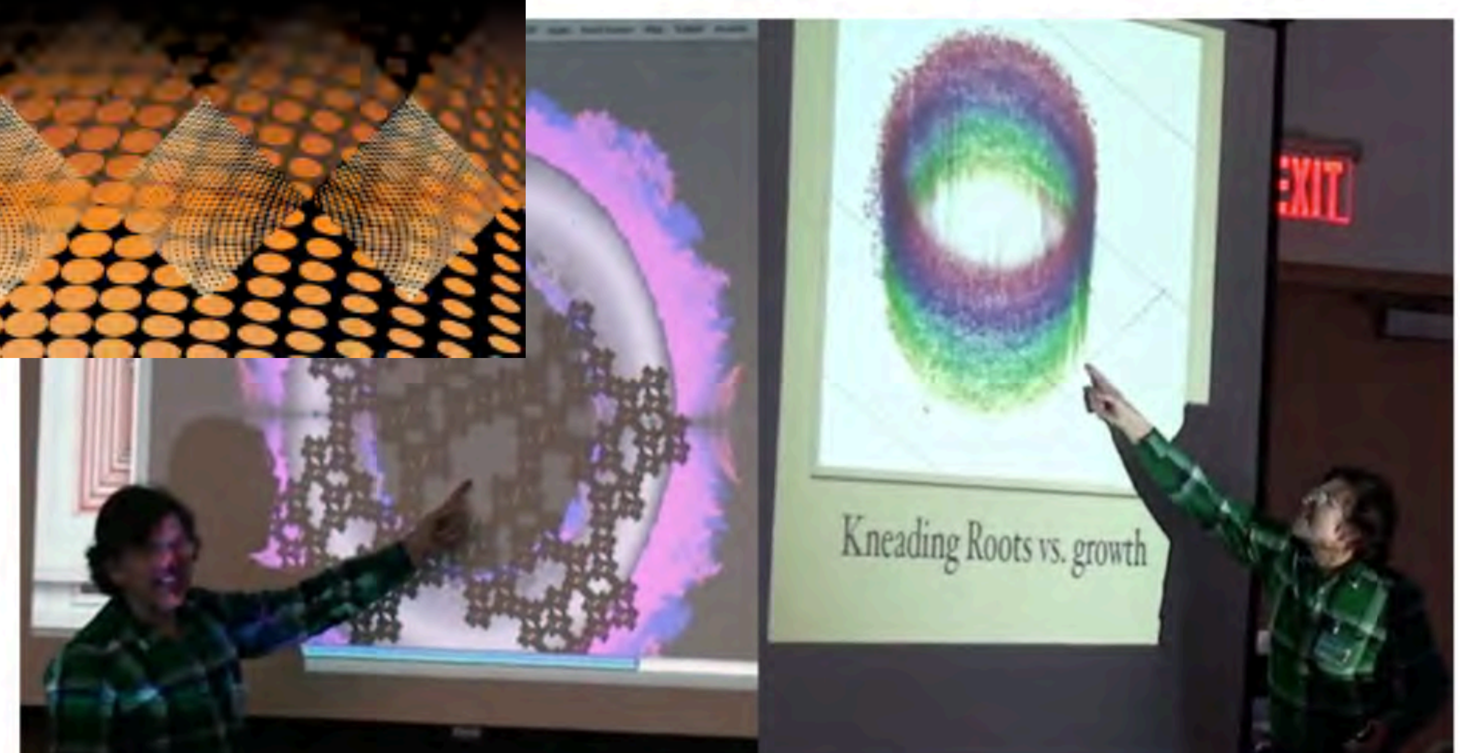
(1946 – 2012)



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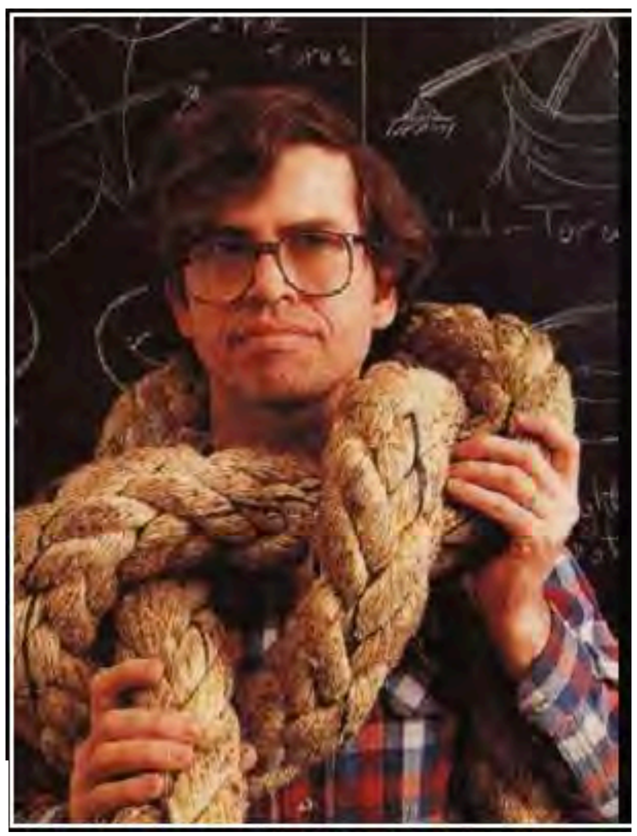

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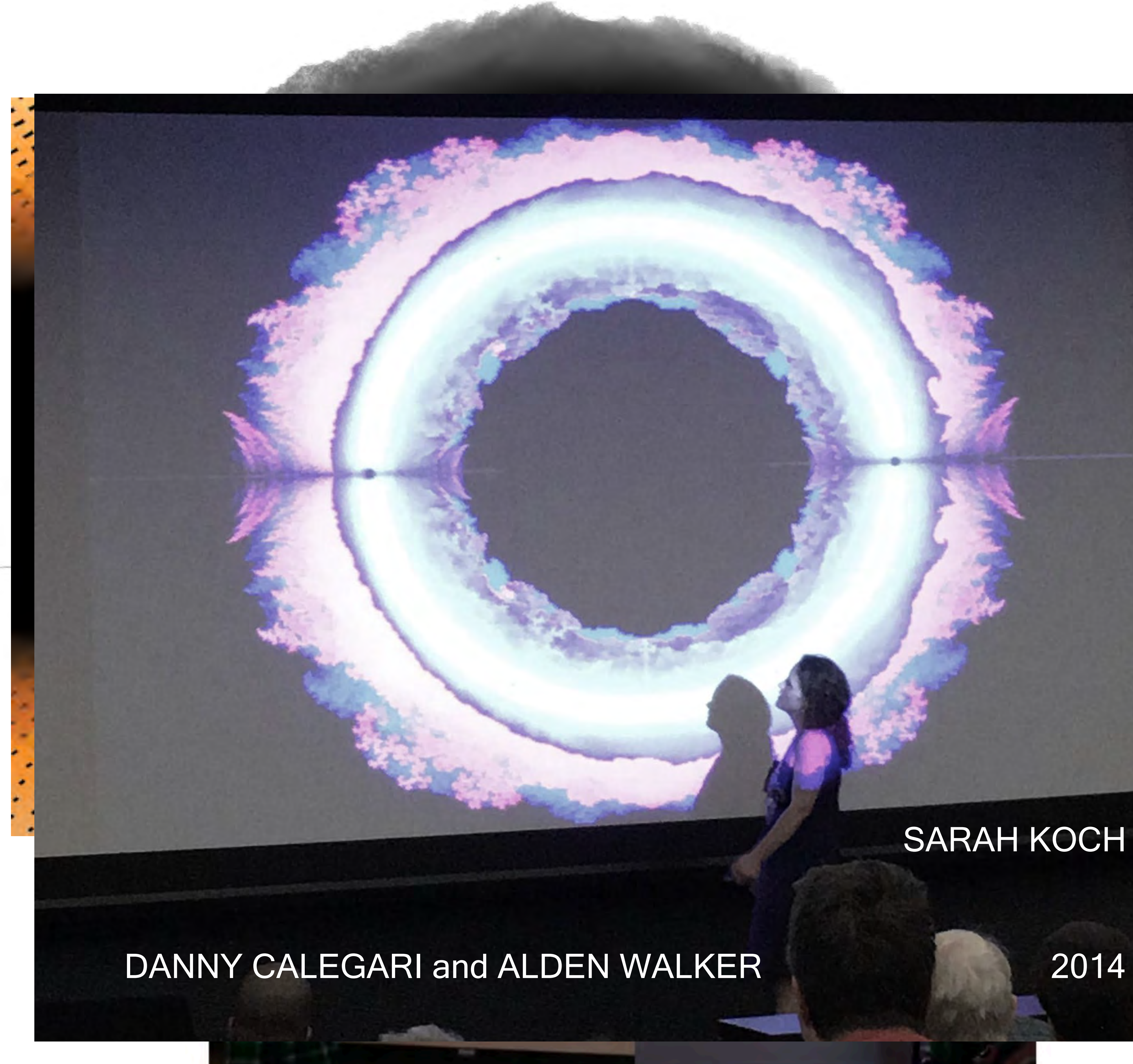

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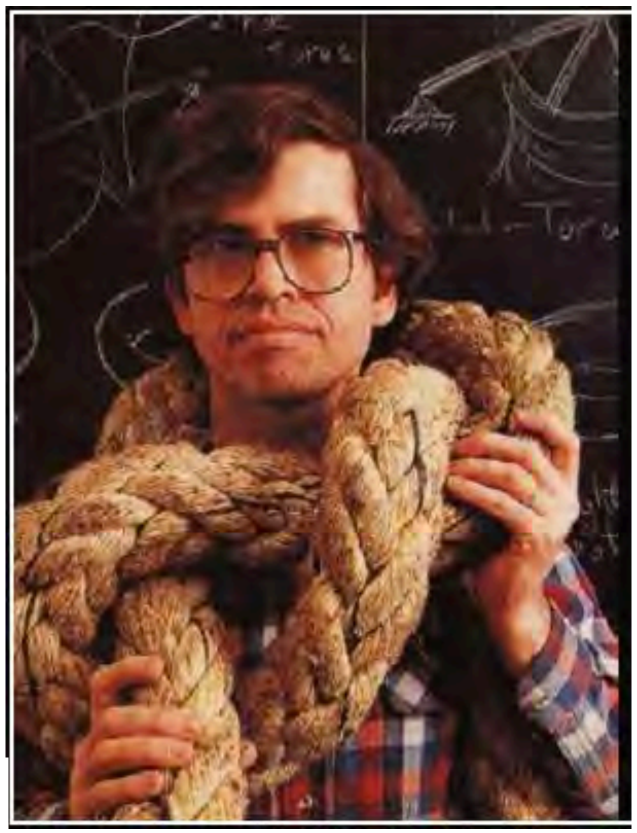
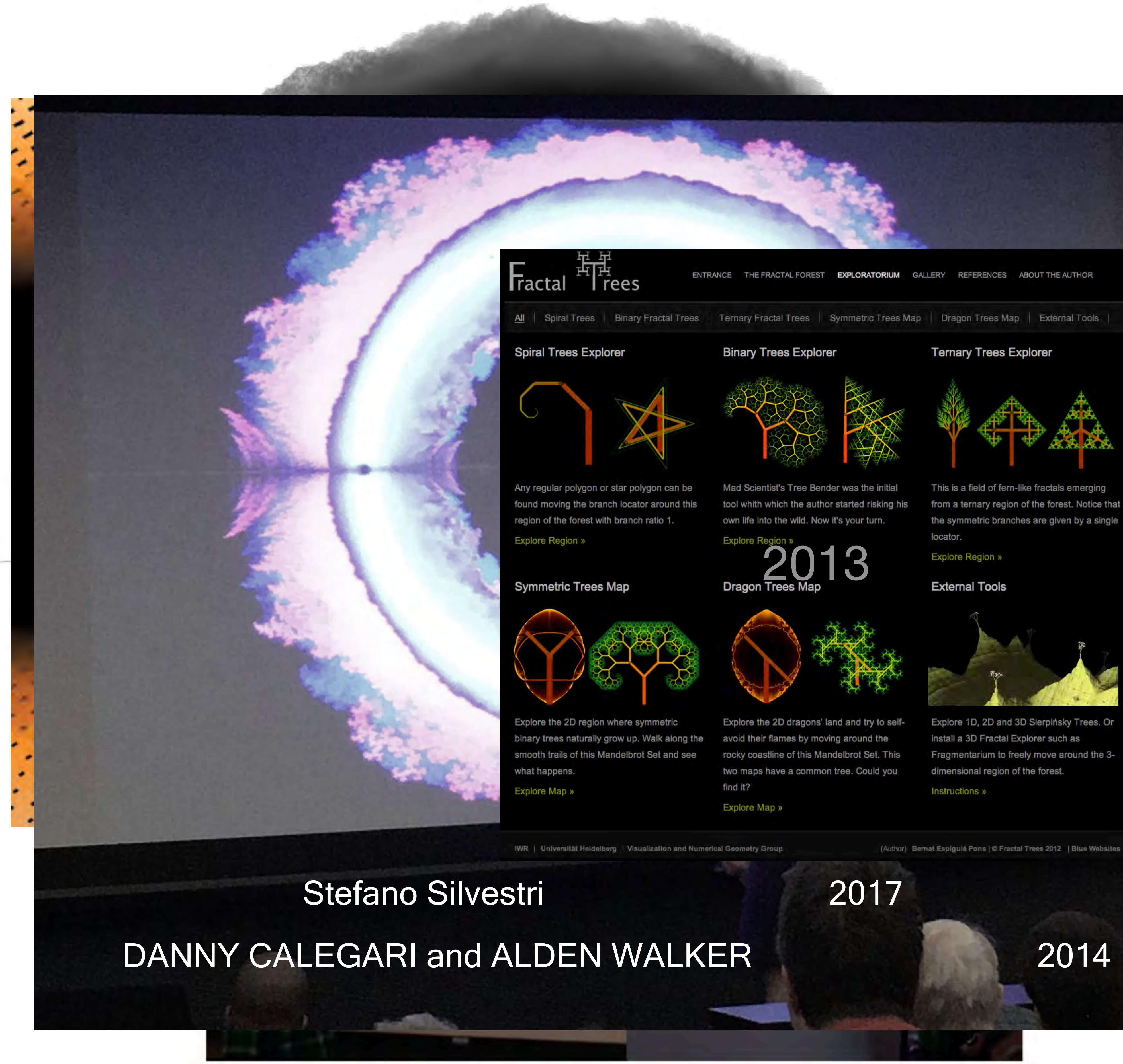


Bill Thurston

(1946 - 2012)



Bill Thurston presenting *M*, *M*_0, and the Thurston Set, Jackfest 2011

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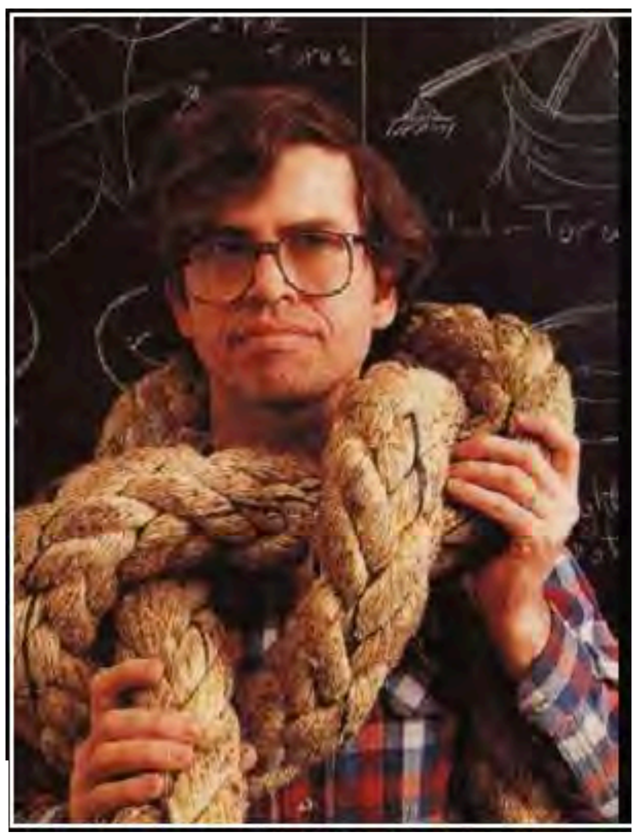

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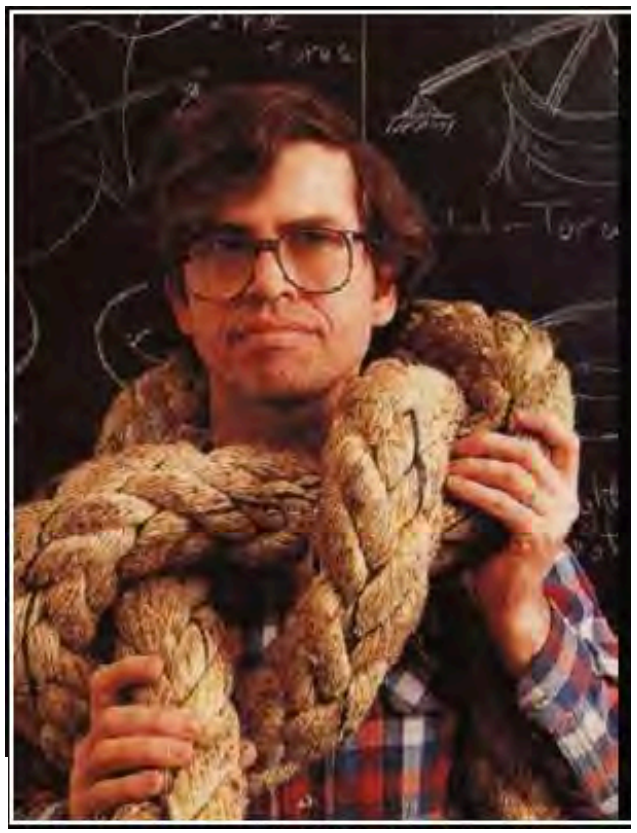

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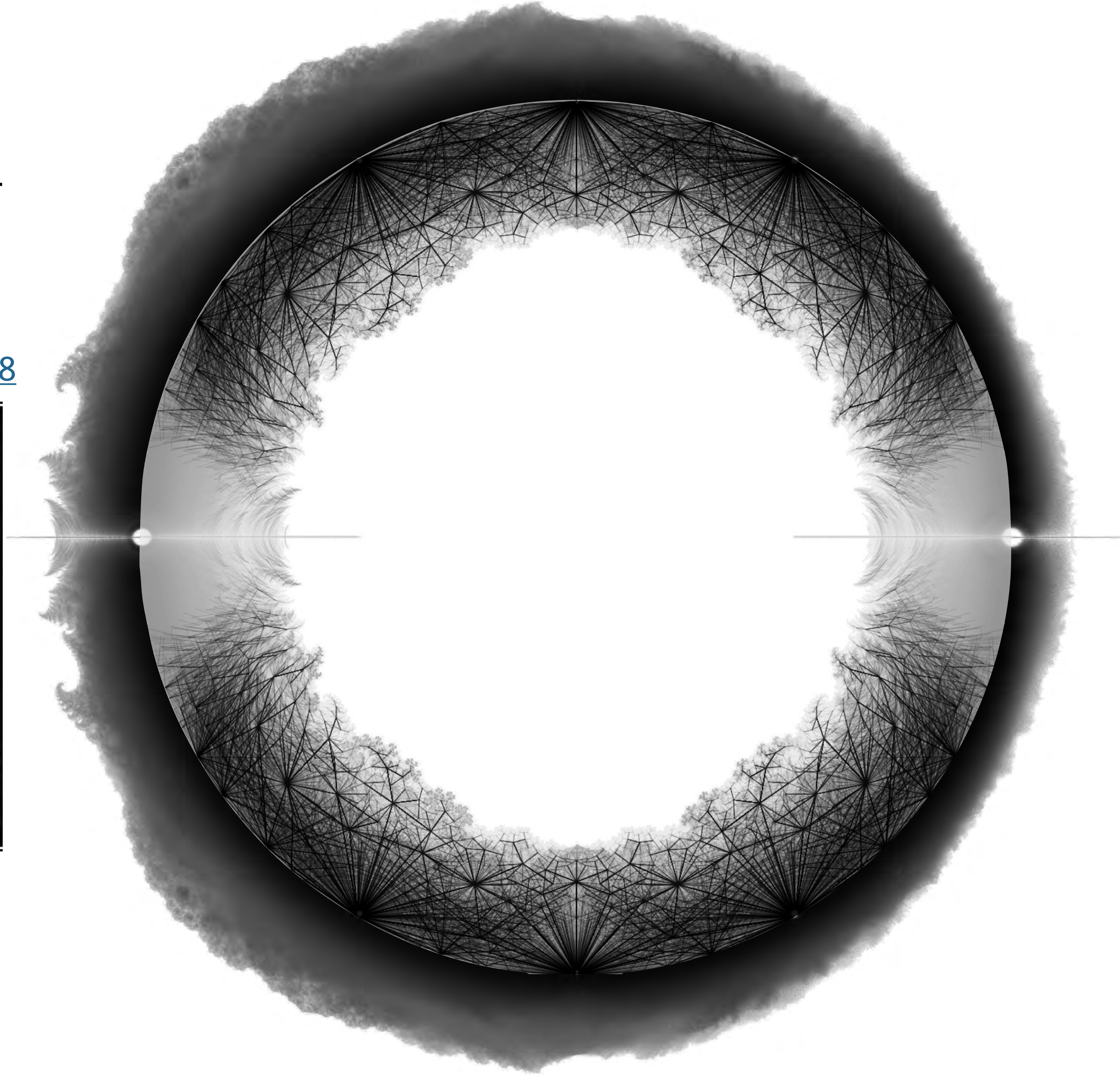
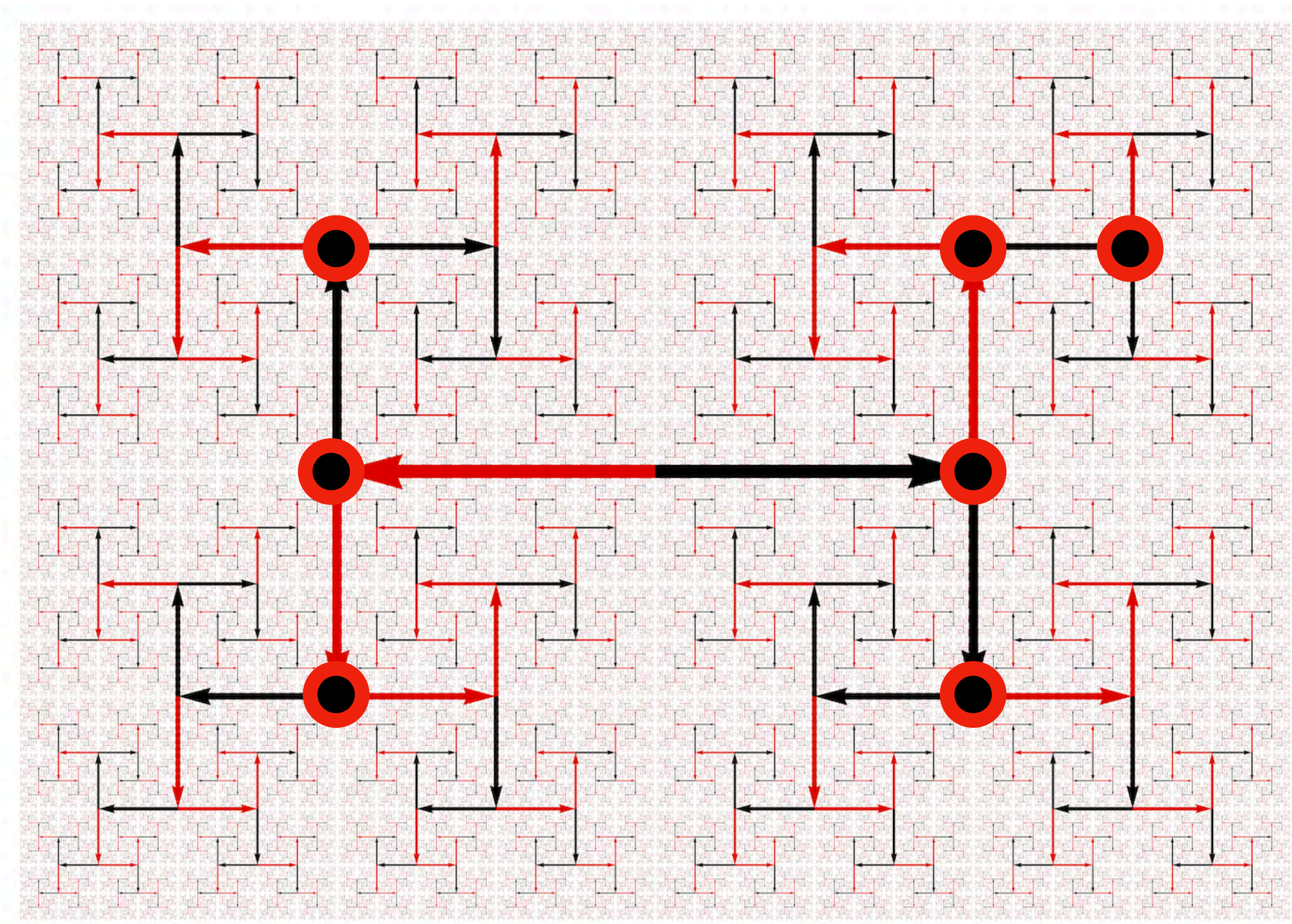


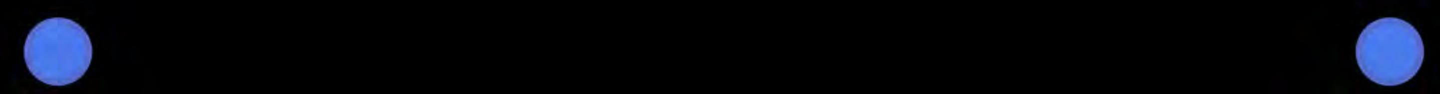


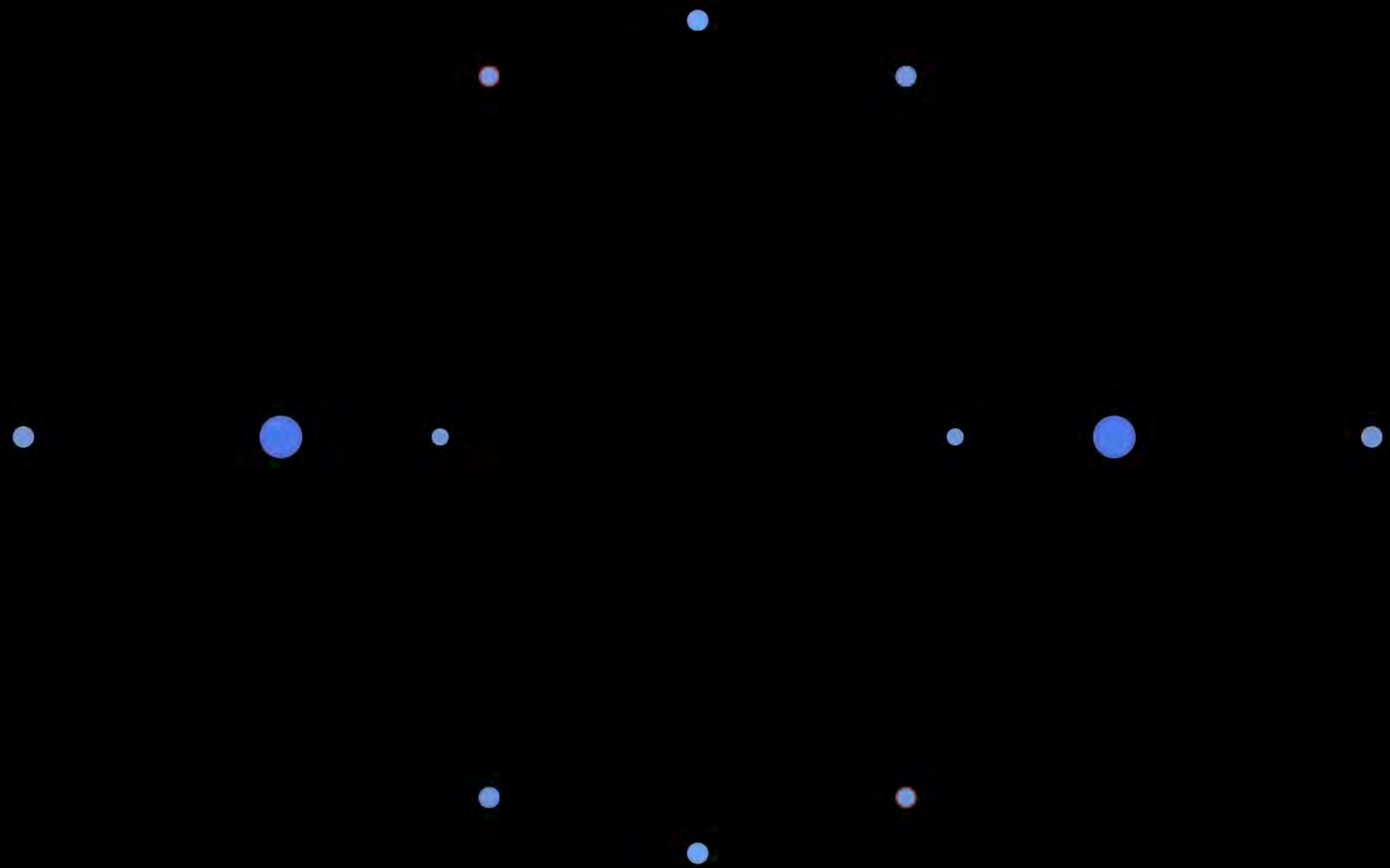
$+1 + x - x^2$

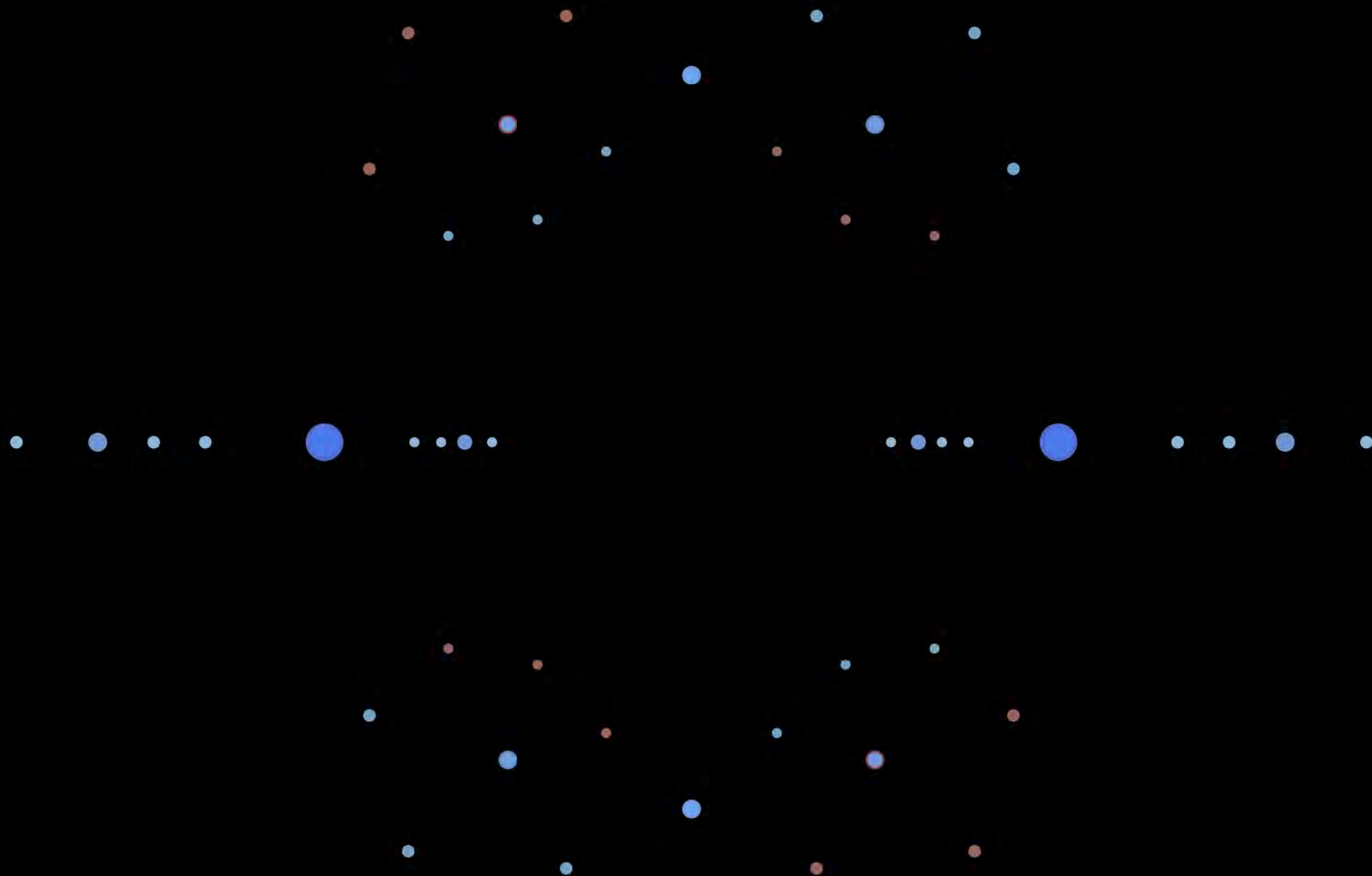
$-1 + x$ $\quad$ $+1 + x$

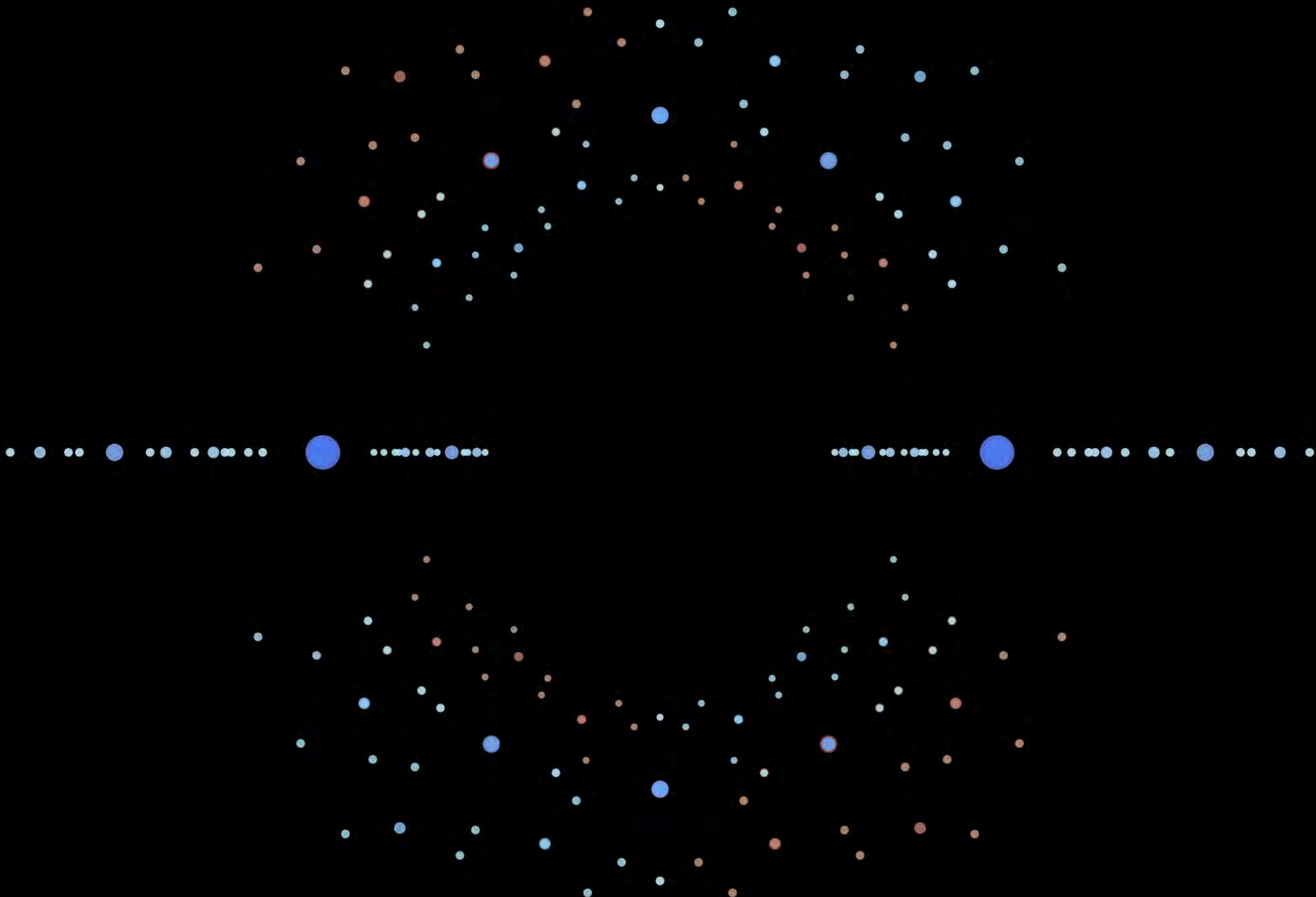
-1 $\quad$ $+1$ $\quad$ $x = i/\sqrt{2}$

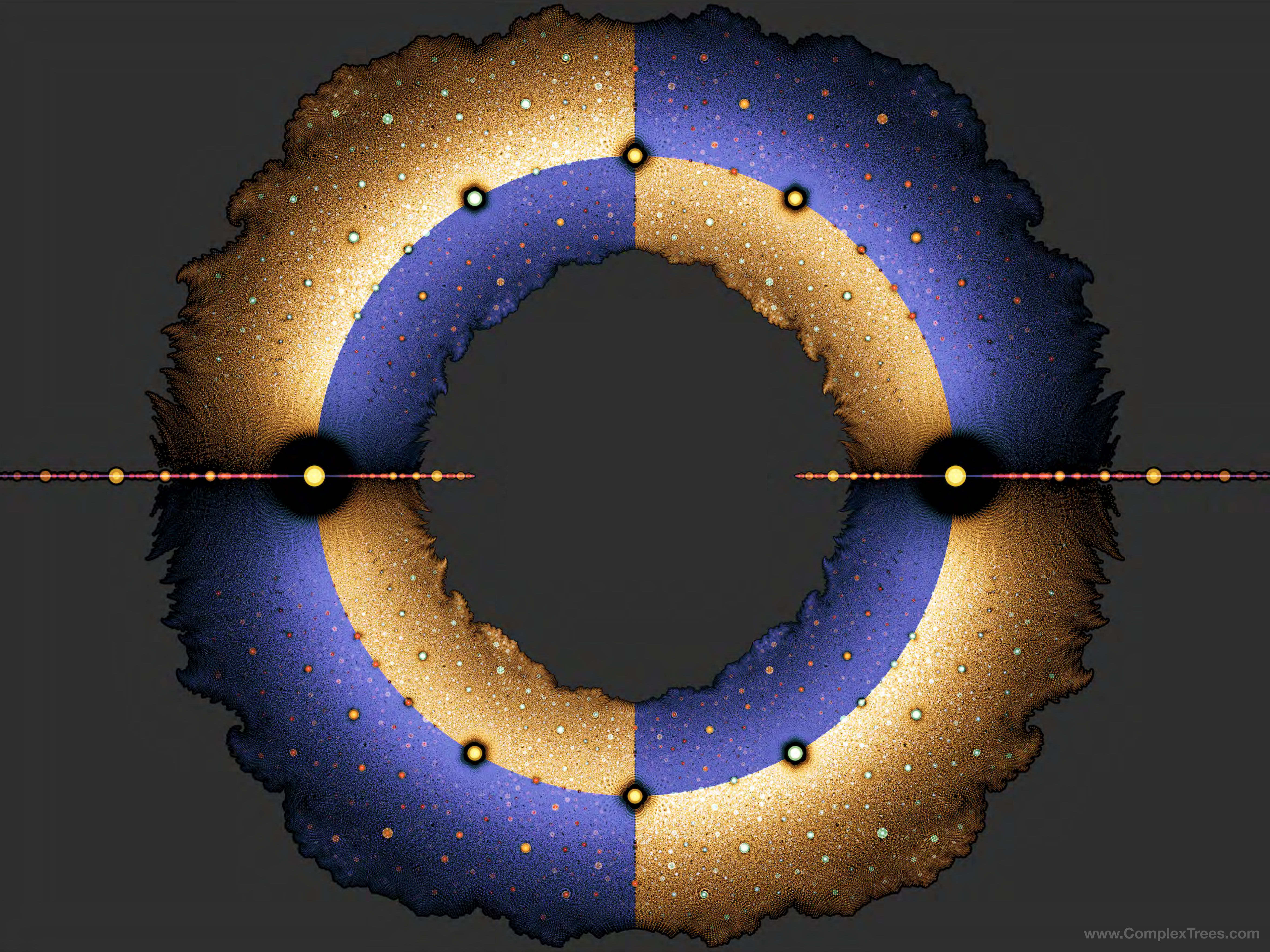
$-1 - x$ $\quad$ $+1 - x$

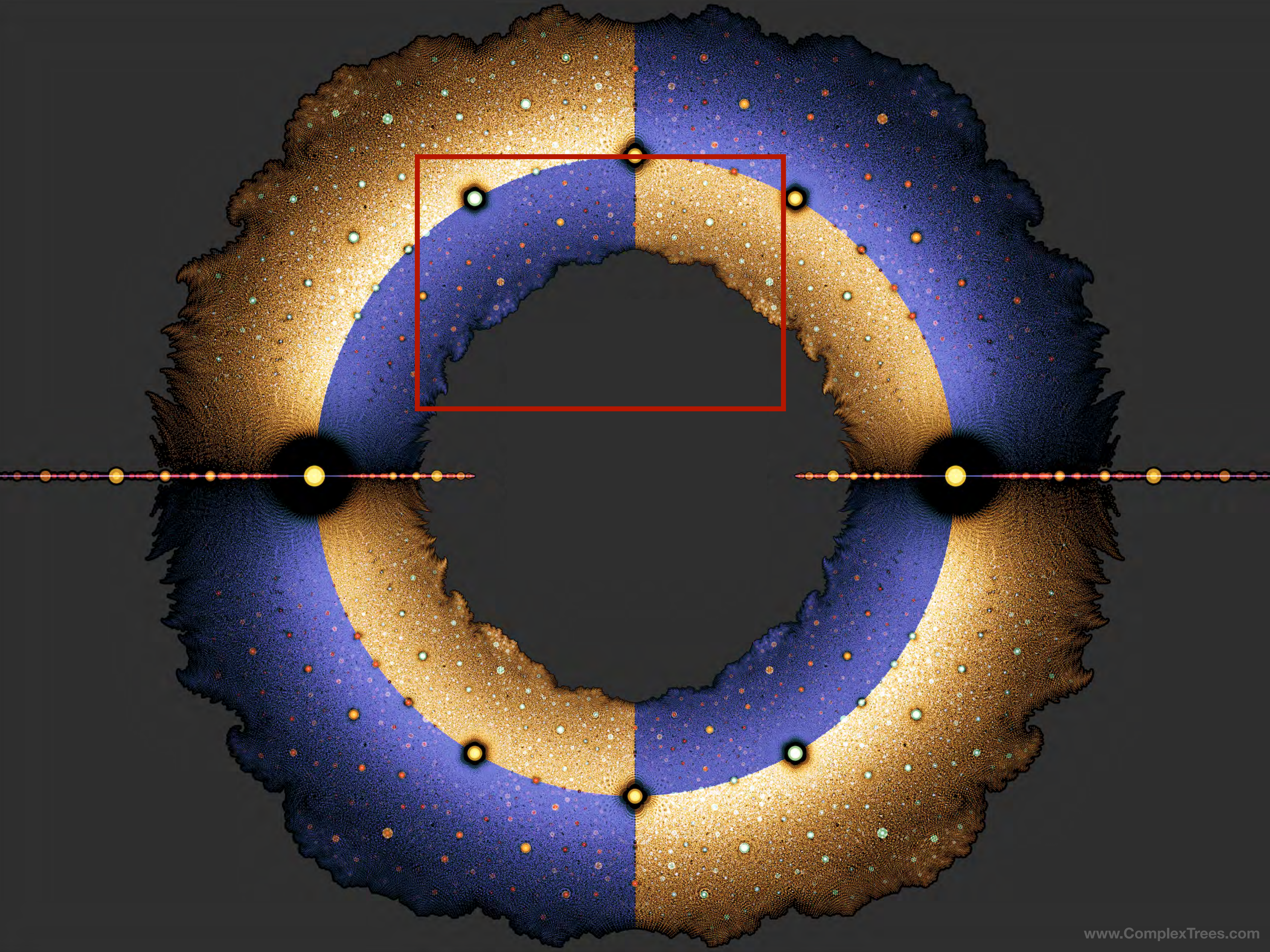


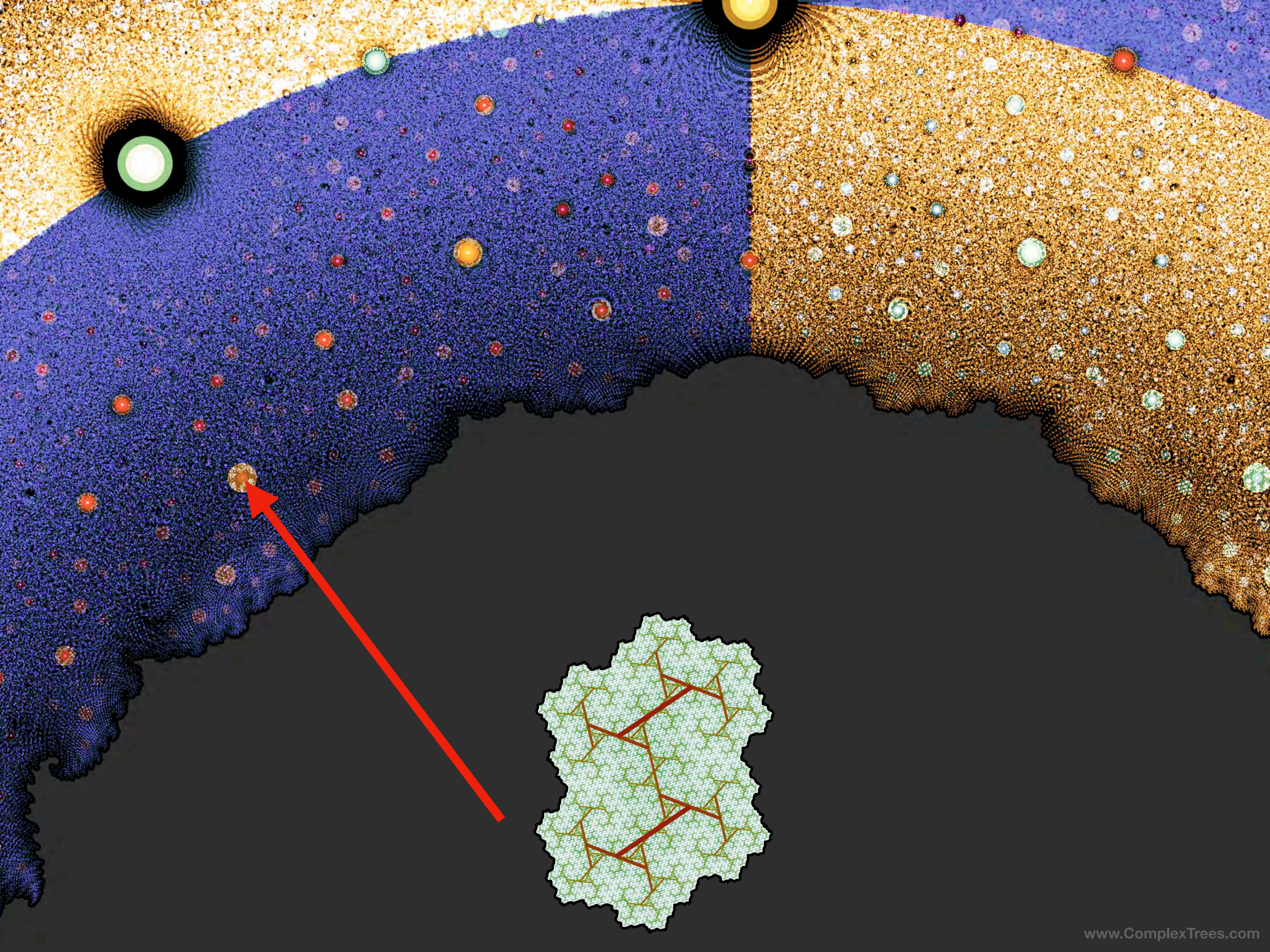


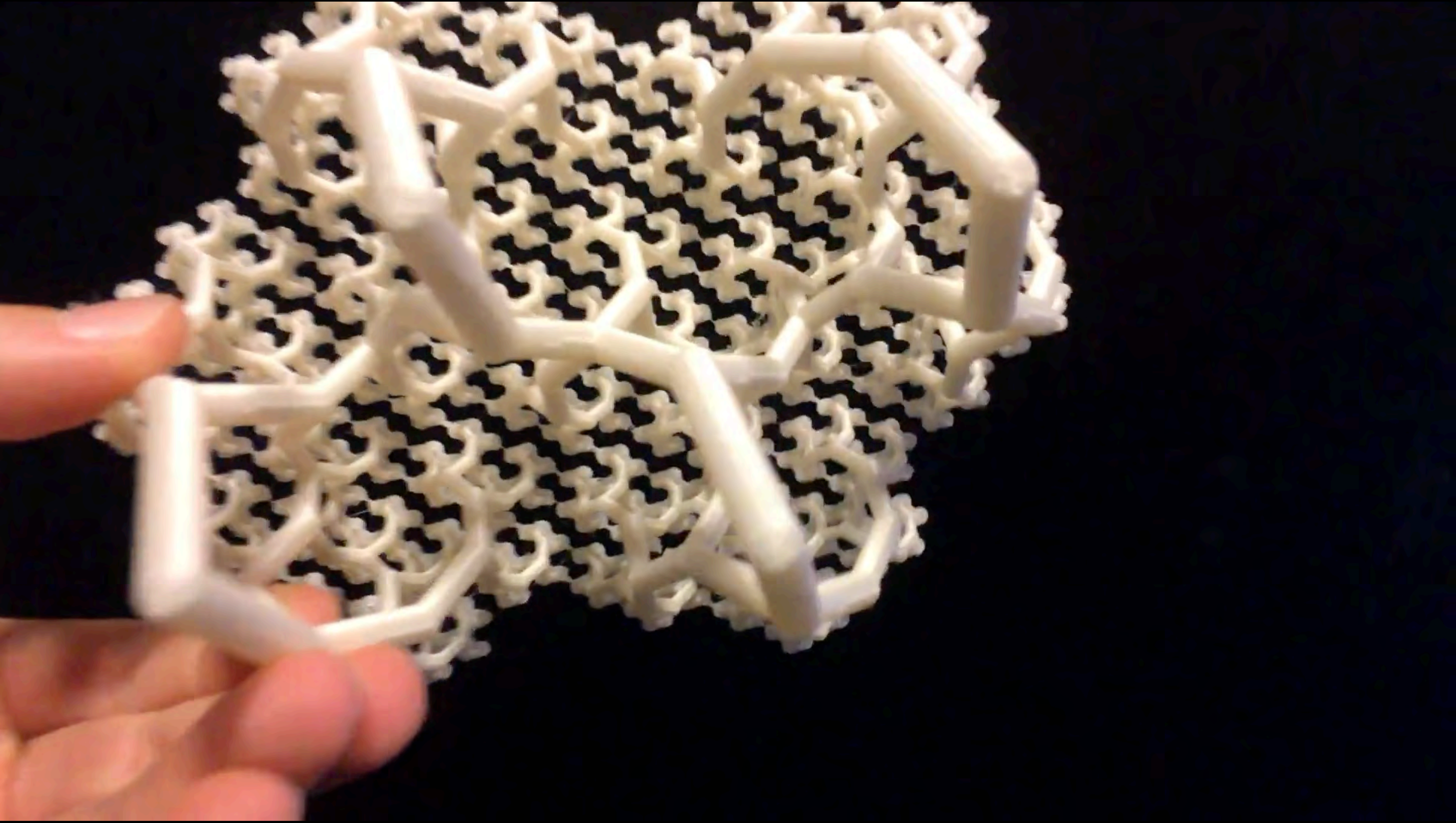






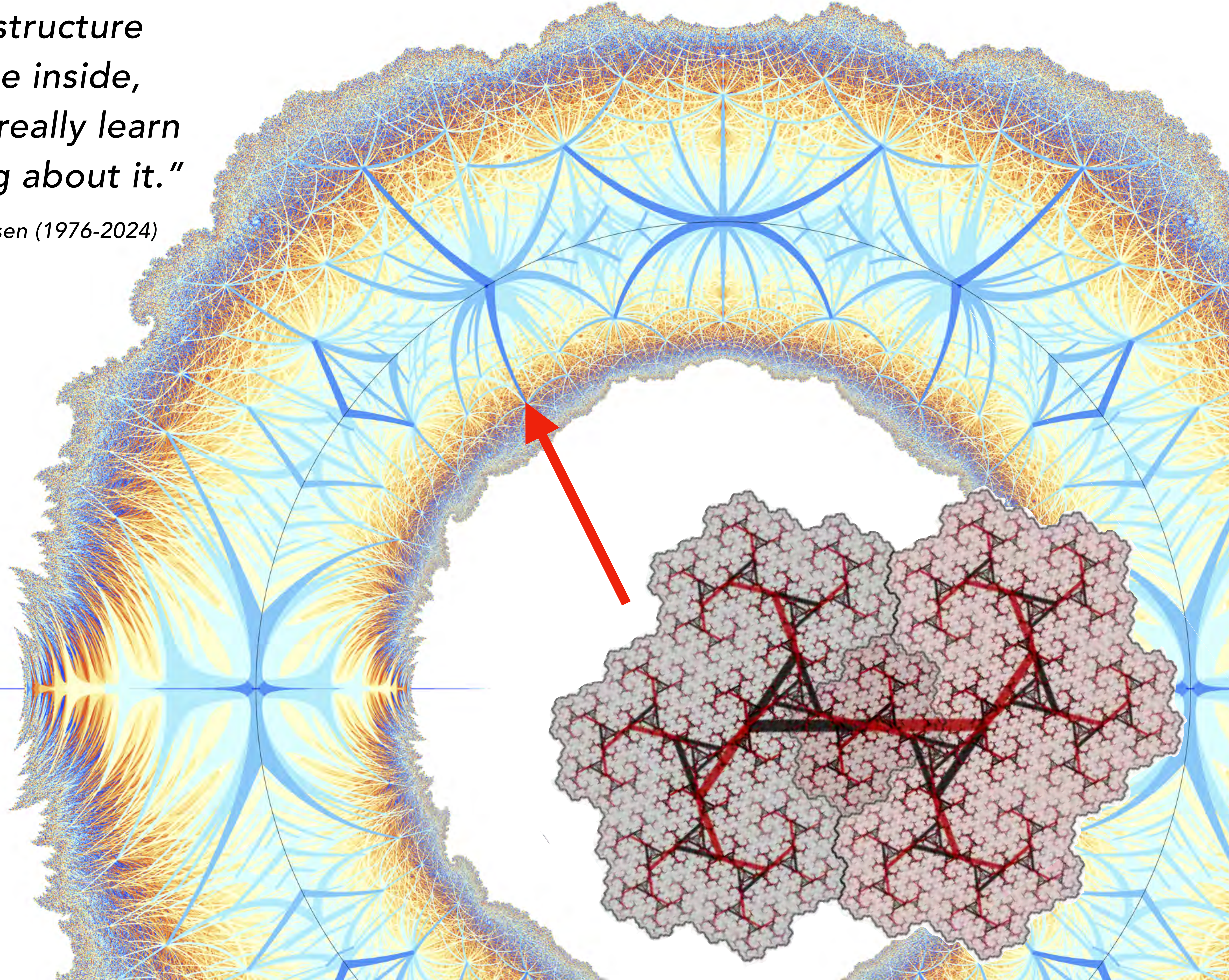

www.ComplexTrees.com


www.ComplexTrees.com



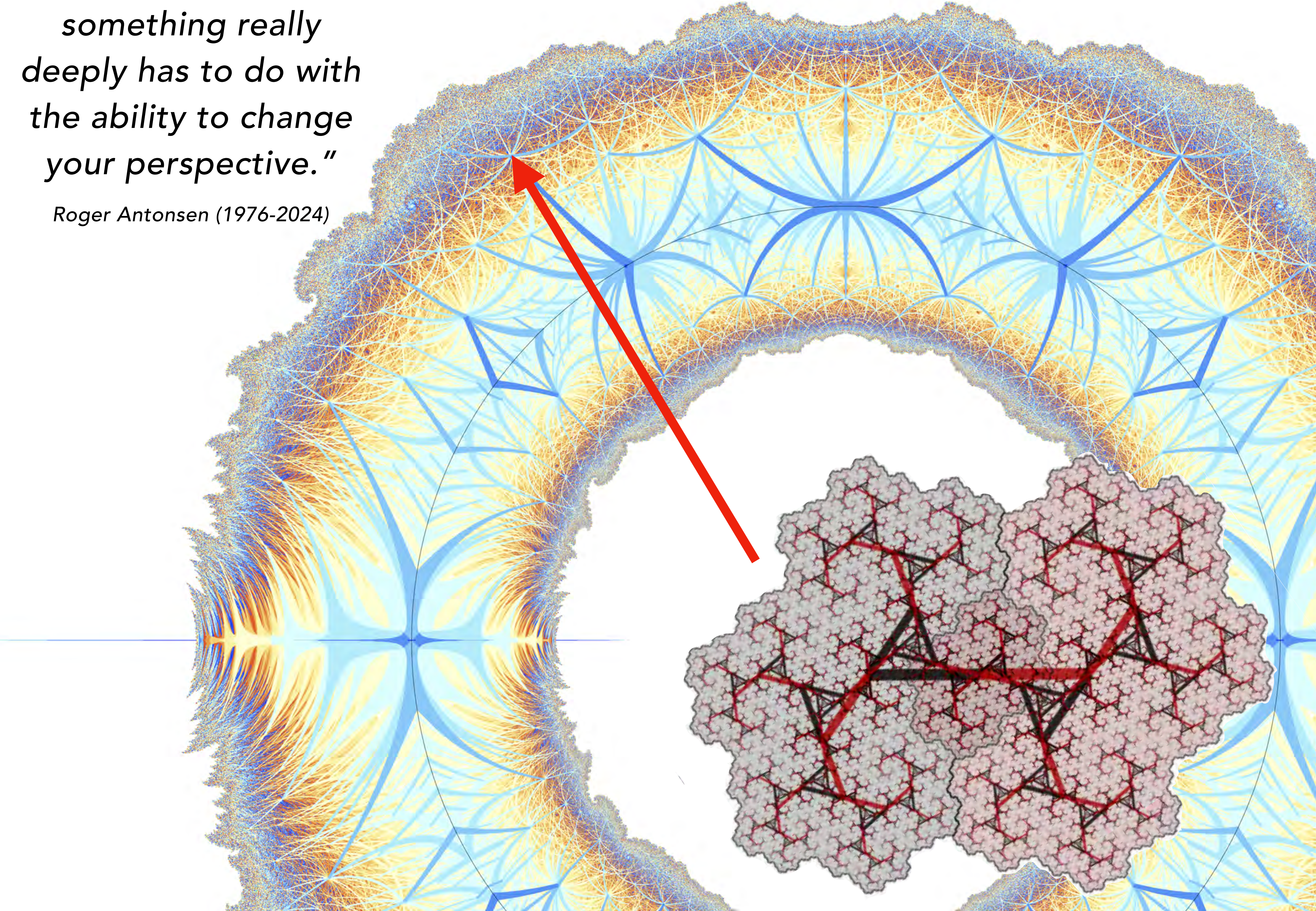
"If you are able to view a structure from the inside, then you really learn something about it."

Roger Antonsen (1976-2024)



"Understanding something really deeply has to do with the ability to change your perspective."

Roger Antonsen (1976-2024)



Definition (collinear digit set)

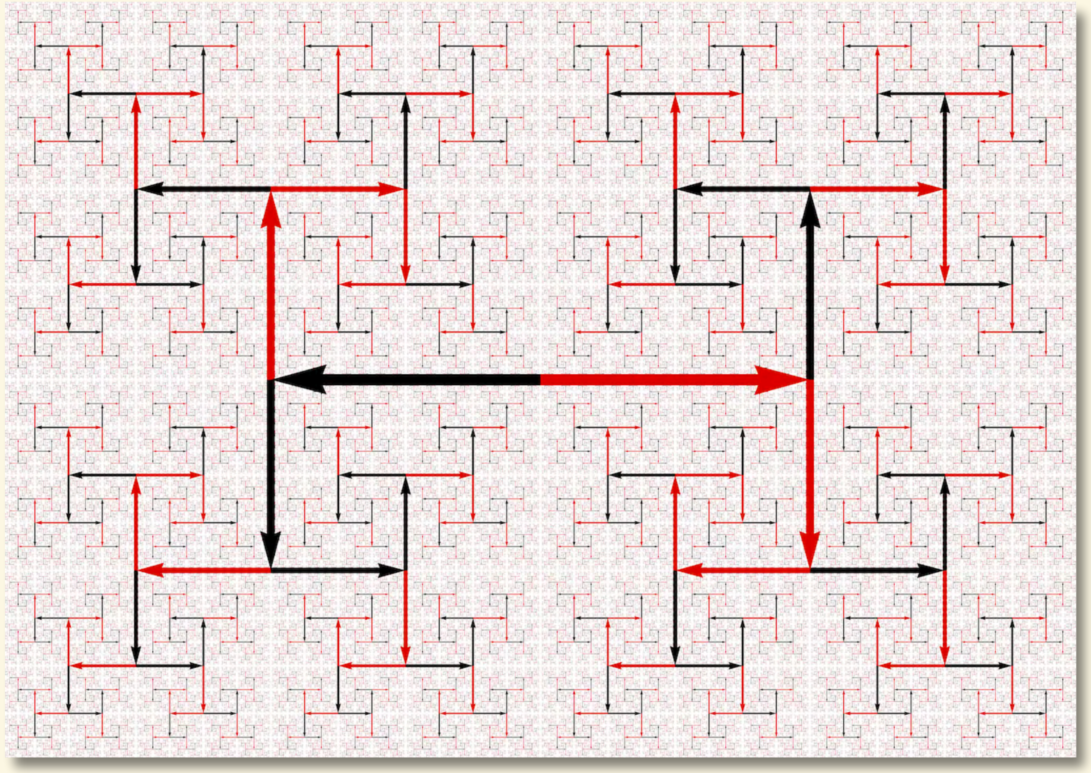
Set of $n \geqslant 2$ integers evenly spaced from $-n+1$ to $n-1$,
$\mathcal{A}_n := \{-n+1, -n+3, \ldots, n-3, n-1\}$.

Definition (collinear fractal)

Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,
$\boldsymbol{E}(c,n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \mathcal{A}_n \right\}$.

$$p^+(c) = +1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$

$$p^-(c) = -1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$



Joint work with my PhD supervisors, Joan Saldaña, and David Juher

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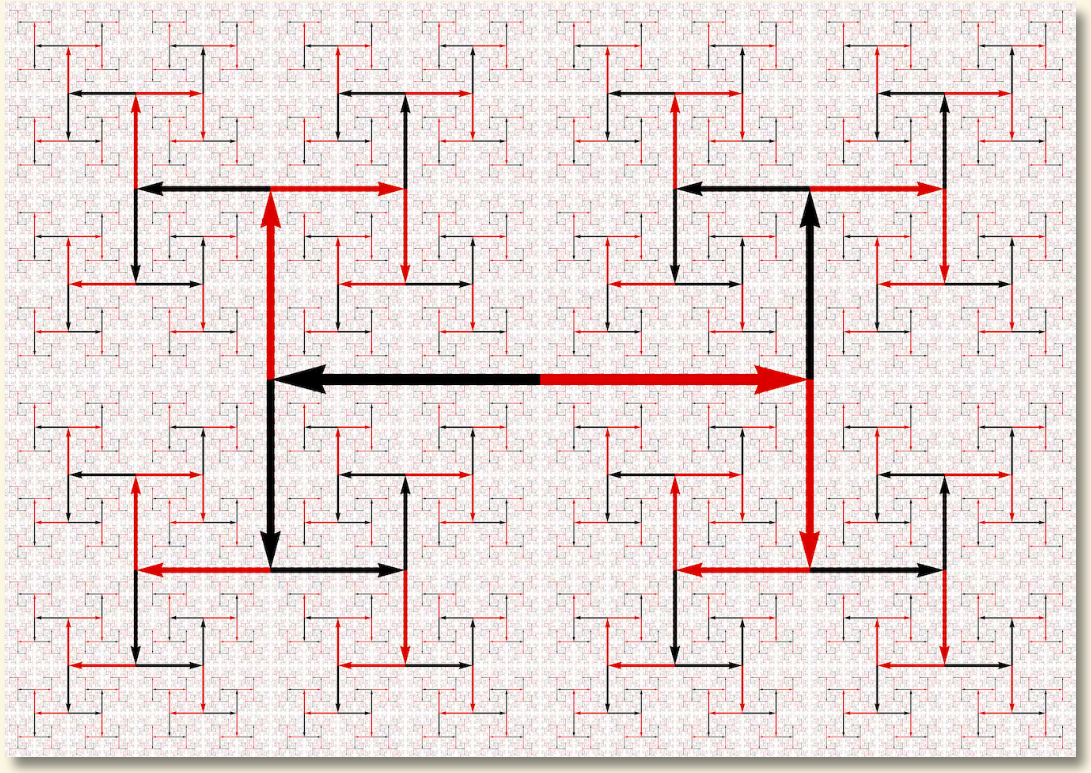

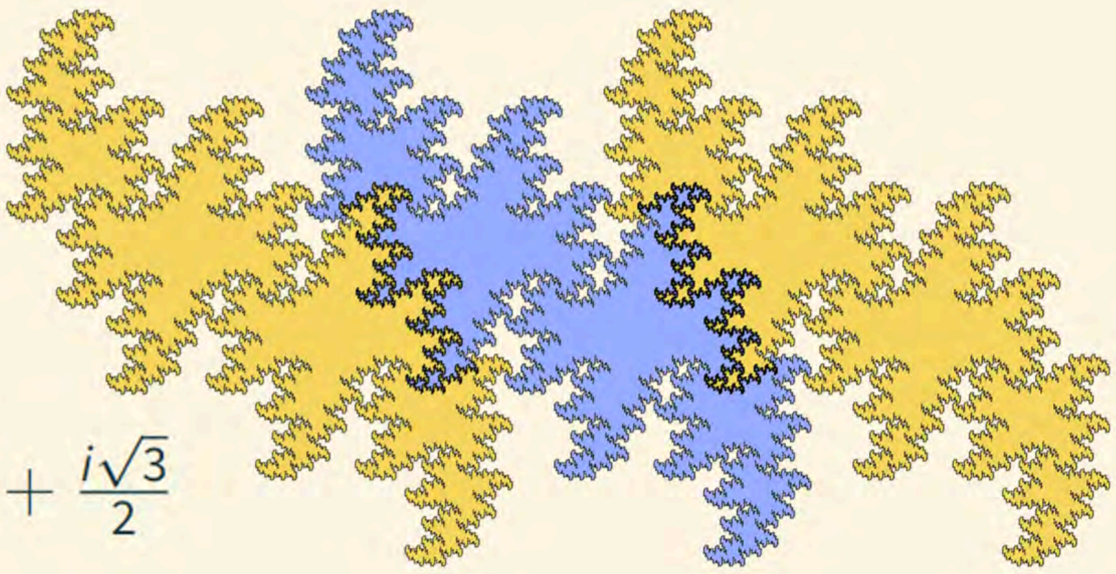

$c = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

Example ($n = 3$)

$$\boldsymbol{E}(c,3) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \{-2, 0, 2\} \right\}$$
$$= \left(\frac{\boldsymbol{E}(c,3)}{c} - 2 \right) \cup \left(\frac{\boldsymbol{E}(c,3)}{c} \right) \cup \left(\frac{\boldsymbol{E}(c,3)}{c} + 2 \right).$$

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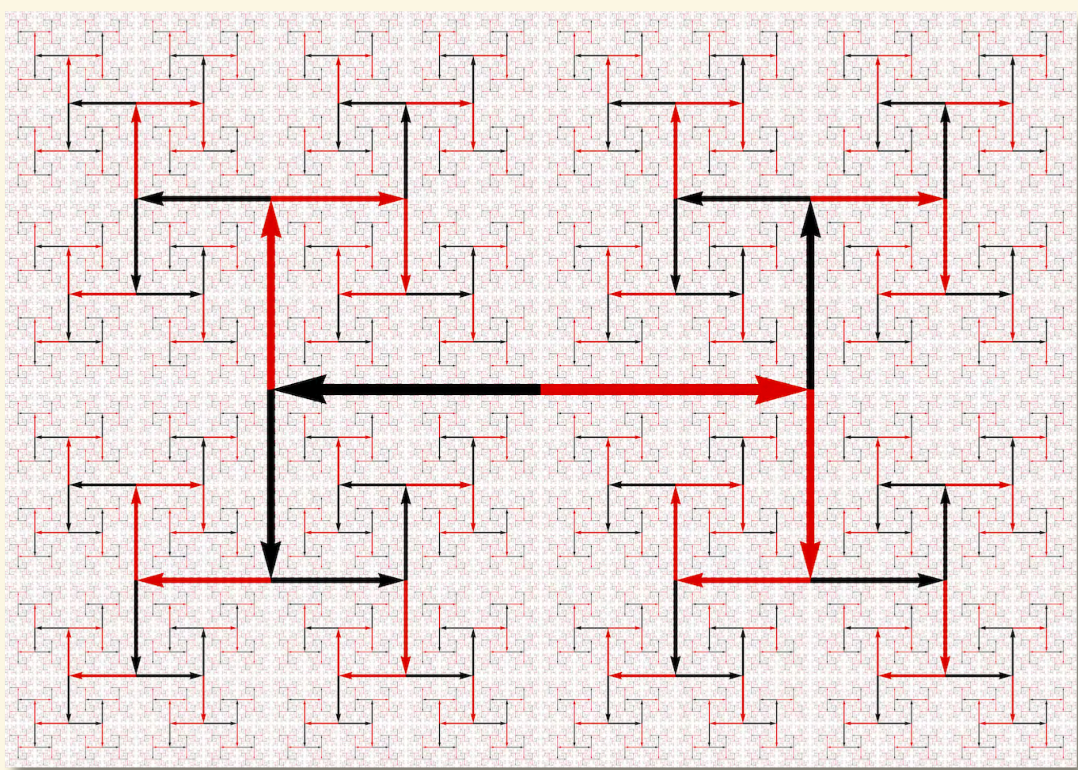
$$\mathcal{M}_n = \left\{ c^{-1} \in \mathbb{D}^* : 2c \in \mathbf{E}(c, 2n-1) \right\}$$

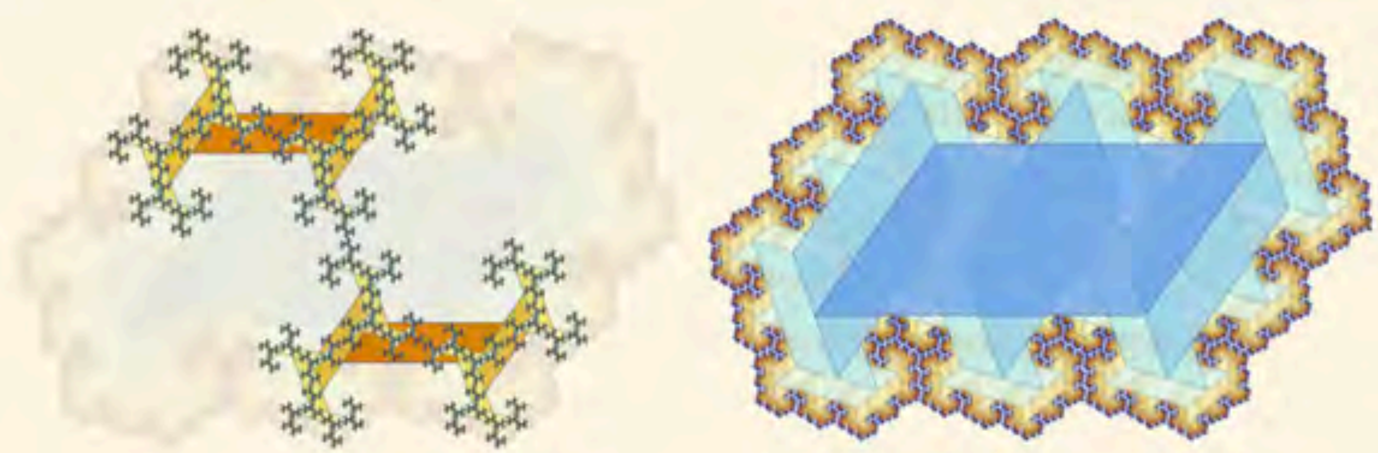
$$p^{+}(c) = +1 \pm c^{-1} \pm c^{-2} \pm \dots \pm c^{-m}$$

$$p^{-}(c) = -1 \pm c^{-1} \pm c^{-2} \pm \dots \pm c^{-m}$$

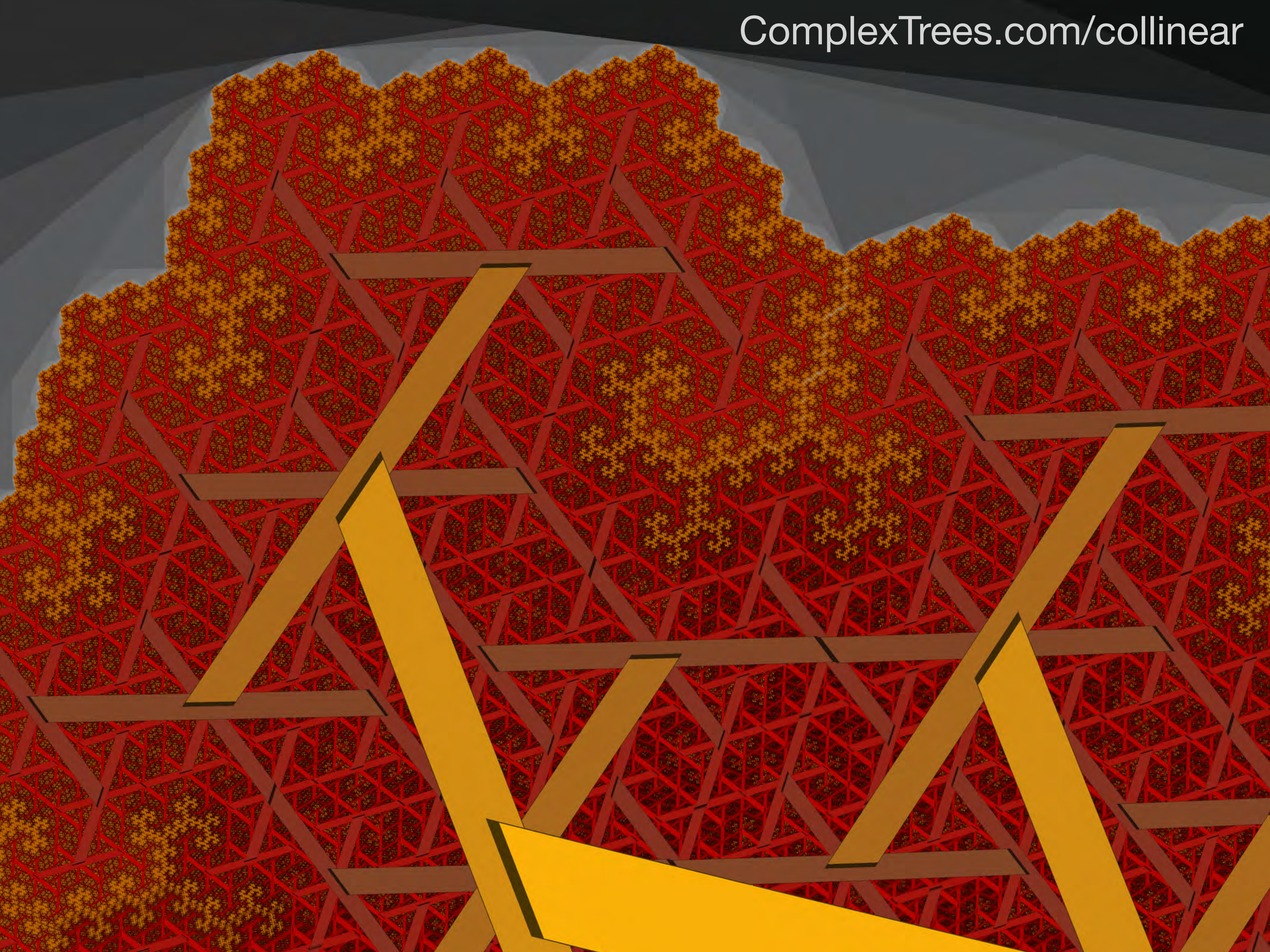


The Minkowski sum and geometric difference

$$\mathbf{E}(c, 2n-1) = \mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$$
$$= \mathbf{E}(c,n) \ominus \mathbf{E}(c,n)$$



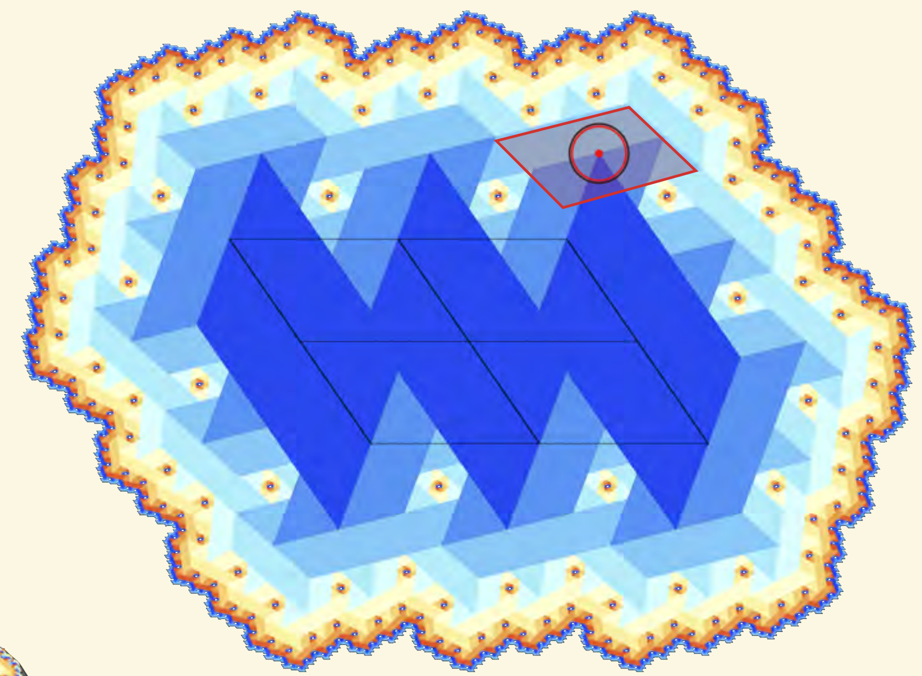
$\mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$ $\quad$ $\mathbf{E}(c, 2n-1)$

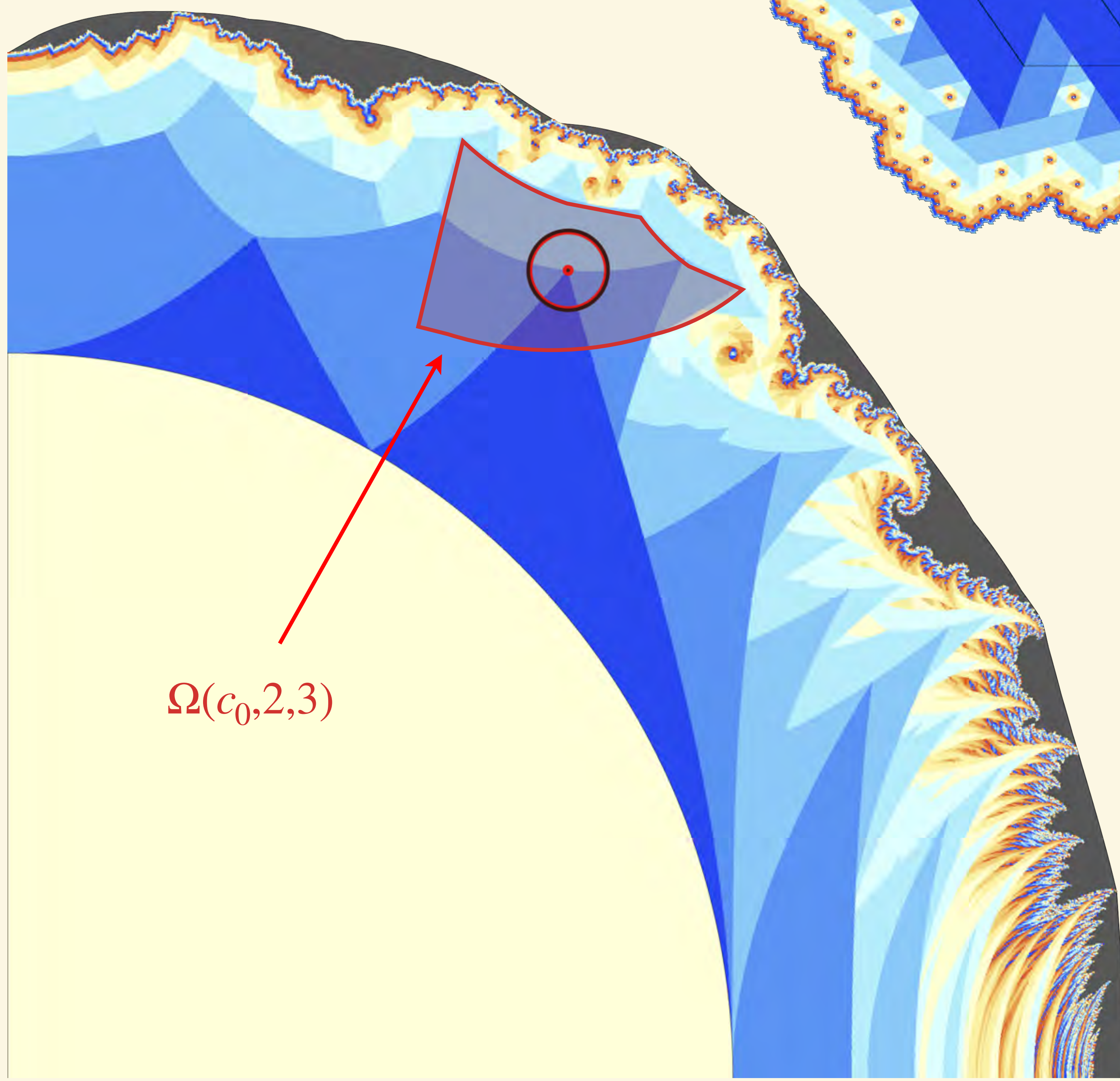

ComplexTrees.com/collinear

$c_0 = 0.7718 + i\,1.1151$

$c^3 - c^2 + c + 1 = 0$

$2c_0 \in \boldsymbol{E}(c, 2n-1)$

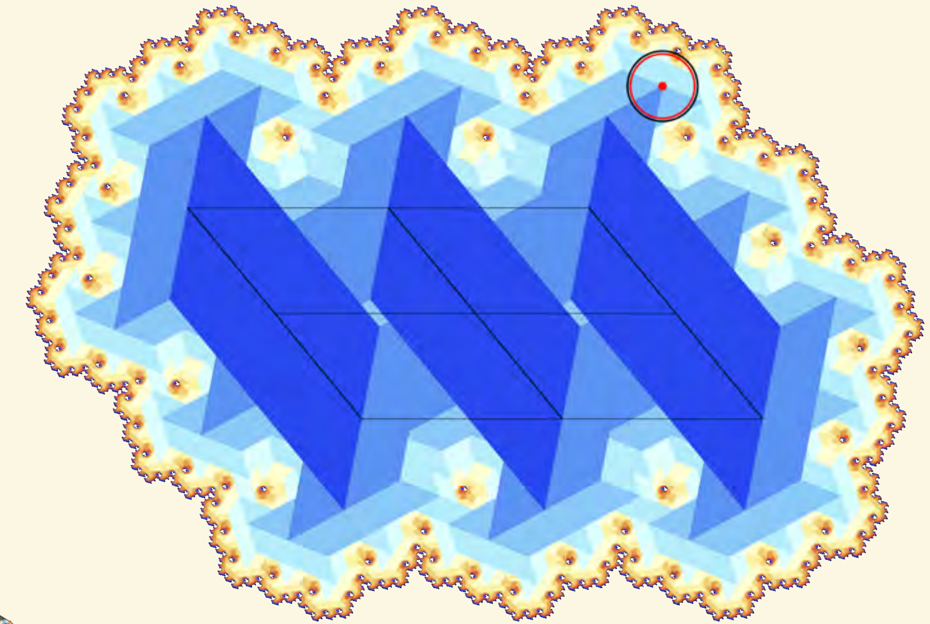


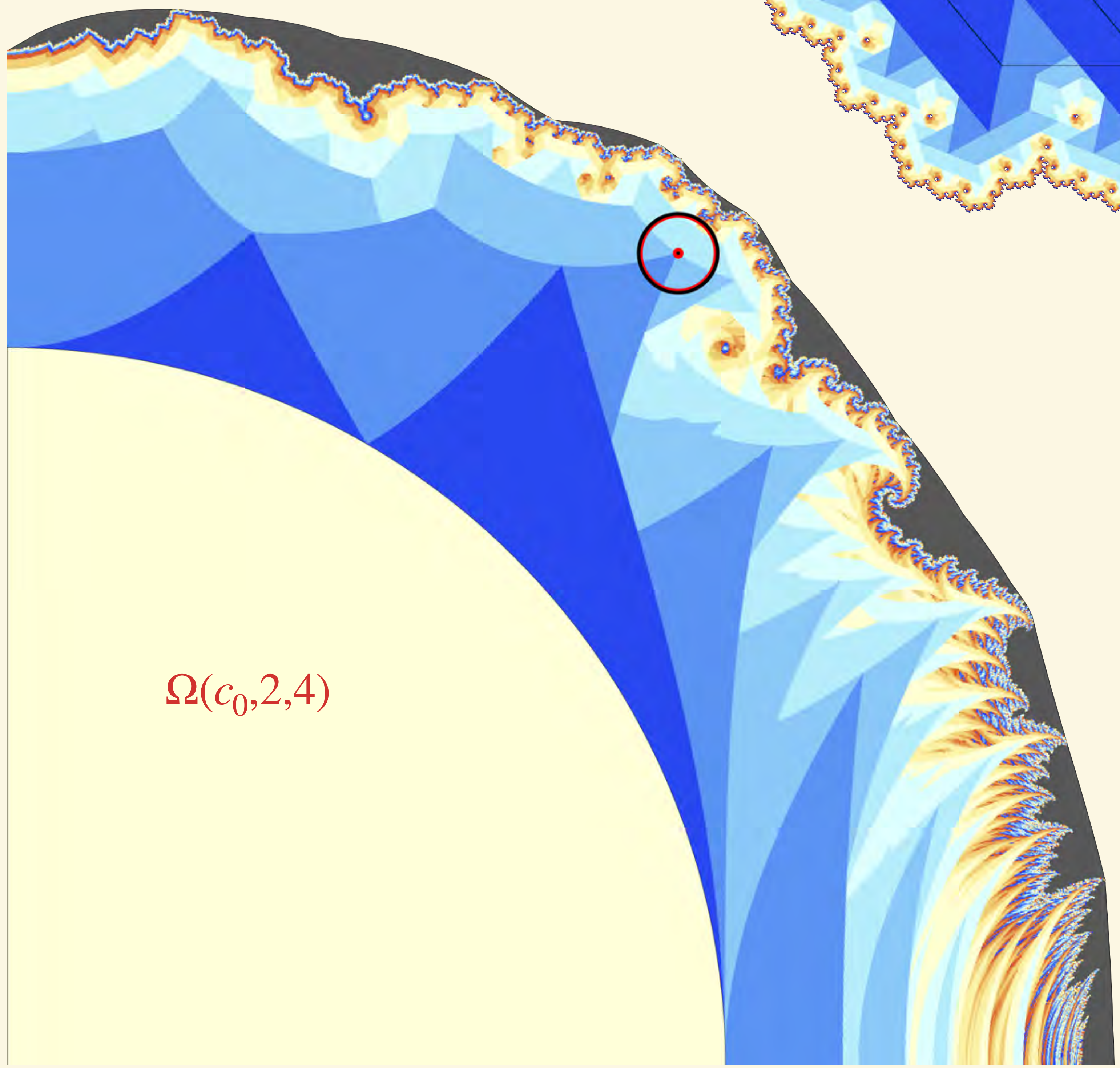


$c_0 = 0.9334 + i\,1.1325$

$2c_0 \in \boldsymbol{E}(c, 2n-1)$

$c^4 - c^3 + c^2 + c + 1 = 0$

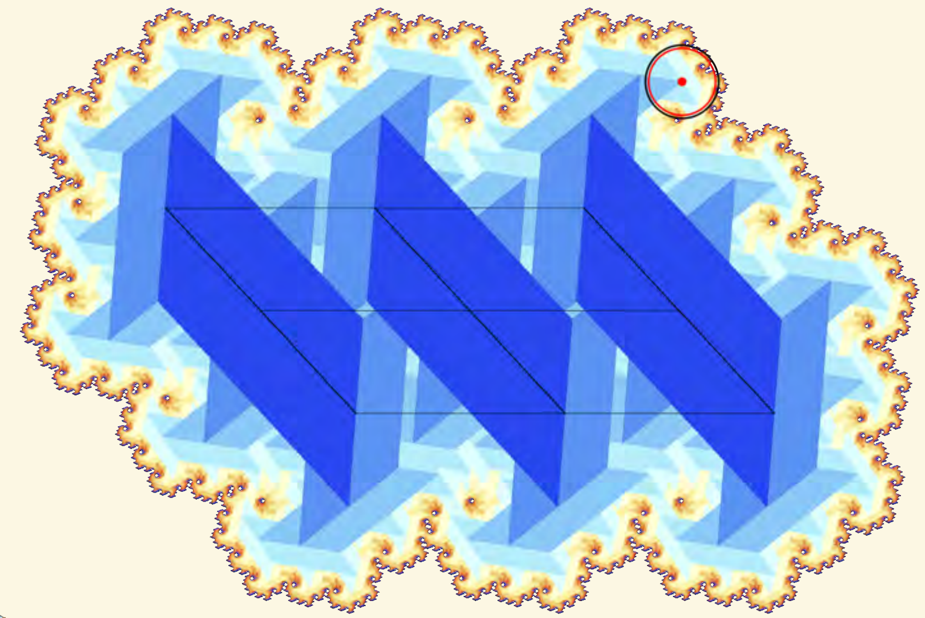


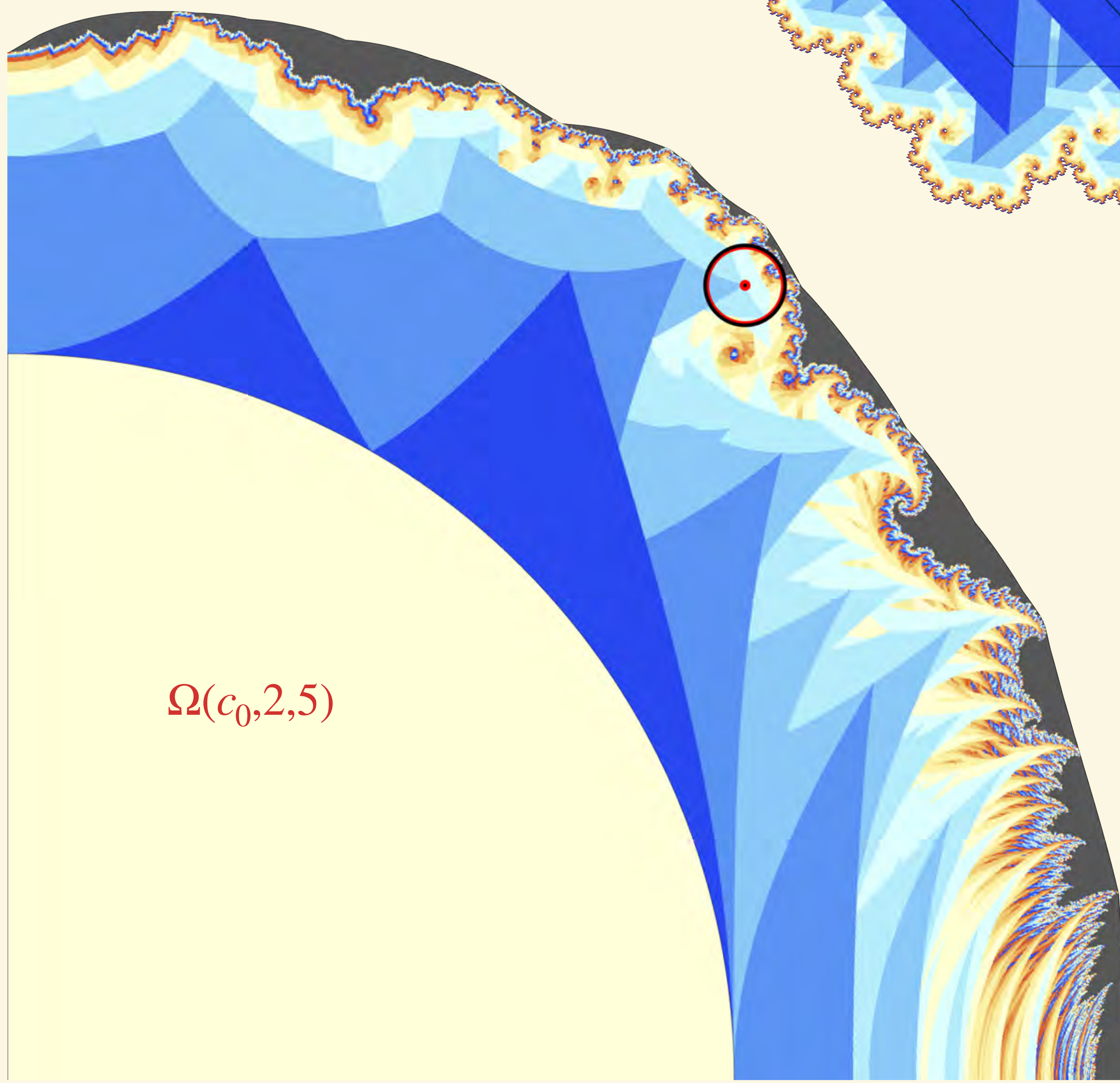


$c_0 = 1.0141 + i\,1.0951$

$2c_0 \in \boldsymbol{E}(c, 2n-1)$

$$c^5 - c^4 + c^3 + c^2 + c + 1 = 0$$



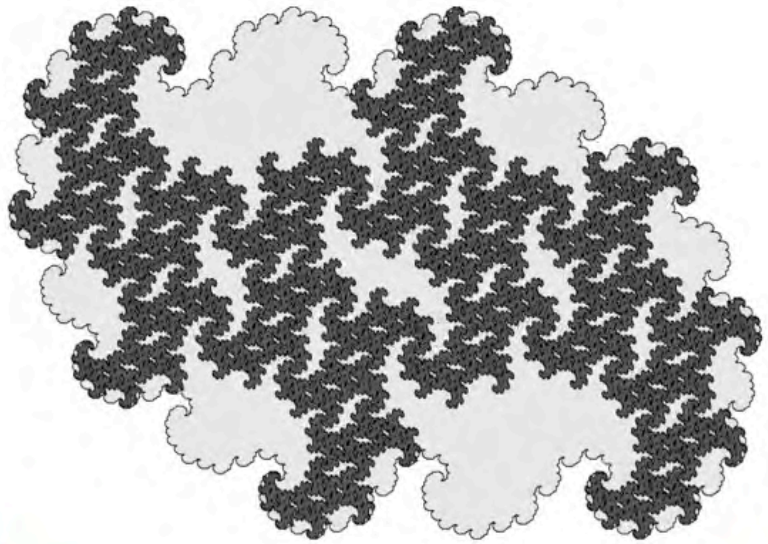


Animation: youtu.be/11NZDHNahJs

$c = 1.2345 + i\,0.7935$

$\mathrm{Arg}(c)=0.5713 = 32.73°$

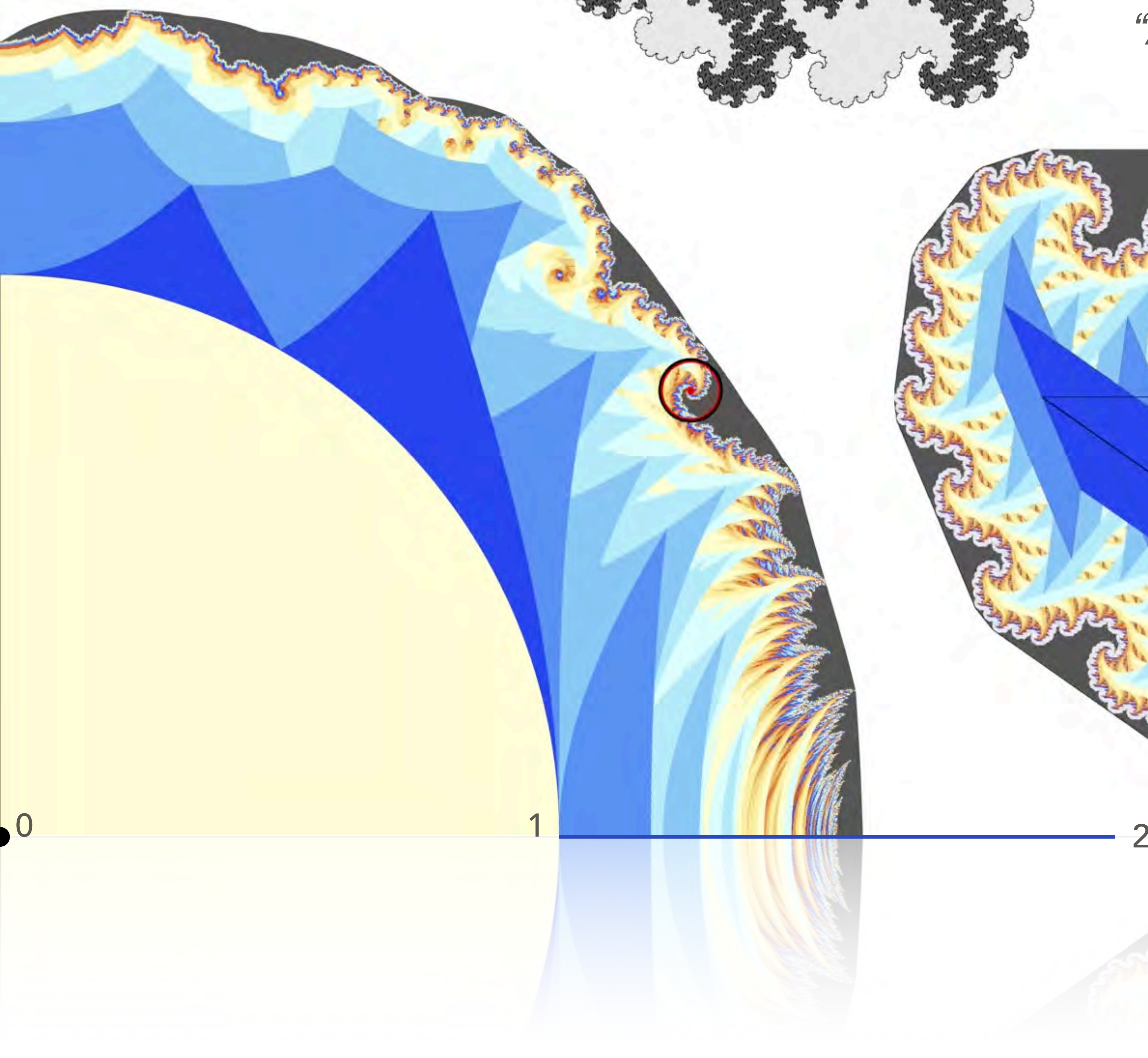
$r=|c^{-1}|=0.6814$

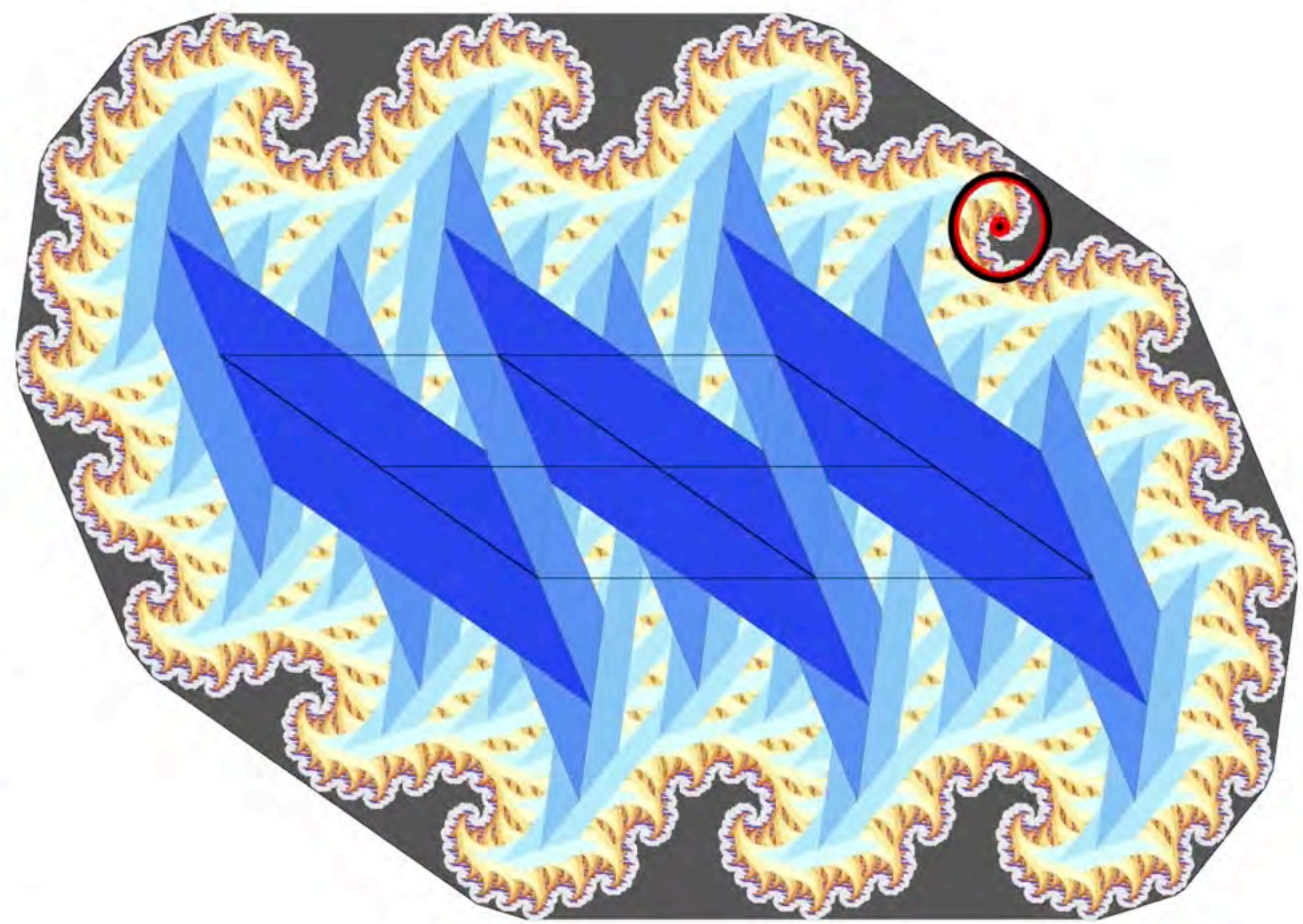


$$\dim_H E(c,2) = \frac{\log 2}{\log |c|} \approx 1.8071$$

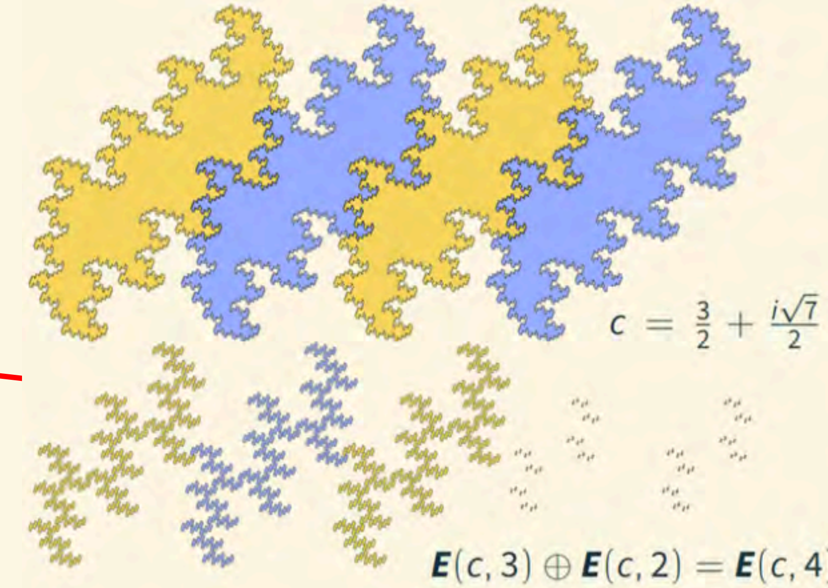
“Asymptotic self-similarity”

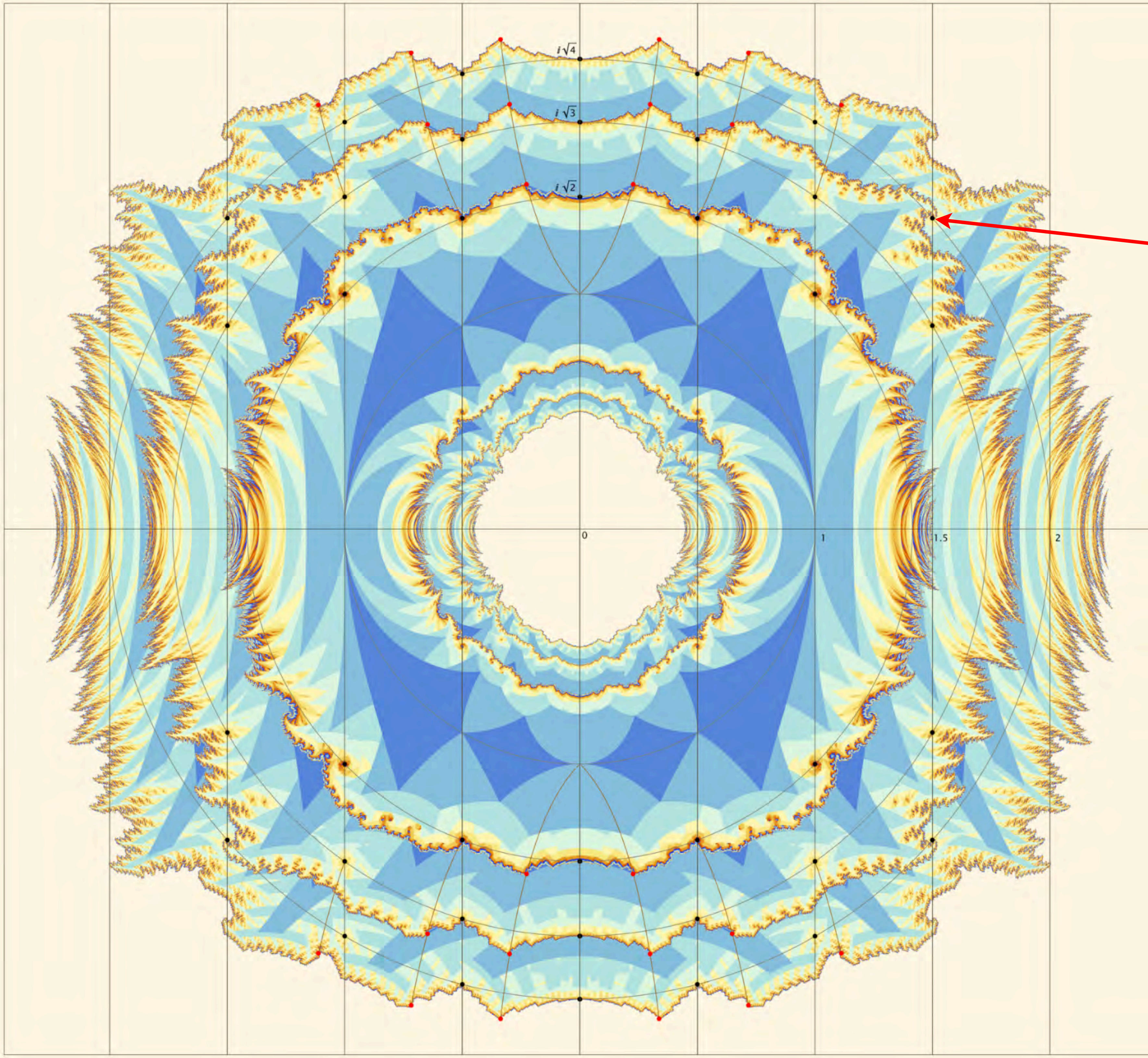
$2c \in E(c,3)$





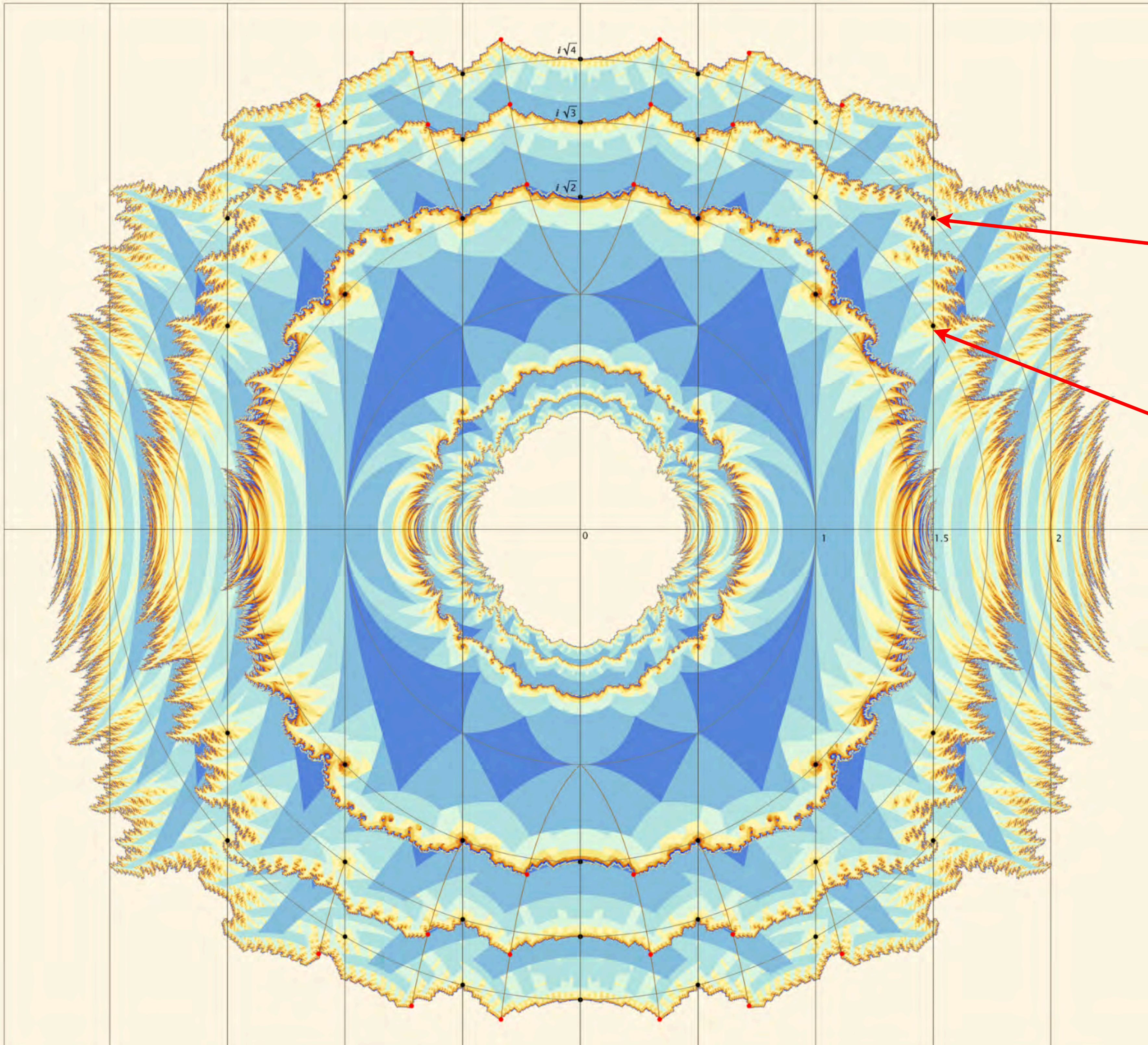
$$\boldsymbol{E}(c, n+1) = \boldsymbol{E}(c, n) \oplus \boldsymbol{E}(c, 2)$$

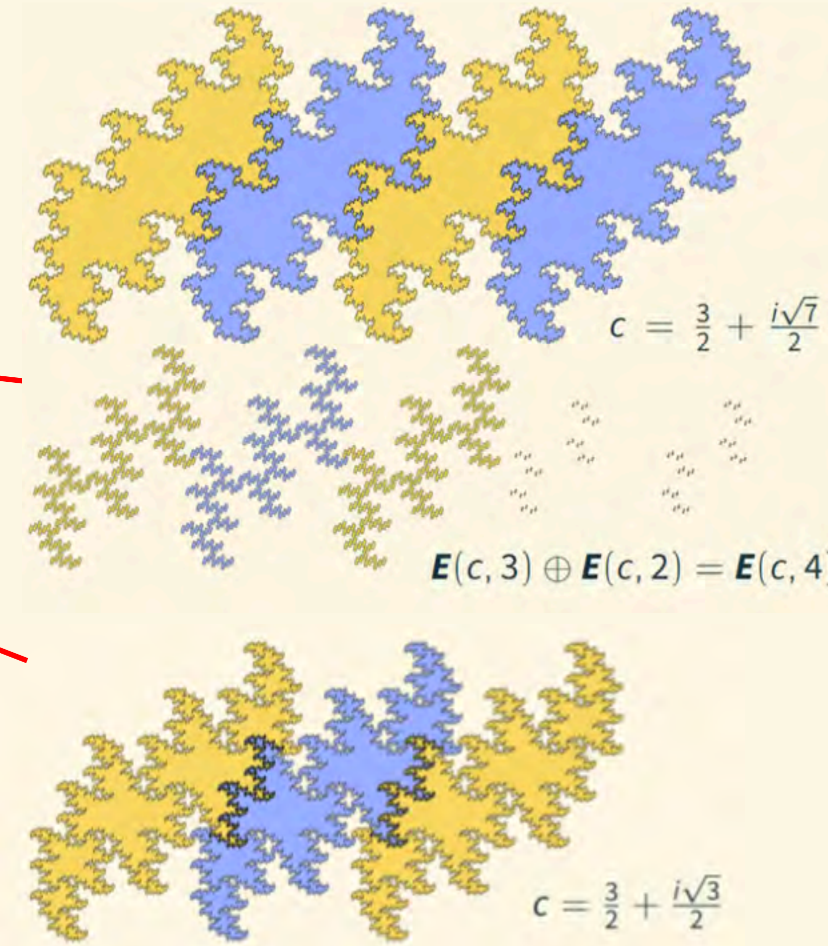




$$\mathcal{M}_n \subset \mathcal{M}_{n+1}$$

a=3

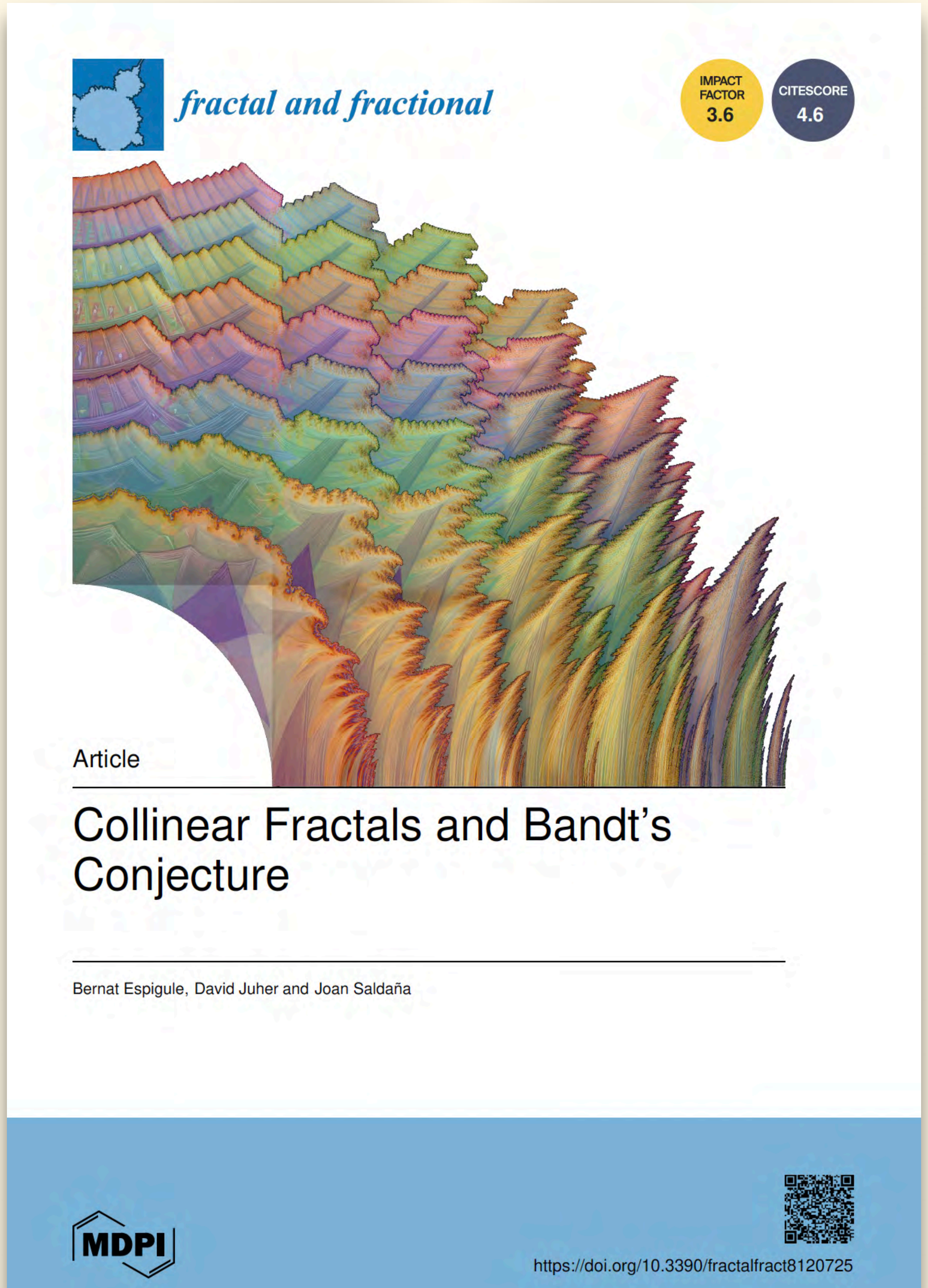




Let

$$c = \frac{a}{2} + \frac{i}{2}\sqrt{4b + a - 2\left\lfloor \frac{a+1}{2} \right\rfloor},$$

where $a \geqslant 0$ and $b \geqslant 1$ are a pair of integers. If $|c| = \sqrt{n}$ then $\boldsymbol{E}(c, n)$ is a planar self-affine tile with a collinear digit set.



fractal and fractional

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Article

Collinear Fractals and Bandt's Conjecture

Bernat Espigule, David Juher and Joan Saldaña

MDPI

https://doi.org/10.3390/fractalfract8120725

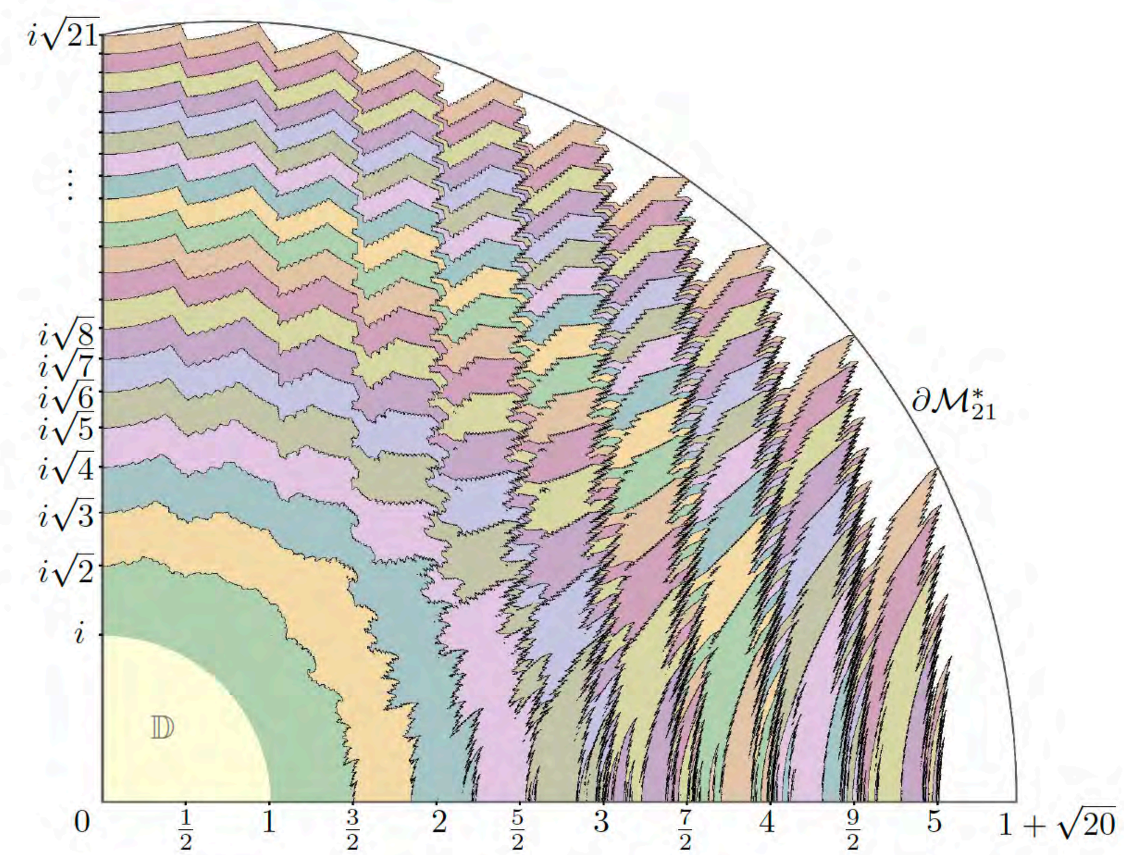


Figure 3. Superimposed arrangement of $\mathcal{M}_2$, $\mathcal{M}_3$, …, $\mathcal{M}_{21}$ constrained within the upper-right section of the complex plane. From Proposition 3, we know that the connectedness loci are nested. The illustration suggests the existence of infinitely many holes, and that for $n \geq 4$, the intersection $\partial\mathcal{M}_n \cap \partial\mathcal{M}_{n+1} \setminus \mathbb{R}$ is nonempty.

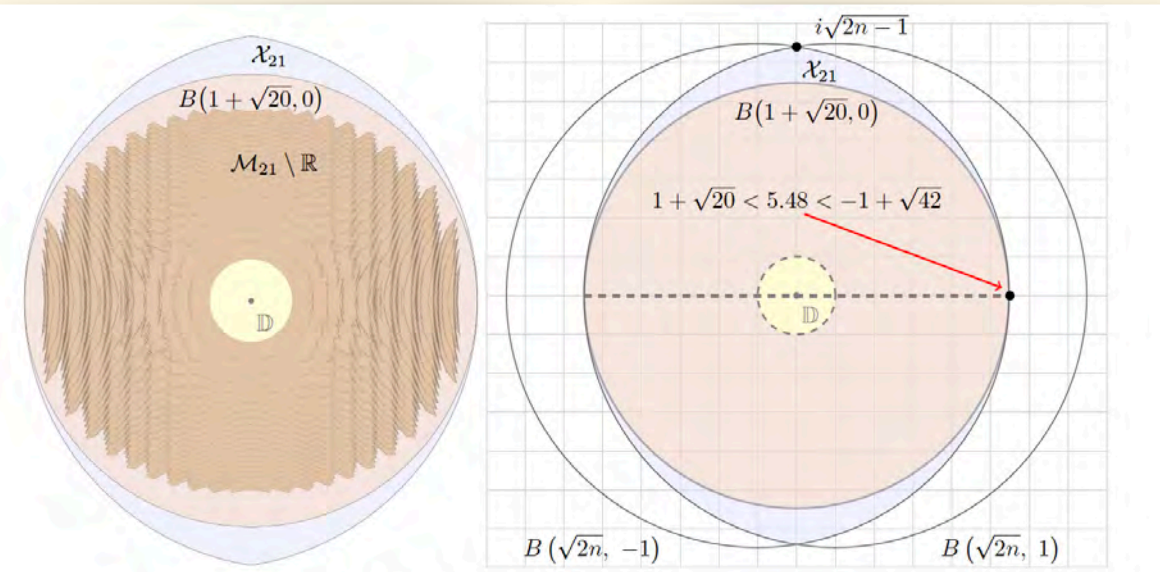
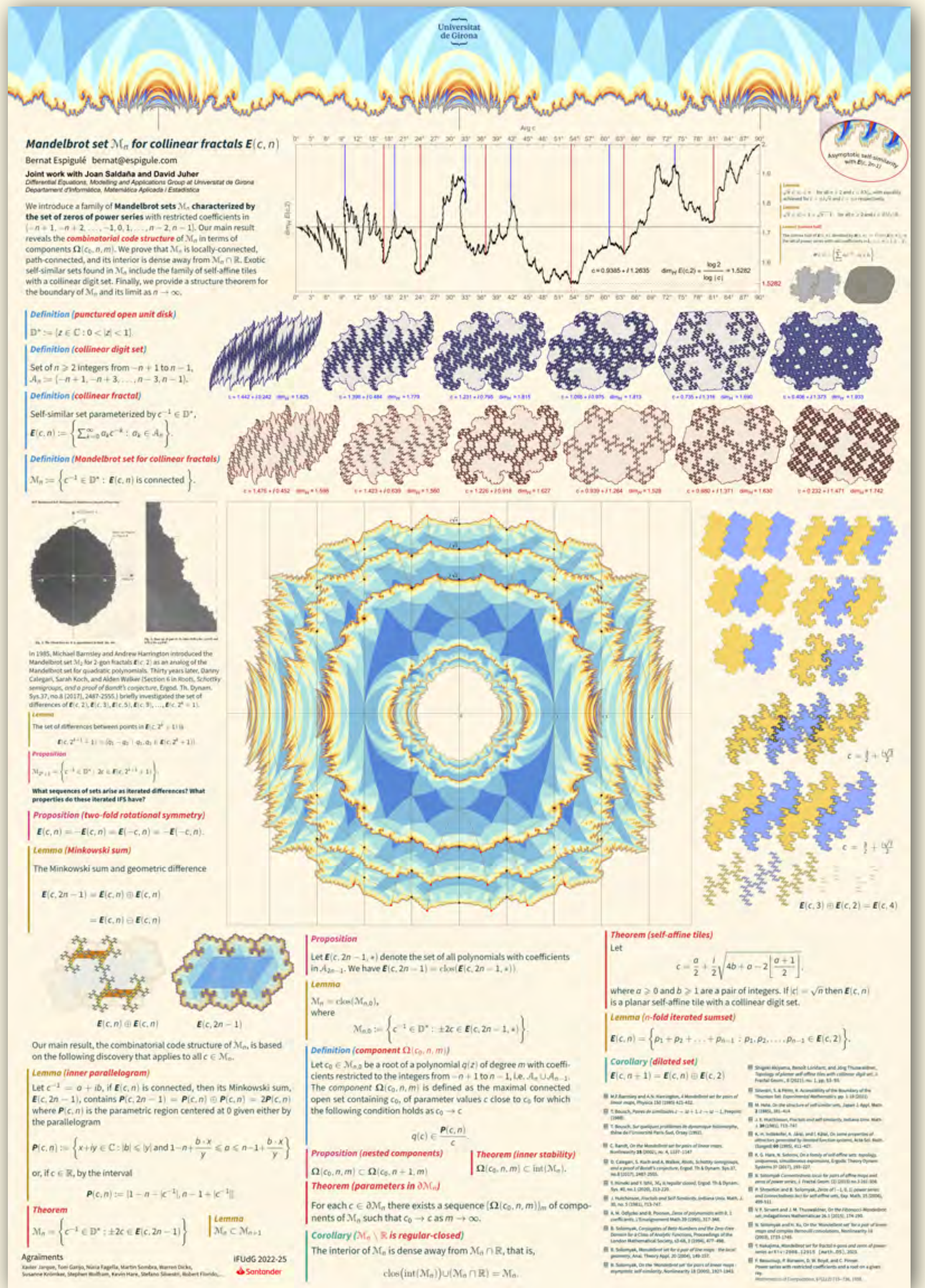
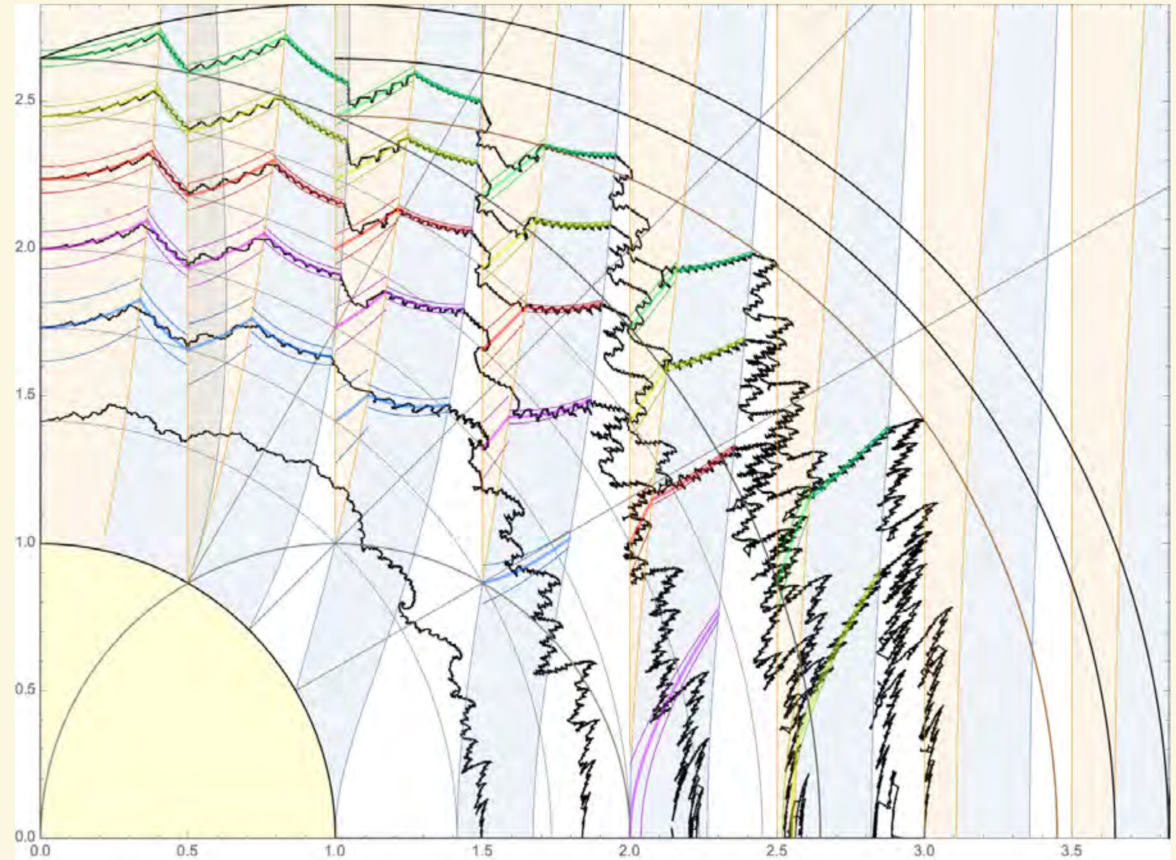


Figure 10. The set $\mathcal{M}_{21} \setminus \mathbb{R}$ contained in $\mathcal{X}_{21}$. Since $1+\sqrt{20} < -1+\sqrt{42}$, the region $\mathcal{X}_{21}$ contains $B\left(1+\sqrt{20},0\right) \setminus (\mathbb{R} \cup \mathbb{D})$ which in turn contains $\mathcal{M}_{21} \setminus \mathbb{R}$ by Proposition 4.





What's next?

Structure theorem for the boundary of $\mathcal{M}_n$

Illustration as a Mathematical Research Technique

Workshop 3: Integrating Research and Illustration in Number Theory

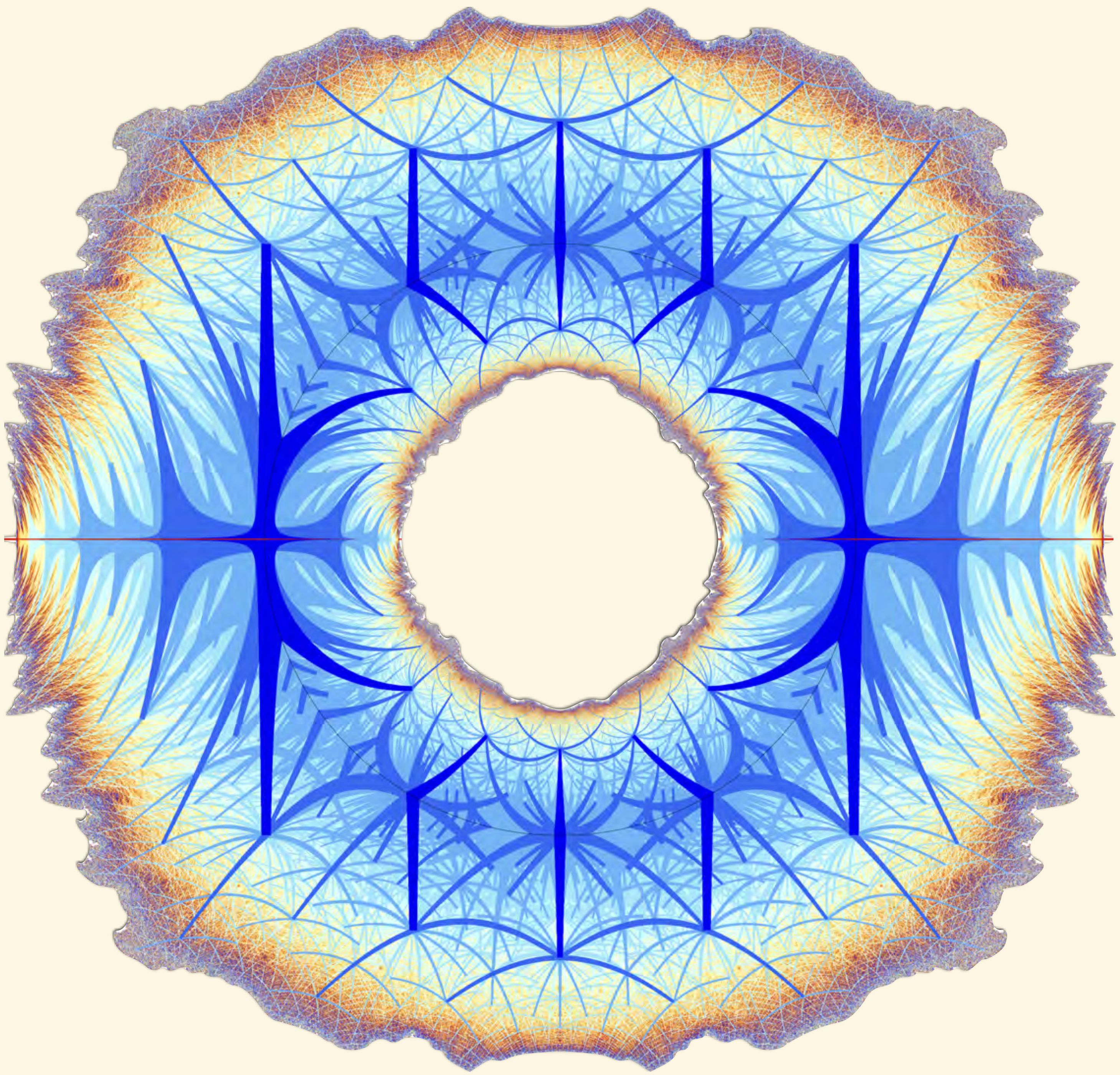
March 23 to 27, 2026 - IHP, Paris

Summary

This workshop will highlight recent research developments in algebra and number theory informed by illustration. It will also provide structured opportunities for participants to engage in new research collaborations that employ illustration. The workshop schedule will include research talks as well as time set aside to establish these collaborations.

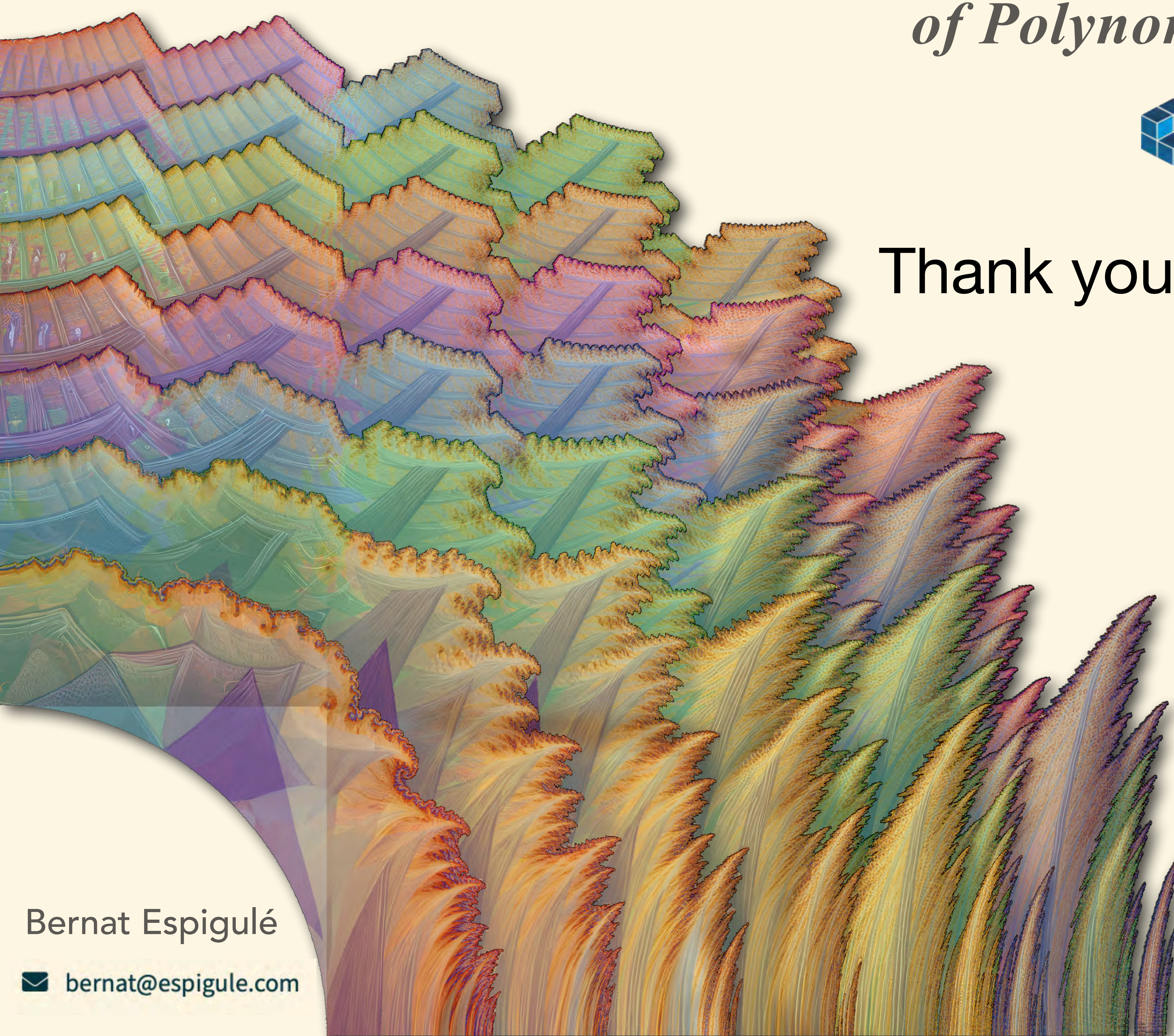
https://indico.math.cnrs.fr/event/13126/

ComplexTrees.com/collinear



- 1886–1971: Paul Lévy
- 1924–2010: Benoit Mandelbrot
- 1985: Michael Barnsley & Andrew Harrington (IFS M2)
- 1985: Masayoshi Hata (fractal connectivity)
- 1991: Christoph Bandt & Karsten Keller (self-similar sets)
- 1992: Thierry Bousch (connectedness in dynamics)
- 1998: Franck Beaucoup et al. (polynomial roots)
- 2002: Christoph Bandt (conjecture on M_2)
- 2004: Boris Solomyak and Xu (partial proof)
- 2006: J. Daniel Christensen (complex root plots)
- 2007: David H. Bailey et al. (experimental math tools)
- 2008: Christoph Bandt & Nguyen Hung (n-gon fractals)
- 2008: Jonathan M. Borwein et al. (polynomial studies)
- 2014: Katherine Stange (visual number theory)
- 2017: Danny Calegari et al. (proof for n=2)
- 2021: Shigeki Akiyama et al. (tiling extensions)
- 2021: Edmund Harriss et al. (algebraic starscapes)
- 2022: Gabriel Dorfsman-Hopkins & Shuchang Xu
- 2023: John C. Baez et al. (beauty of roots)
- 2024: Bernat Espigulé et al. (generalized resolutions)
- 1959–2025: Michael Trott

Collinear Fractals - Unveiling the Hidden Geometry of Polynomial Roots



Thank you!

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IFUdG 2022–2024 Santander

Low dimensional dynamical systems: topology, geometry, combinatorics and bifurcations

PID2023-146424NB-I00

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
AGENCIA ESTATAL DE INVESTIGACIÓN

Bernat Espigulé

bernat@espigule.com

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought.

SHORT STORIES

The Beauty of Roots

John C. Baez, J. Daniel Christensen, and Sam Derbyshire

Figure 1. Roots of all polynomials of degree 23 whose coefficients are ± 1. The brightness shows the number of roots per pixel.

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought. For example, if we plot the roots of all polynomials of degree 23 whose coefficients are all 1 or -1, we get an astounding picture, shown in Figure 1.

More generally, define a **Littlewood polynomial** to be a polynomial $p(z) = \sum_{i=0}^{d} a_i z^i$ with each coefficient a_i equal to 1 or -1. Let $\mathbf{X}_n$ be the set of complex numbers that are roots of some Littlewood polynomial with n nonzero terms (and thus degree $n-1$). The 4-fold symmetry of Figure 1 comes from the fact that if $z \in \mathbf{X}_n$ so are $-z$ and $\overline{z}$. The set $\mathbf{X}_n$ is also invariant under the map $z \mapsto 1/z$, since if z is the root of some Littlewood polynomial then $1/z$ is a root of the polynomial with coefficients listed in the reverse order.

It turns out to be easier to study the set

$$\mathbf{X} = \bigcup_{n=1}^{\infty} \mathbf{X}_n = \{z \in \mathbb{C} \mid z \text{ is the root of some Littlewood polynomial}\}.$$

If n divides m then $\mathbf{X}_n \subseteq \mathbf{X}_m$, so $\mathbf{X}_n$ for a highly divisible number n can serve as an approximation to $\mathbf{X}$, and this is why we drew $\mathbf{X}_{24}$.

Some general properties of $\mathbf{X}$ are understood. It is easy to show that $\mathbf{X}$ is contained in the annulus $1/2 < |z| < 2$. On the other hand, Thierry Bousch showed [2] that the closure of $\mathbf{X}$ contains the annulus $2^{-1/4} \leq |z| \leq 2^{1/4}$. This means that the holes near roots of unity visible in the sets $\mathbf{X}_n$ must eventually fill in as we take the union over all n. More surprisingly, Bousch showed in 1993 that the closure $\overline{\mathbf{X}}$ is connected and locally path-connected [3]. It is worth comparing the work of Odlyzko and Poonen [7], who previously showed similar result for roots of polynomials whose coefficients are all 0 or 1.

John C. Baez is a professor of mathematics at UC Riverside. His email address is john.baez@ucr.edu.

J. Daniel Christensen is a professor of mathematics at the University of Western Ontario. His email address is jdc@uwo.ca.

Sam Derbyshire is a Haskell consultant at Well-Typed. His email address is sam@well-typed.com.

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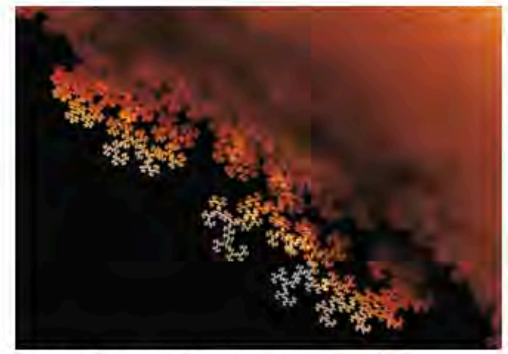

Figure 2. The region of $\mathbf{X}_{24}$ near the point $z = \frac{1}{2}e^{i/8}$.

The big challenge is to understand the diverse, complicated and beautiful patterns that appear in different regions of the set $\mathbf{X}$. There are websites that let you explore and zoom into this set online [4, 5, 8]. Different regions raise different questions.

For example, what is creating the fractal patterns in Figure 2 and elsewhere? An anonymous contributor suggested a fascinating line of attack which was further developed by Greg Egan [5]. Define two functions from the complex plane to itself, depending on a complex parameter q:

$$f_{+q}(z) = 1 + qz, \qquad f_{-q}(z) = 1 - qz.$$

When $|q| < 1$ these are both contraction mappings, so by a theorem of Hutchinson [6] there is a unique nonempty compact set $D_q \subseteq \mathbb{C}$ with

$$D_q = f_{+q}(D_q) \cup f_{-q}(D_q).$$

We call this set a **dragon**, or the **q-dragon** to be specific. And it seems that *for $|q| < 1$, the portion of the set $\mathbf{X}$ in a small neighborhood of the point q tends to look like a rotated version of D_q.*

Figure 3 shows some examples. To precisely describe what is going on, much less prove it, would take real work. We invite the reader to try. A heuristic explanation is known, which can serve as a starting point [1, 5]. Bousch [3] has also proved this related result:

Theorem. *For $q \in \mathbb{C}$ with $|q| < 1$, we have $q \in \overline{\mathbf{X}}$ if and only if $0 \in D_q$. When this holds, the set D_q is connected.*

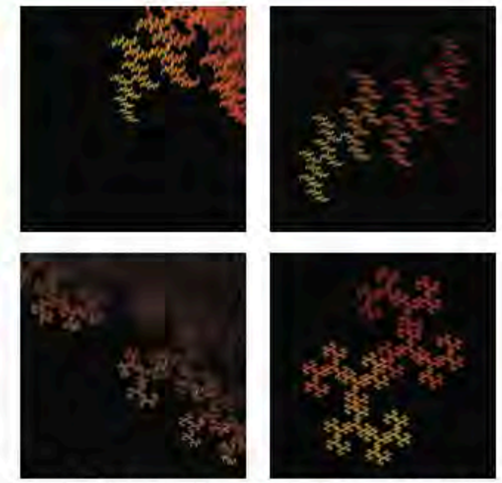

Figure 3. Top: the set $\mathbf{X}$ near $q = 0.594 + 0.254i$ at left, and the set D_q at right. Bottom: the set $\mathbf{X}$ near $q = 0.375453 + 0.544825i$ at left, and the set D_q at right.

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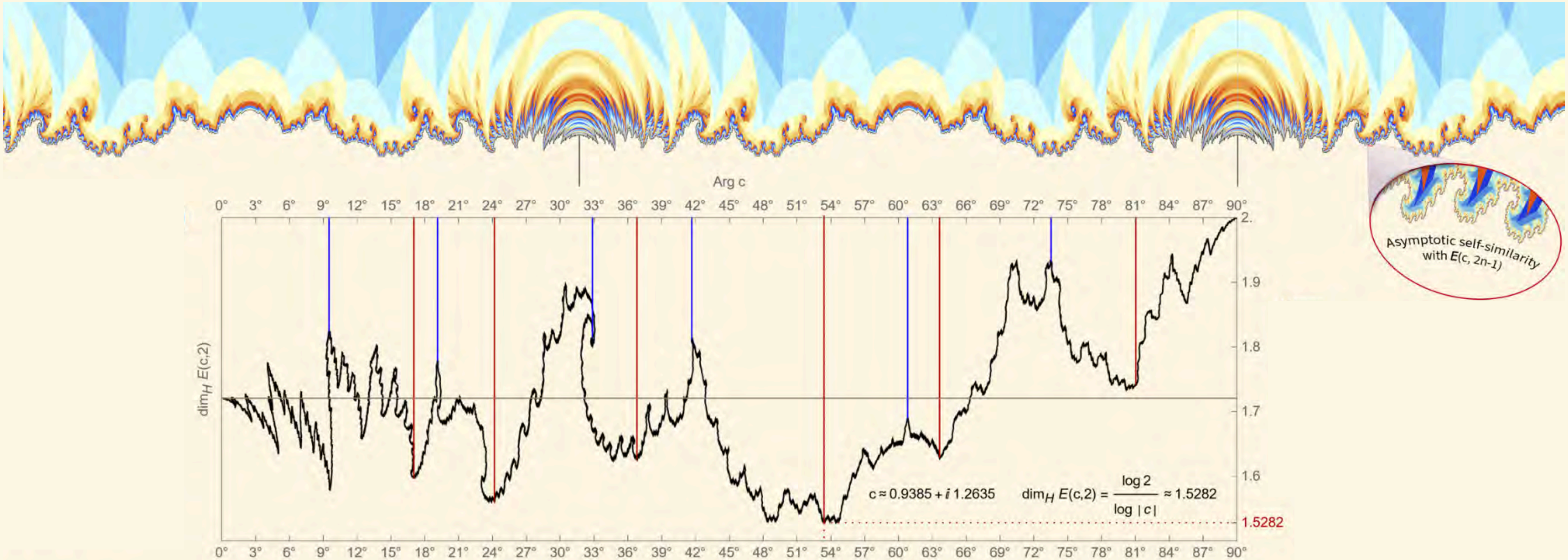
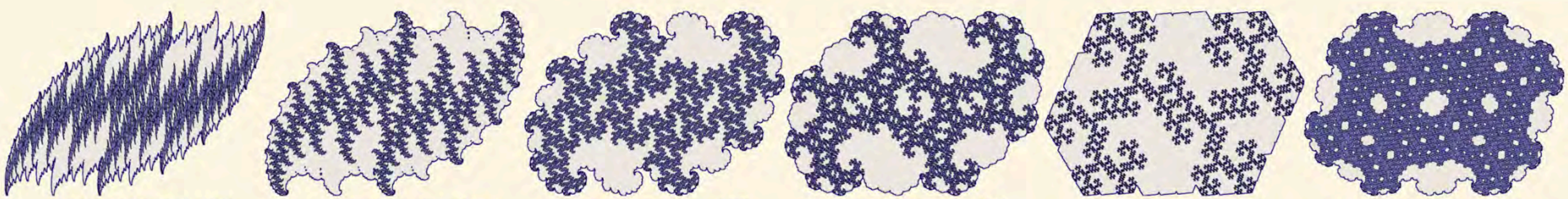
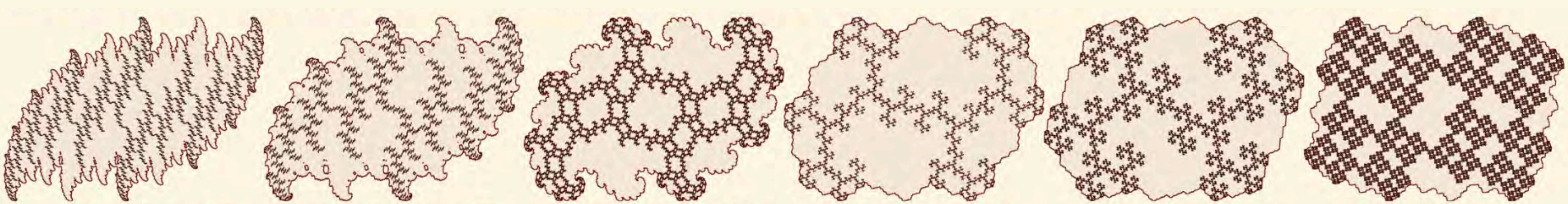




c = 1.442 + *i* 0.242 dim$_H$ ≈ 1.825 c = 1.395 + *i* 0.484 dim$_H$ ≈ 1.779 c = 1.231 + *i* 0.795 dim$_H$ ≈ 1.815 c = 1.095 + *i* 0.975 dim$_H$ ≈ 1.813 c = 0.735 + *i* 1.316 dim$_H$ ≈ 1.690 c = 0.406 + *i* 1.373 dim$_H$ ≈ 1.933



c = 1.476 + *i* 0.452 dim$_H$ ≈ 1.598 c = 1.423 + *i* 0.639 dim$_H$ ≈ 1.560 c = 1.226 + *i* 0.918 dim$_H$ ≈ 1.627 c = 0.939 + *i* 1.264 dim$_H$ ≈ 1.528 c = 0.680 + *i* 1.371 dim$_H$ ≈ 1.630 c = 0.232 + *i* 1.471 dim$_H$ ≈ 1.742

M.F. Barnsley and A.N. Harrington / A Mandelbrot set for pairs of linear maps

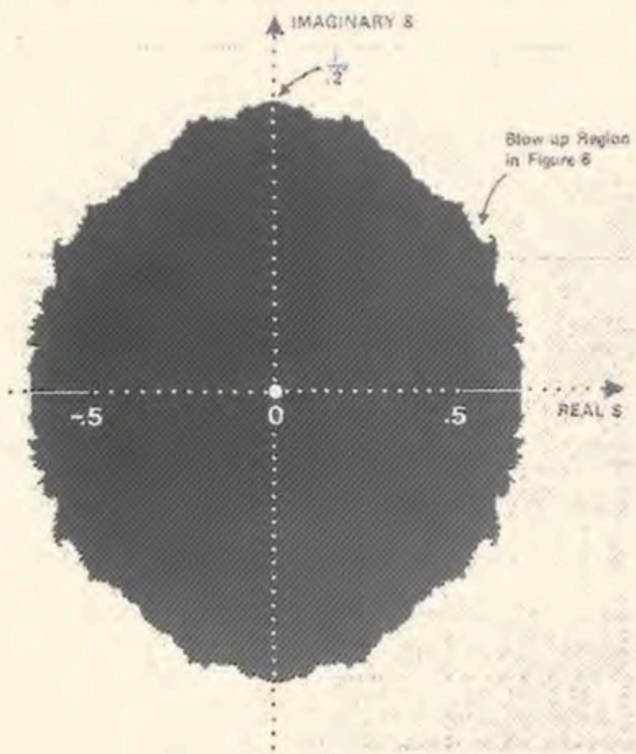
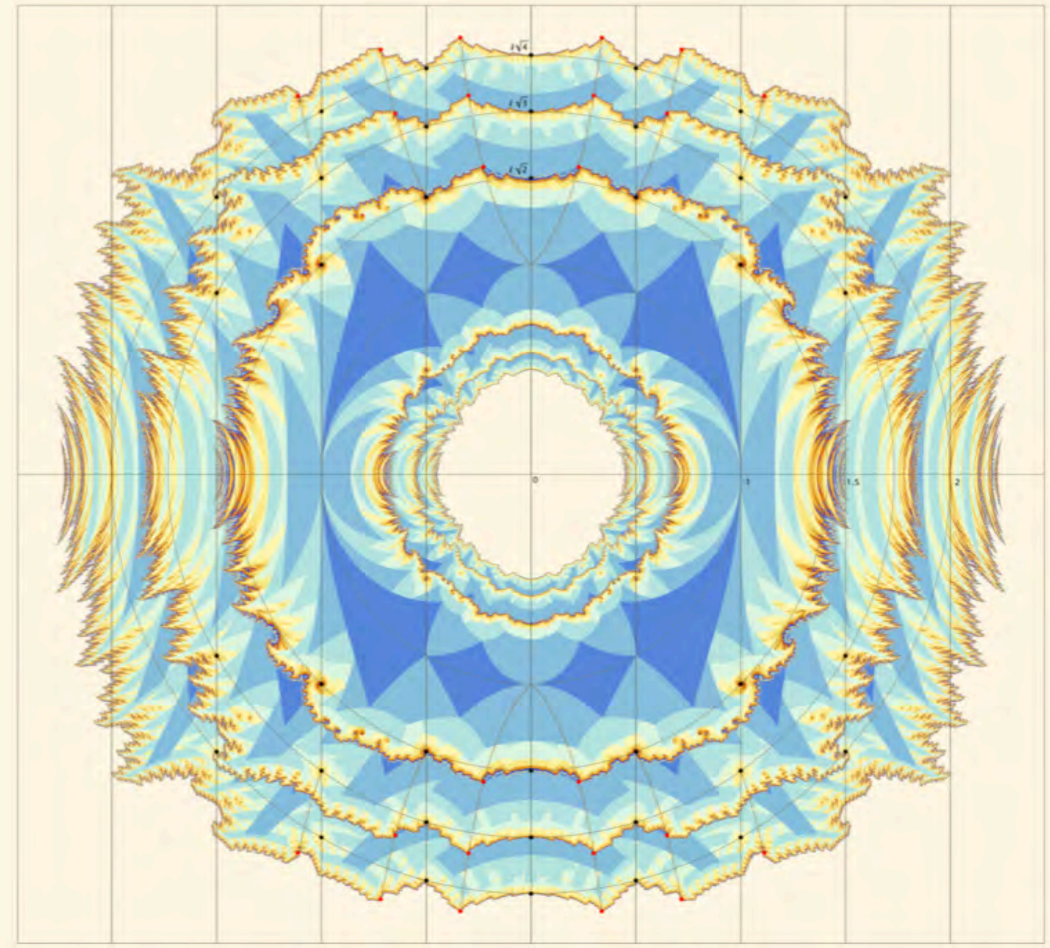


Fig. 5. The Mandelbrot set D is approximated in black. See text.

Fig. 6. Blow up of part of D, where $0.49 \le \mathrm{Re}\ s \le 0.55$ and $0.35 \le \mathrm{Im}\ s \le 0.45$.



Our main contributions: combinatorial code structure, inner stability, components for any n>1, collinear tiles, structure theorem for the boundary of $\mathcal{M}_n$

In 1985, Michael Barnsley and Andrew Harrington introduced the Mandelbrot set $\mathcal{M}_2$ for 2-gon fractals $\boldsymbol{E}(c, 2)$ as an analog of the Mandelbrot set for quadratic polynomials. Thirty years later, Danny Calegari, Sarah Koch, and Alden Walker (Section 6 in *Roots, Schottky semigroups, and a proof of Bandt's conjecture*, Ergod. Th. Dynam. Sys.37, no.8 (2017), 2487-2555.) briefly investigated the set of differences of $\boldsymbol{E}(c,2), \boldsymbol{E}(c,3), \boldsymbol{E}(c,5), \boldsymbol{E}(c,9), \dots, \boldsymbol{E}(c, 2^k+1)$.

Lemma

The set of differences between points in $\boldsymbol{E}(c, 2^k+1)$ is

$$\boldsymbol{E}(c, 2^{k+1}+1) = \{q_1 - q_2 \,:\, q_1, q_2 \in \boldsymbol{E}(c, 2^k+1)\}.$$

Proposition

$$\mathcal{M}_{2^k+1} = \left\{ c^{-1} \in \mathbb{D}^* \,:\, 2c \in \boldsymbol{E}(c, 2^{k+1}+1) \right\}.$$

What sequences of sets arise as iterated differences? What properties do these iterated IFS have?

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