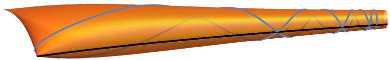
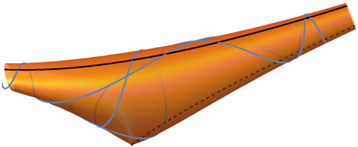
Visualizing domains and surfaces to study continued fractions





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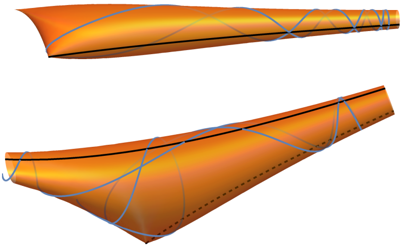
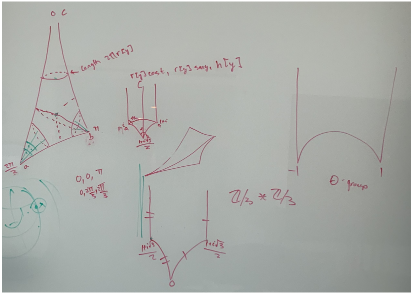
Geodesics on surfaces





Continued fraction crash course

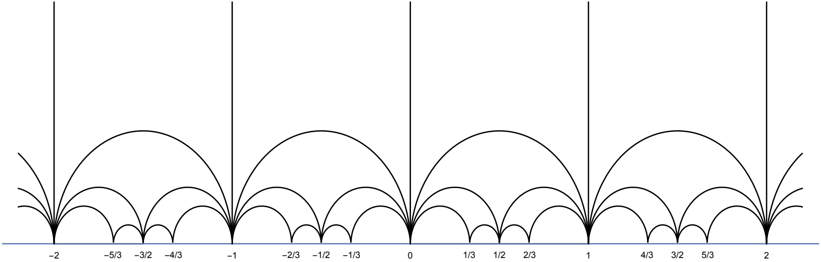
Continued fractions represent (real) numbers using rational approximates.

Real	Regular continued fraction	Even continued fraction
π	$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cdots}}}$	$4-\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{8+\cfrac{1}{2-\cdots}}}}$
$x>0$	$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}}$ $a_i \in \mathbb{Z}_+$, a_0 can be 0	$2k_0 \pm \cfrac{1}{2k_1 \pm \cfrac{1}{2k_2+\cdots}}$ $k_i \in \mathbb{Z}_+$, k_0 can be 0

Regular continued fractions on the upper half plane

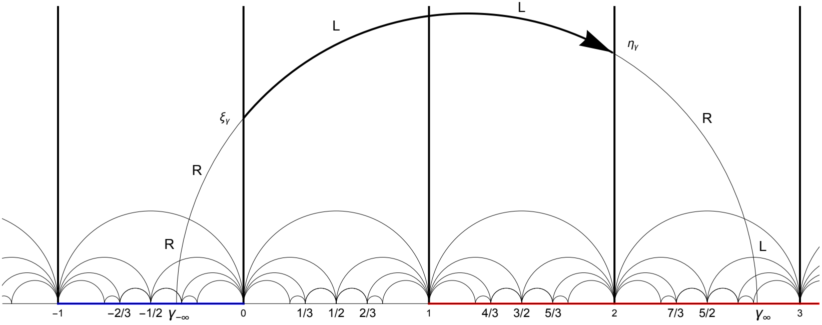
The upper half plane $\mathbb{H} := \{x + iy \mid y > 0\}$, the shortest distance between two points (*geodesic*) is on a vertical line or a semicircle centered on the x-axis.

Connect two rational numbers $\frac{p}{q}, \frac{p'}{q'}$ if and only if $pq' - p'q = \pm 1$



Geodesics on the upper half plane

Pick a geodesic that starts in $(-1,0)$ and ends in $(1,\infty)$ or that starts in $(0,1)$ and ends in $(-\infty,-1)$.

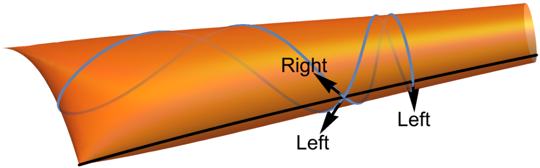


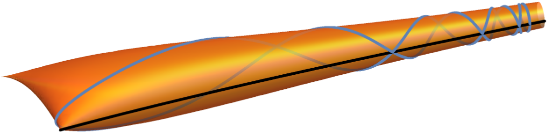
$$\gamma_{-\infty} = \cfrac{-1}{2 + \cfrac{1}{1 + \cdots}} \qquad \gamma_{\infty} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cdots}}$$

Geodesics on the modular surface

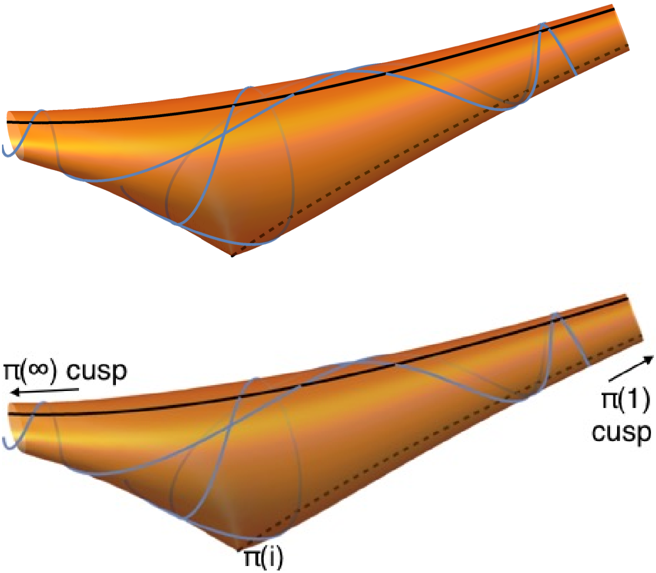
Modular surface $\mathcal{M} = \mathrm{PSL}(2, \mathbb{Z})\backslash\mathbb{H}$

The longer the cusp stays on the same side of the geodesic, the farther the geodesic has traveled into the cusp

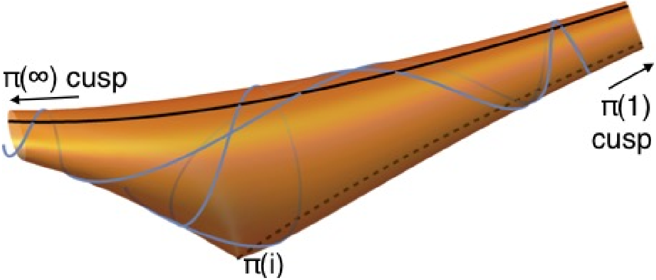




Two cusp surface

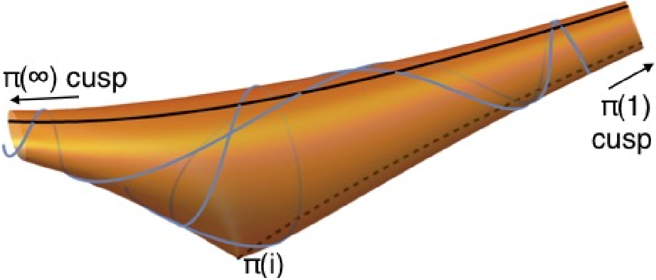


Two cusp surface



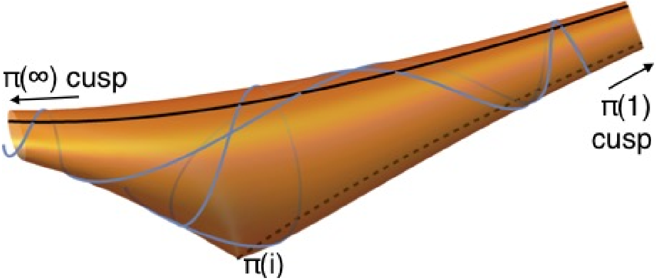
- Coding using Farey tessellation is tracking the $\pi(\infty)$ cusp.

Two cusp surface



- Coding using Farey tessellation is tracking the $\pi(\infty)$ cusp.
- Large even digit is farther into the $\pi(\infty)$ cusp.
- Digit followed by subtraction corresponds to crossing the dotted line without changing whether the cusp is on the right or left; addition is crossing the dotted line and changing.

Two cusp surface

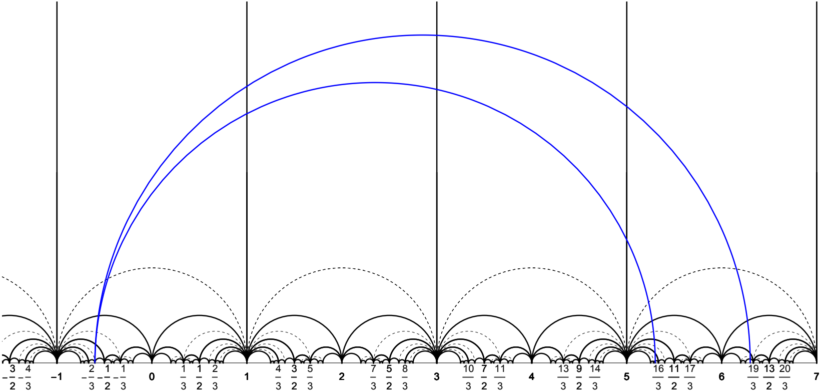


- Coding using Farey tessellation is tracking the $\pi(\infty)$ cusp.
- Large even digit is farther into the $\pi(\infty)$ cusp.
- Digit followed by subtraction corresponds to crossing the dotted line without changing whether the cusp is on the right or left; addition is crossing the dotted line and changing.
- The more occurances of digit $2 - \frac{1}{\cdots}$ in a row corresponds to how far the geodesic travels into the $\pi(1)$ cusp.

A new tessellation

Keep the edges of the Farey tessellation that connect two rationals $\frac{p}{q}, \frac{p'}{q'}$, where exactly one of p, q, p', q' is even (sides that travel between cusps).

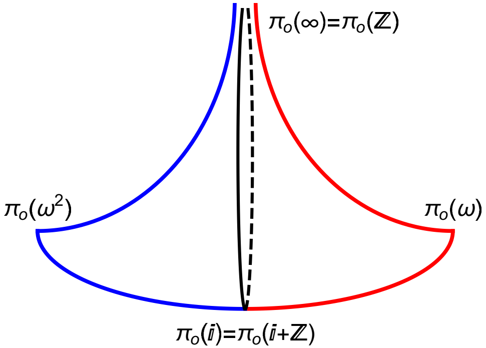
Add dotted edges between adjacent rationals where p, q, p', q' are all odd (runs from $\pi(1)$ cusp to $\pi(i)$ and back).



Odd continued fractions

The new tessellation allows for a much better coding (especially notationally!) for the odd continued fractions. However, I still cannot prove the results for the slowdown map.

My first question is: how does the coding of geodesics on $\mathbb{H}$ correspond to the geodesics on this modular surface:



Natural extension domains for α-odd continued fractions

Here's a gif of natural extesion domains for α-odd continued fractions on the left and the dynamical system on the left.

Animating these domains was originally done with undergraduate students Xavier Ding, Gustav Jennetten, and Joel Rozhon in the Illinois Geometry Lab (now Math Lab) in Spring 2019.

Dropbox link