Visualizing genus zero dessins

Mike Musty ICERM 2025 August 12, 2025

Thanks!

To:

- The organizers
- Sam Schiavone
  `https://math.mit.edu/~sschiavo/`
- Everett Sullivan
  `https://facultyprofiles.clayton.edu/faculty/esullivan4`
- `https://faraday.ai/`

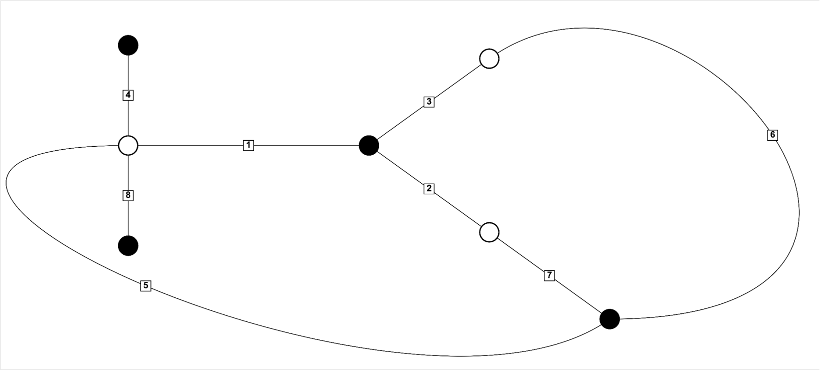
Objective of this work

Draw topologically correct dessins to help visualize the Galois orbits of all (genus zero) passports in the LMFDB.

- `https://www.lmfdb.org/Belyi/`
- `https://michaelmusty.github.io/dessins/`

Permutation triple ⟶ dessin

σ_0 = (1,4,5,8)(2,7)(3,6), σ_1 = (1,2,3)(5,6,7), σ_∞ = (8,7,1)(6,2)(5,4,3)
Base field: 1 - 3T^2
Embedding: -1.732051



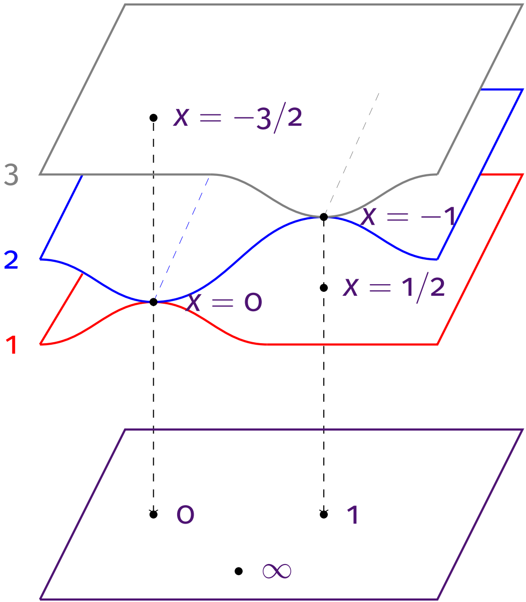
8T38-4.2.2_3.3.1.1_3.3.2

Degree 3 example

$$\big((1\,2)(3),\ (1)(2\,3),\ (1\,3\,2)\big)$$

$$\updownarrow$$

Degree 3 example



$$X \xrightarrow{\varphi(x)} \mathbb{P}^1$$

$$\varphi(x) = \underbrace{2x^3 + 3x^2}_{z} = x^2(2x+3) = (2x-1)(x+1)^2 + 1$$

Belyi maps, dessins, etc.

- This branched covering is an example of a **Belyi map**, a map of Riemann surfaces unbranched outside of $0, 1, \infty$.
- A Belyi map determines a **permutation triple** via monodromy.
- A permutation triple determines a **triangle subgroup** Γ which acts on a spherical, Euclidean, or hyperbolic space $\mathcal{H}$ such that $\Gamma \backslash \mathcal{H} \to \Delta \backslash \mathcal{H}$ is the corresponding Belyi map.
- The preimage of $[0, 1]$ is a (connected) bipartite graph embedded on a surface called a **dessin d'enfant**.

Equivalent categories

$$
\begin{tikzcd}
\left\{\begin{array}{l}\text{Transitive permutation triples}\\ (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3 \text{ with } \sigma_\infty\sigma_1\sigma_0 = \\ 1 \text{ up to simultaneous conjuga-}\\ \text{tion}\end{array}\right\} \arrow[dr, leftrightarrow, "\sim"] & & \left\{\begin{array}{l}\text{Conjugacy classes of in-}\\ \text{dex } d \text{ triangle subgroups}\\ \Gamma \leq \Delta(a, b, c)\end{array}\right\} \arrow[dl, leftrightarrow, "\sim"'] \\
& \left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{dessins with } d \text{ edges}\end{array}\right\} & \\
\left\{\begin{array}{l}\text{Isomorphism classes of degree } d \text{ function}\\ \text{field extensions } K \supseteq \mathbb{C}(z) \text{ with discrimi-}\\ \text{nant supported above } 0, 1, \infty\end{array}\right\} \arrow[ur, leftrightarrow, "\sim"] & & \left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{Belyi maps of degree } d\end{array}\right\} \arrow[ul, leftrightarrow, "\sim"']
\end{tikzcd}
$$

Degree 3 example

$$\big((1\,2)(3),(1)(2\,3),(1\,3\,2)\big)$$

$$[\Delta(2,2,3):\Gamma]=3$$

$$\mathbb{C}\Big(\underbrace{2x^3+3x^2}_{z}\Big)\subseteq\mathbb{C}(x)$$

$$\varphi(x)=\underbrace{2x^3+3x^2}_{z}$$

Galois action

Theorem (Belyi, 1979)

A smooth projective curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a nonconstant morphism of algebraic curves $\varphi : X \to \mathbb{P}^1$ unramified outside $\{0, 1, \infty\}$.

In the 1980s Grothendieck described an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of (isomorphism classes of) dessins.

Passports

A **passport** consists of the data (G, λ) where

- $G \leq S_d$ is a transitive subgroup
- $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d

A permutation triple $\sigma \in S_d^3$ **belongs** to (G, λ) if

- $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle = G$
- $\sigma_0, \sigma_1, \sigma_\infty$ have the cycle types specified by $\lambda_0, \lambda_1, \lambda_\infty$

Note

- The **size** of a passport is the number of permutation triples belonging to it (up to simultaneous conjugacy).
- A **refined passport** replaces λ with $C = (C_0, C_1, C_\infty)$ a triple of conjugacy classes of G.
- Passports and refined passports are stable under the Galois action and therefore are unions of Galois orbits.

7T6-5.1.1_4.2.1_3.2.2

- `https://www.lmfdb.org/Belyi/7T6/5.1.1/4.2.1/3.2.2/`
- `https://michaelmusty.github.io/dessins/passports/7T6-5.1.1_4.2.1_3.2.2/index.html`
- All drawings together

9T34-6.1.1.1_5.2.2_6.1.1.1

- `https://www.lmfdb.org/Belyi/9T34/6.1.1.1/5.2.2/6.1.1.1/`
- `https://michaelmusty.github.io/dessins/passports/9T34-6.1.1.1_5.2.2_6.1.1.1/index.html`
- All drawings together

Motivating future questions

The ultimate goal is to learn something about the absolute Galois group through its action on dessins. To that end:

- Can we tell when 2 dessins correspond to Galois conjugate Belyi maps?
- Is there topological or combinatorial data that can tell us when a passport splits into multiple Galois orbits?
- Can we determine the Galois orbits given a set of representatives for a passport?
- For every passport in the LMFDB with multiple Galois orbits, can we give an explicit reason for that splitting?

Completeness of the data

https://www.lmfdb.org/Belyi/stats

$d \backslash g$	0	1	2	3	≥ 4	total
1	1/1	0	0	0	0	1/1
2	1/1	0	0	0	0	1/1
3	2/2	1/1	0	0	0	3/3
4	6/6	2/2	0	0	0	8/8
5	12/12	6/6	2/2	0	0	20/20
6	38/38	29/29	7/7	0	0	74/74
7	89/89	50/50	7/13	2/3	0	148/155
8	243/261	83/217	0/84	0/11	0	326/573
9	410/583	33/427	0/163	0/28	0/6	443/1207
total	802/993	204/732	16/269	2/42	0/6	1024/2042

Completeness of the data

https://www.lmfdb.org/Belyi/stats

$d \backslash g$	0	1	2	3	≥ 4	total
1	1/1	0	0	0	0	1/1
2	1/1	0	0	0	0	1/1
3	2/2	1/1	0	0	0	3/3
4	6/6	2/2	0	0	0	8/8
5	12/12	6/6	2/2	0	0	20/20
6	38/38	29/29	7/7	0	0	74/74
7	89/89	50/50	7/13	2/3	0	148/155
8	243/261	83/217	0/84	0/11	0	326/573
9	410/583	33/427	0/163	0/28	0/6	443/1207
total	802/993	204/732	16/269	2/42	0/6	1024/2042

Thanks for listening!