Enumerating Truchet tiles on the faces of polyhedra

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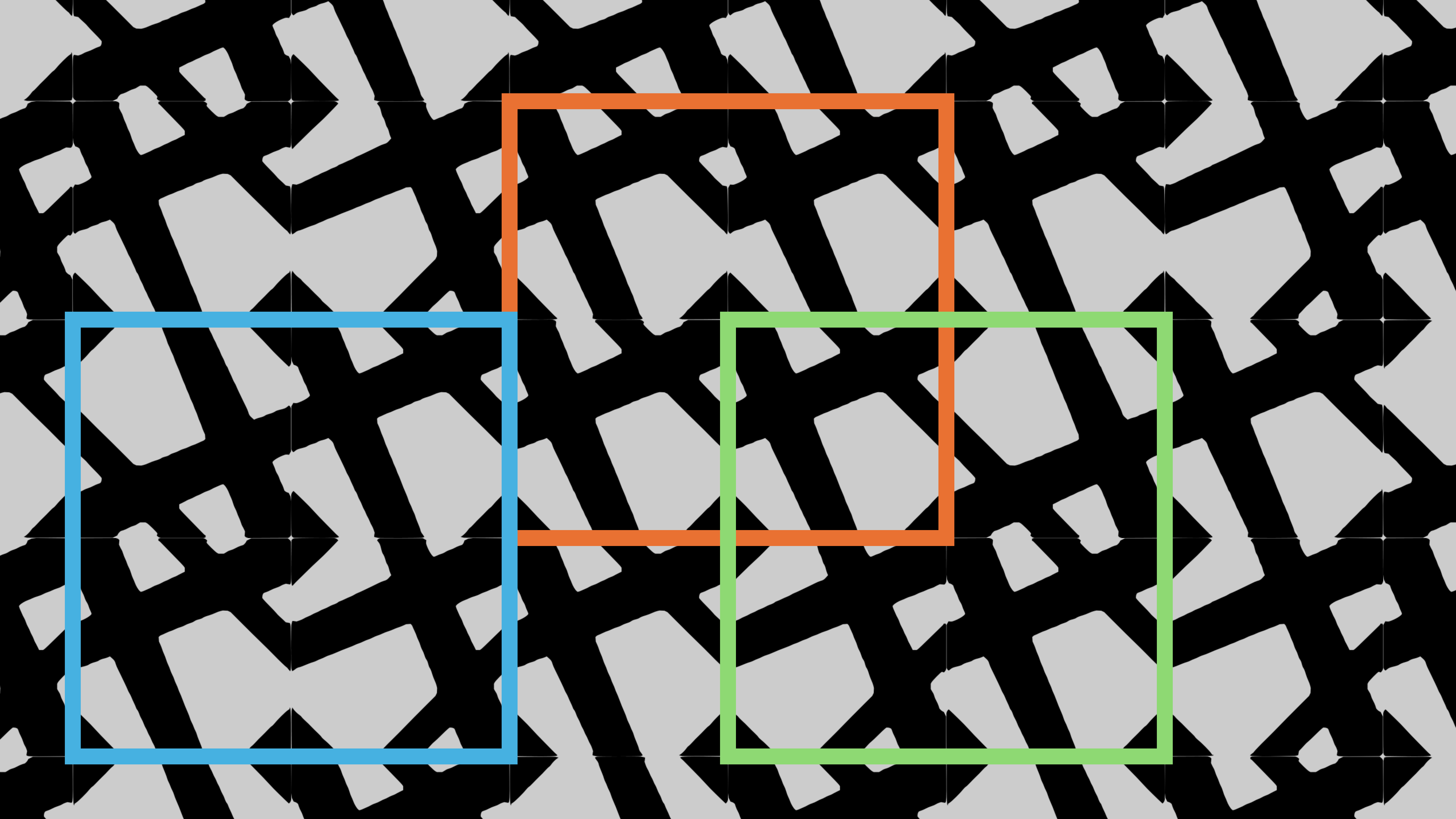
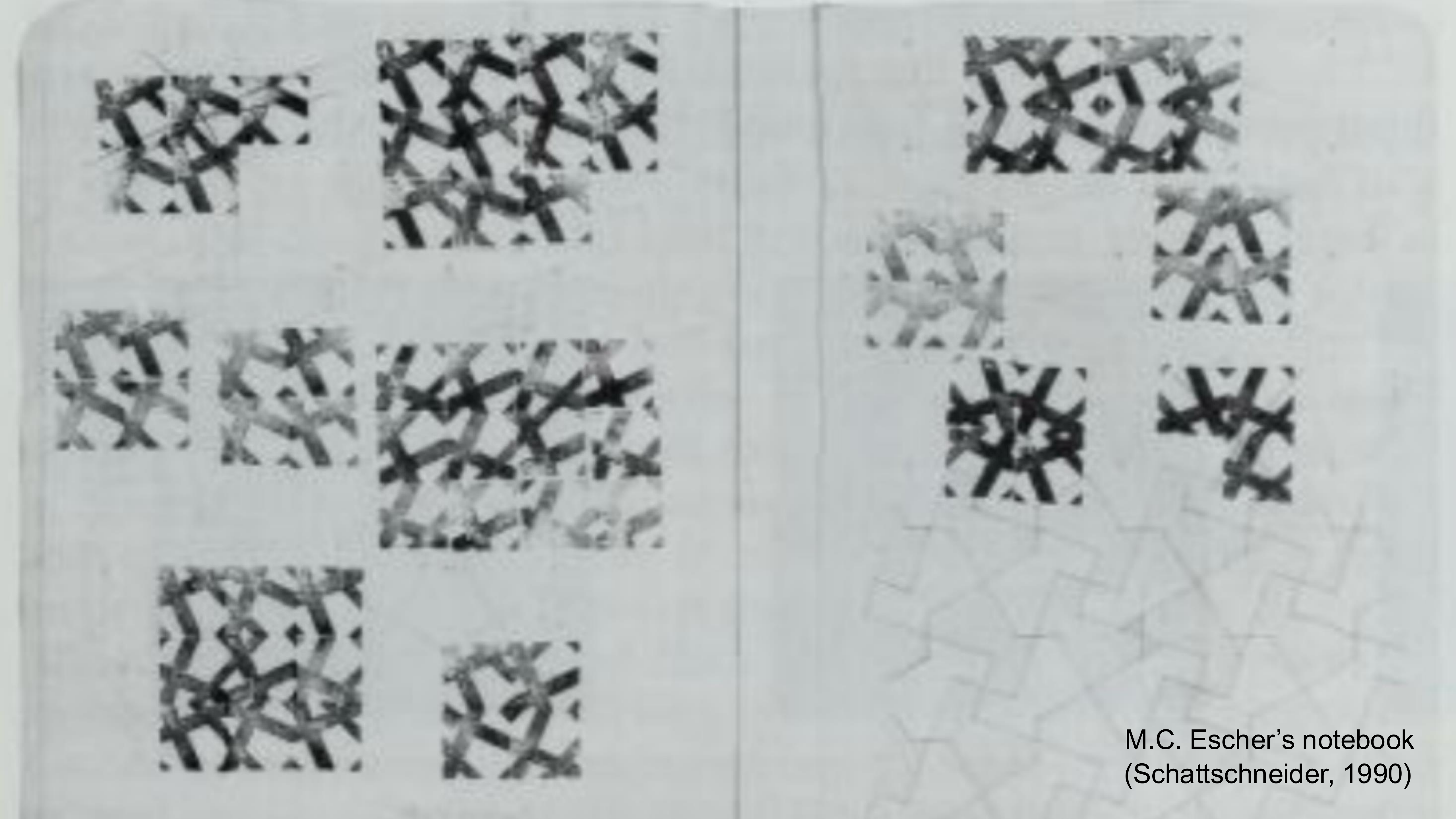
@ ICERM Illustrating Mathematics: Reunion/Expansion
2025-08-11



Escher’s “goal was […] to classify and enumerate all possible patterns created by a square tile.”

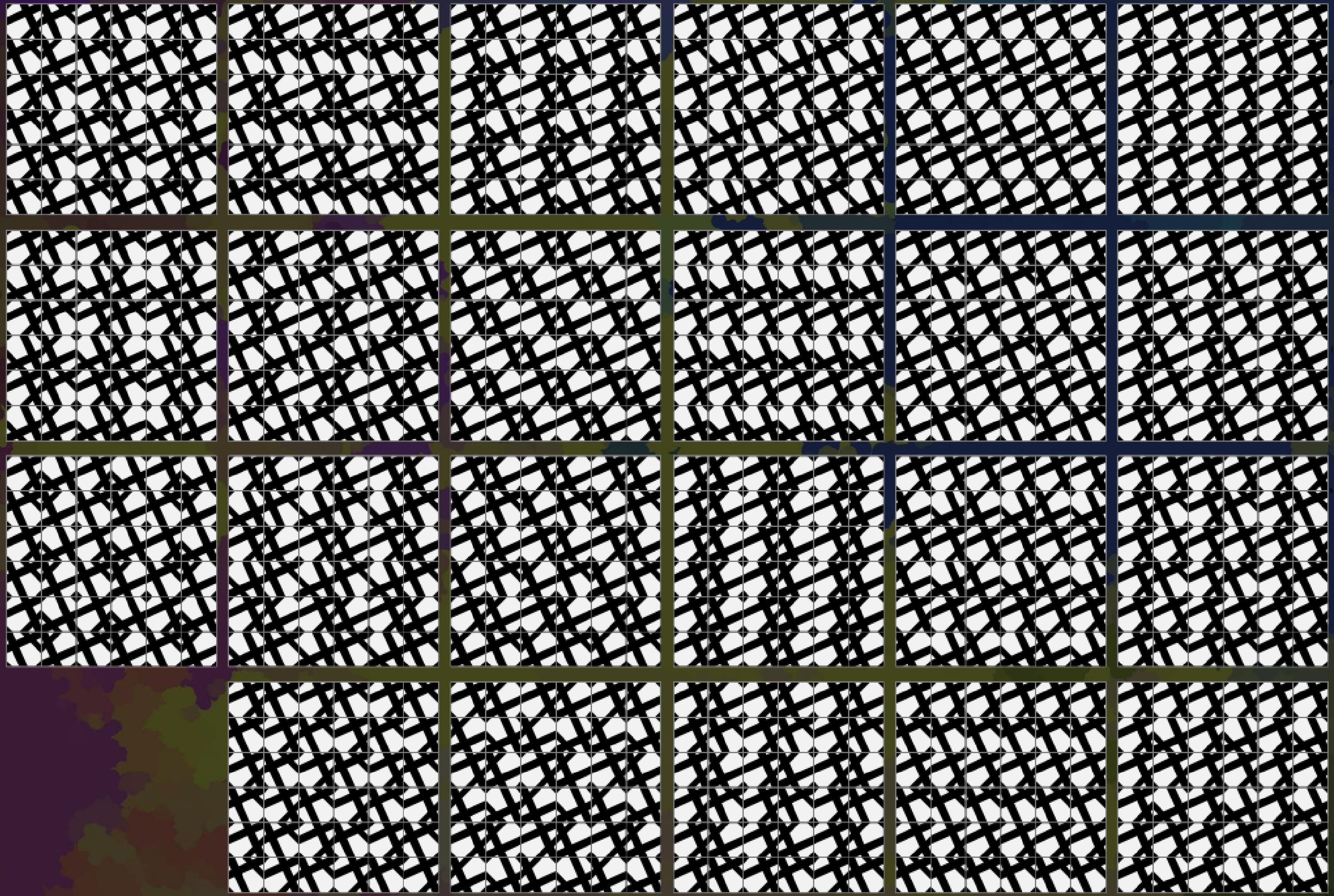
(Schattschneider, 1990)

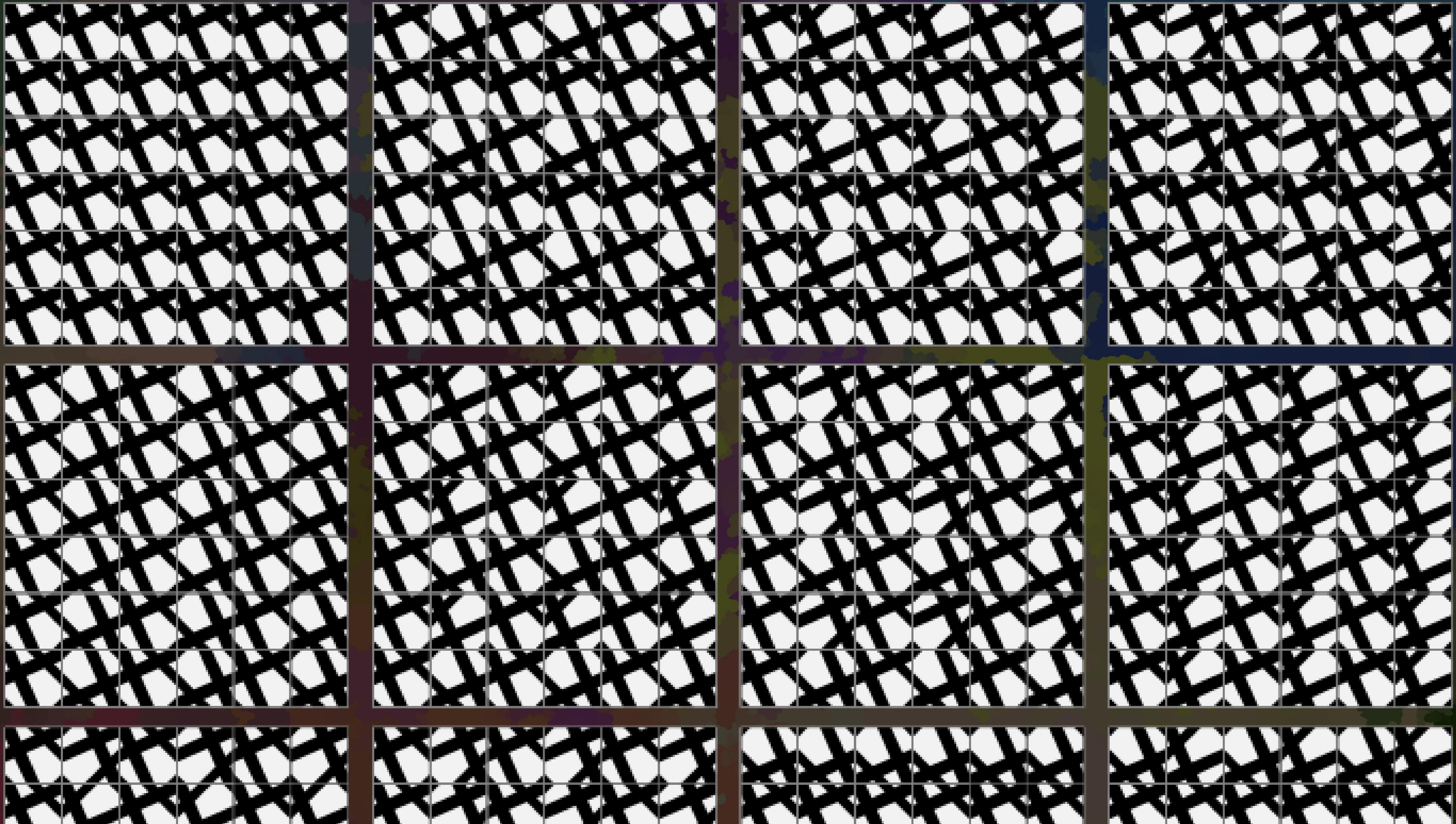


M.C. Escher’s notebook
(Schattschneider, 1990)

M.C. Escher (1942)

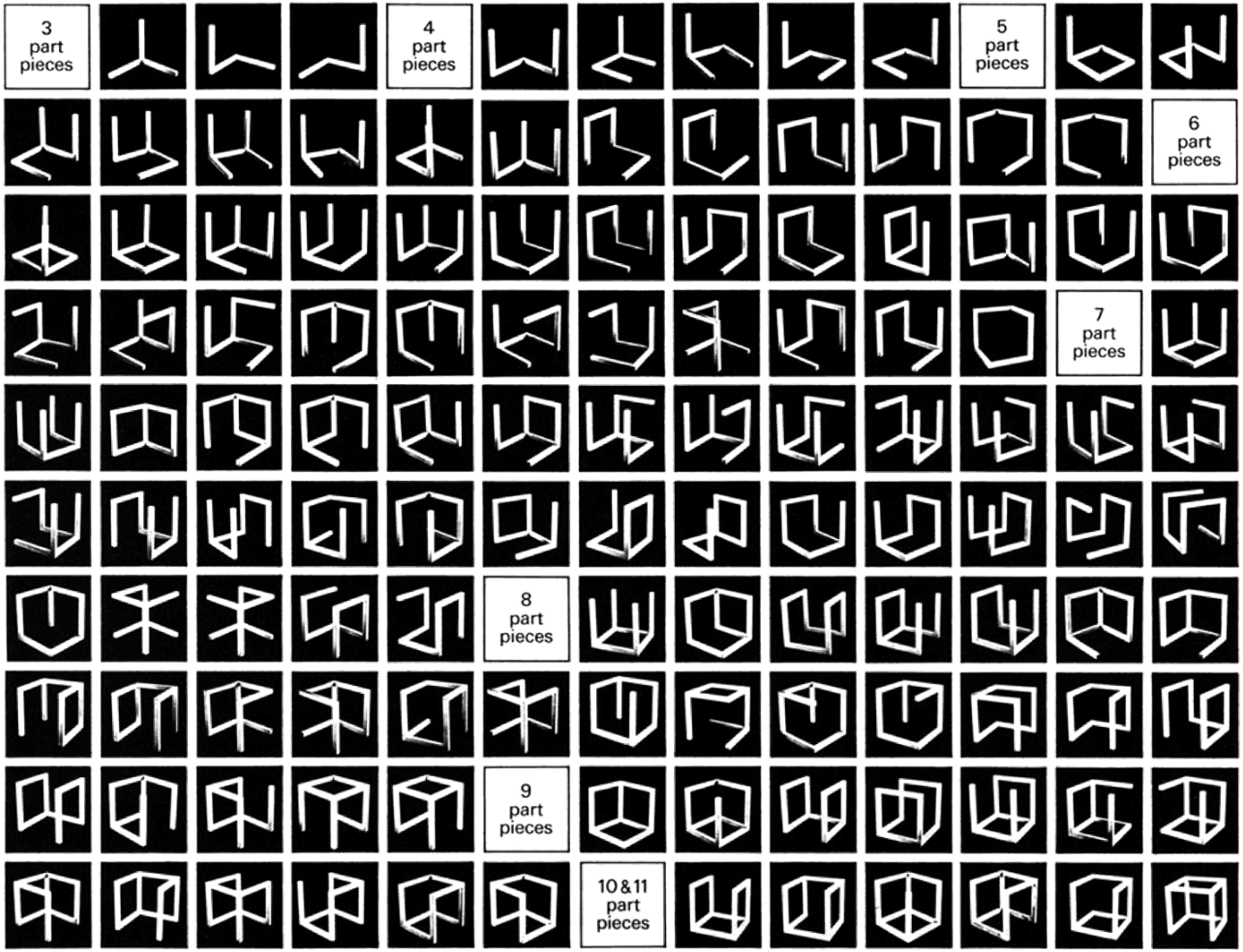
In May 1942 Escher created printed designs [which] illustrate each of his 23 distinct possible types of patterns requiring just one printing block.





Incomplete Open Cubes

Sol Lewitt



Problem 28.

Consider tilings of the $n \times n$ grid up to D_8 action where the tiles are diagonals.

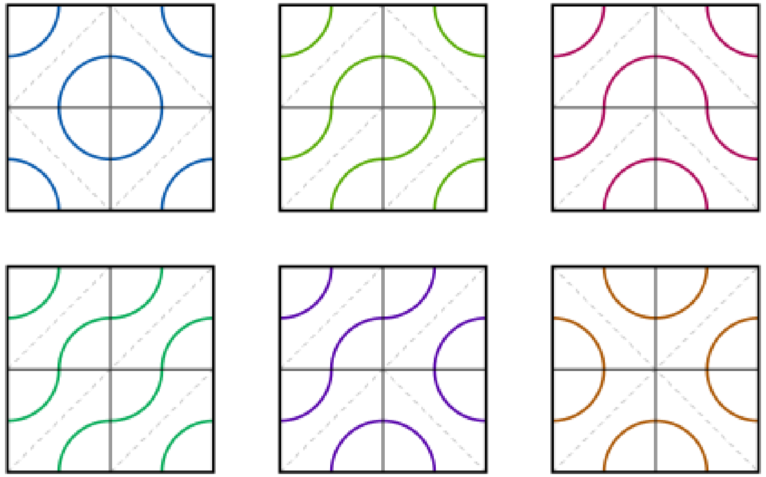


Figure 1: An example of the $a(2) = 6$ different ways to fill the 2×2 grid with diagonal tiles (up to dihedral action).

Question. How many such tilings exist?

Related.

1. What if grids are only counted up to C_4 (rotation) action?
2. What if this is counted on the torus/cylinder/Möbius strip?
3. What if each tile can have no diagonals or both diagonals?
4. What if tiles are black or white?
5. Is there an obvious bijection between the results on the $2n \times 2n$ grid for black/white versus diagonal tile types?

References.

https://oeis.org/A295229

Problem 31.

Consider square, triangular, and hexagonal grids that are filled in with with tiles of different patterns.

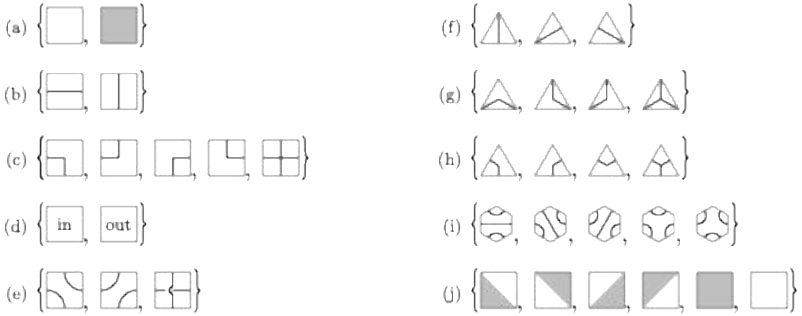


Figure 1: Ten examples of different tiles.

Question. How many essentially different grids of size n exist with these tiles? (Up to dihedral action? Up to cyclic action?)

Related.

1. The square grid can be $n \times n$ or $n \times m$.
2. The hexagonal grid can have triangles with side length n or hexagons with side length n.
3. The triangular grid can have triangles with side length n or hexagons with side length n.
4. The square grid can be quotiented to be a cylinder, torus, or Möbius strip.
5. What if shapes have to "match-up" (e.g. the lines in the third example or colors in the last example have to be "smooth".)
6. How many distinct regions, as in Problem 2?

References.

Problem 2.

Problem 28.

https://en.wikipedia.org/wiki/Burnside%27s_lemma

https://en.wikipedia.org/wiki/Palago

Journal of Integer Sequences
Vol. 27 (2024), Article 24.6.1

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Counting Tilings of the $n \times m$ Grid, Cylinder, and Torus

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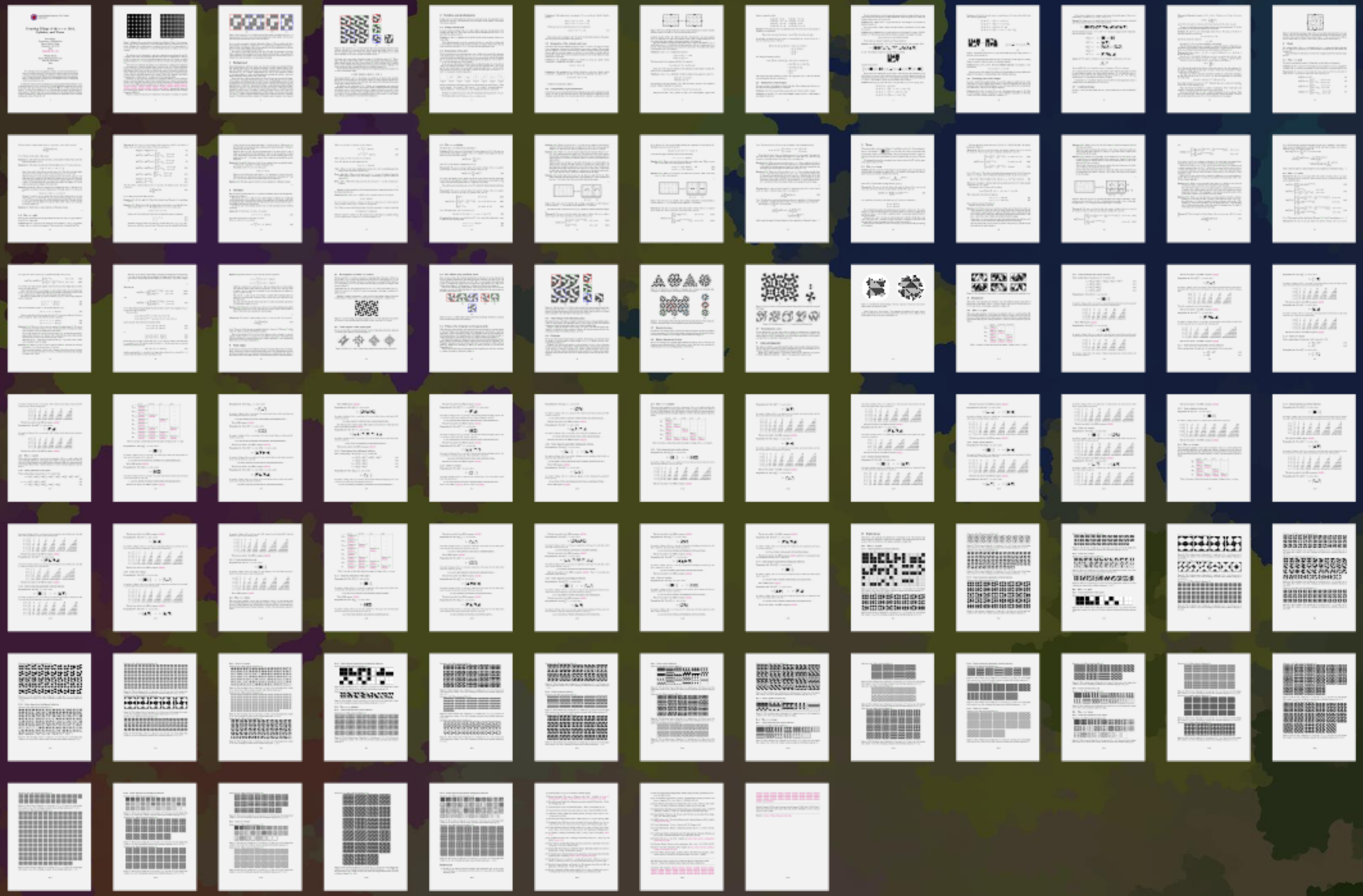
Abstract

We count tilings of the rectangular grid, cylinder, and torus with arbitrary tile designs up to arbitrary symmetries of the square and rectangle, along with cyclic shifting of rows and columns, generalizing and classifying a a tiling problem first enumerated by M. C. Escher in May 1942. This provides a unifying framework for understanding a family of counting problems, expanding on the work by Ethier and Lee counting tilings of the torus by tiles of two colors.

In 1704 the Dominican priest, mathematician, and typographer Sébastien Truchet, wrote a manuscript *Mémoire sur les combinaisons* [29], which illustrates designs that can be made from many copies and rotations of the "Truchet tile", one of which is reproduced in Figure 1. In 1722, Douat published a book containing futher analysis and illustrations of these tilings [9]. Truchet's and Douat's work resurfaced in Cyril Stanley Smith and Pauline Boucher's translation [27], which also introduced another tile design which is also, somewhat ambiguously, called a Truchet tile:.



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"I envisioned something along the lines of a typical prison pen pal service ... what I found was completely different. I joined a community and I discovered I was welcome," writes math lover Bill Keehn. "It is truly humbling for someone who thought he was alone."

When I first entered the system, as part of my group therapy programming, I was advised to find meaning and purpose in my life. For someone looking at a long sentence, what meaning and purpose can one find? The system is designed to remove you from society and make you irrelevant. For the rest of the world we have no meaning, no purpose. Realistically, what is one to do?

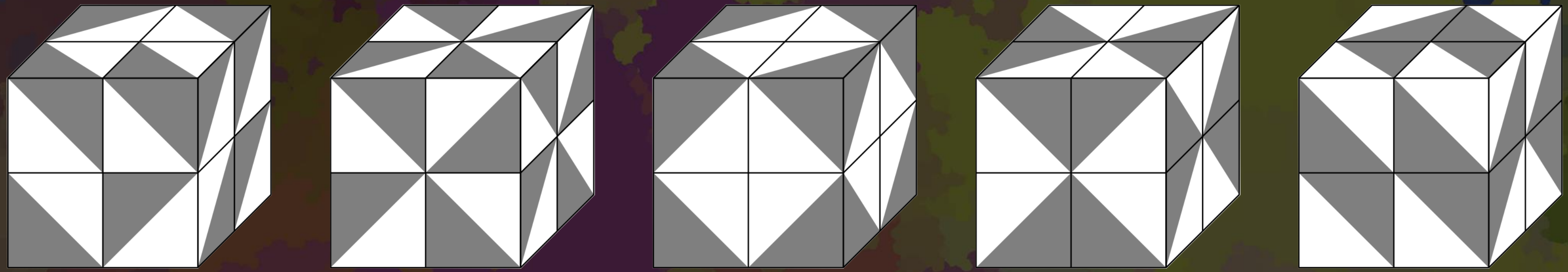
ALT

This month marks the end of my first year with my mentor, Peter. I couldn't have asked for a better mentor than Peter. Not only have I learned a lot from Peter, but he's helped me reach a goal I never thought I would reach. We are currently finishing a paper we've co authored and it should be posted to Arxiv soon. Thank you Peter.

ALT

7:58 AM · Nov 5, 2023 · **1,956** Views

“We are also interested in settings related to polyhedra. For instance, one could use various tile designs to count the number of distinct tilings of a 2×2×2 Rubik’s cube-like object.”



Five illustrations of $2 \times 2 \times 2$ cubes tiled with Truchet tiles.

heavy metal squiggle orb

By Matt Zucker

shadertoy.com/view/wsGfD3

Truchet Tile Ball

by Jon-Paul Wheatley

Tantrix Rock

Truchet Cubes at MoMath

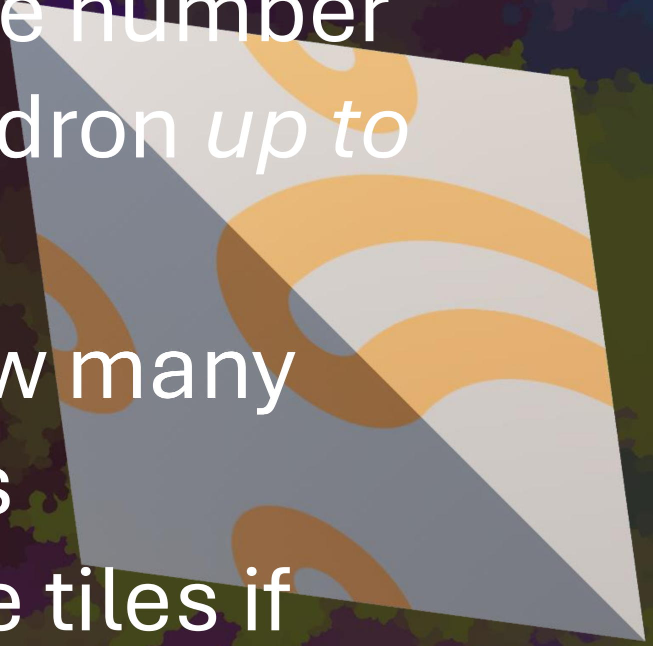
David A. Reimann

Counting tilings of polyhedra

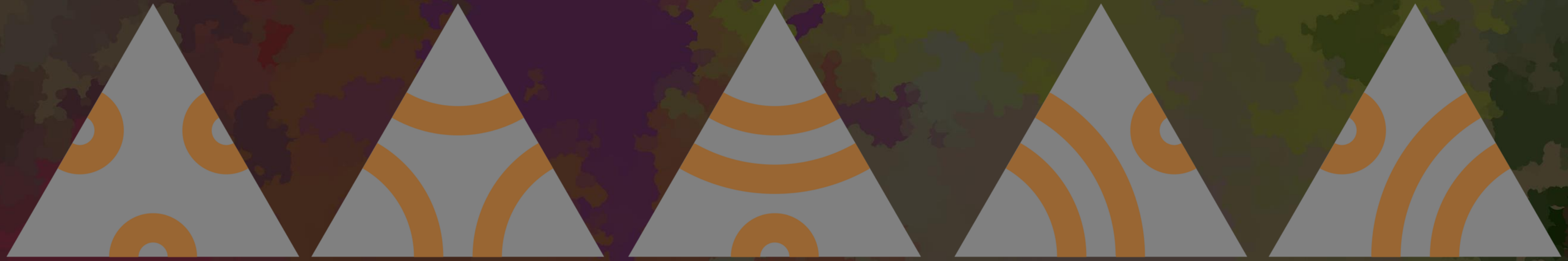
By Matt Zucker

We want to count the number of tilings of a polyhedron *up to rotation*.

Easier question: How many ways can we tile this tetrahedron by these tiles if it's fixed in place?



$$5^4 = 625$$



We want to count the number of tilings of a polyhedron *up to rotation*.

Harder question: How many ways can we tile this tetrahedron if rotations are considered equivalent?

$$> \frac{5^4}{12} > 52.08$$

Burnside's Lemma

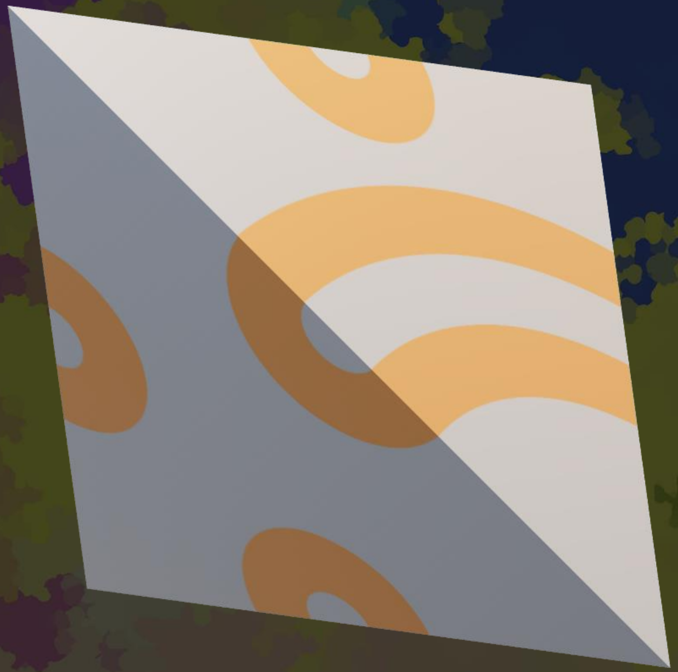
$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Number of tilings X up to the symmetries of the object G

Average over all symmetries

Number of tilings that look identical when g is applied.

Example: Zucker’s tetrahedra



Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

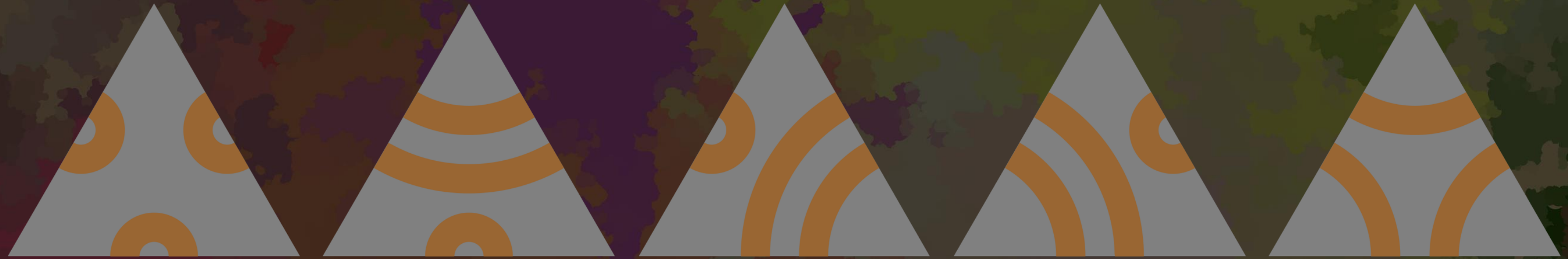
$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let G be the group of **rotational** symmetries of the tetrahedron. What is $|G|$?

$$|G| = 12$$

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

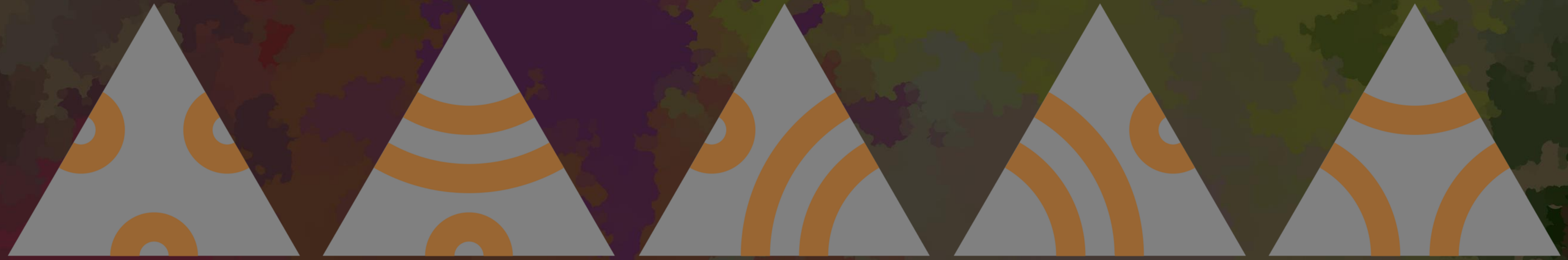


Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let G be the group of **rotational** symmetries of the tetrahedron. What is G?

- Rotations of $\pm 120°$ about a face. (8)
- Rotations of $180°$ about an edge (3)
- Identity. (1)

$$|X/G| = \frac{1}{|G|}\sum_{g \in G} |X^g|$$



Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates a face by $120°$. What is $|X^g|$?

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates a face by $120°$.
What is $|X^g|$?

What choices do we have for the rotating face?

The **two** rotationally symmetric designs.

$$|X/G| = \frac{1}{|G|}\sum_{g \in G} |X^g|$$

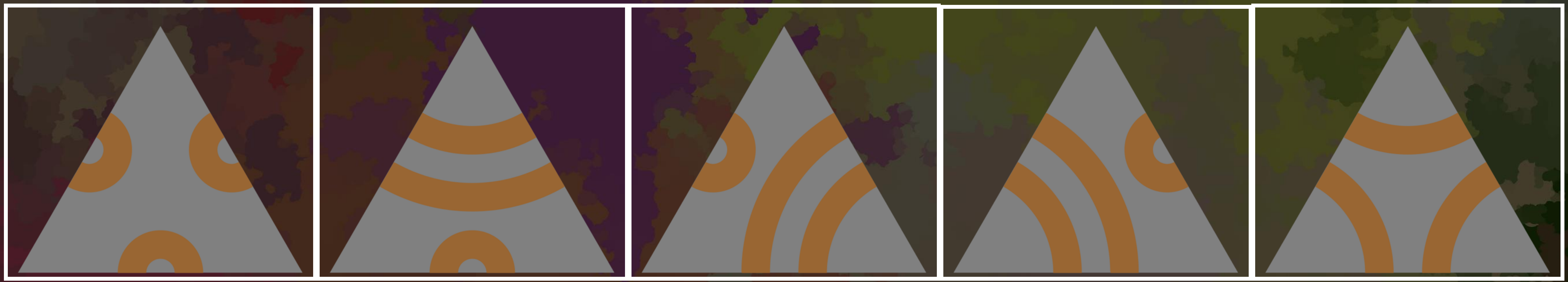
Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates an edge by $120°$. What is $|X^g|$?

What choices do we have for the other faces?

Any of the **five** designs work, but all three remaining faces must be the same.

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$



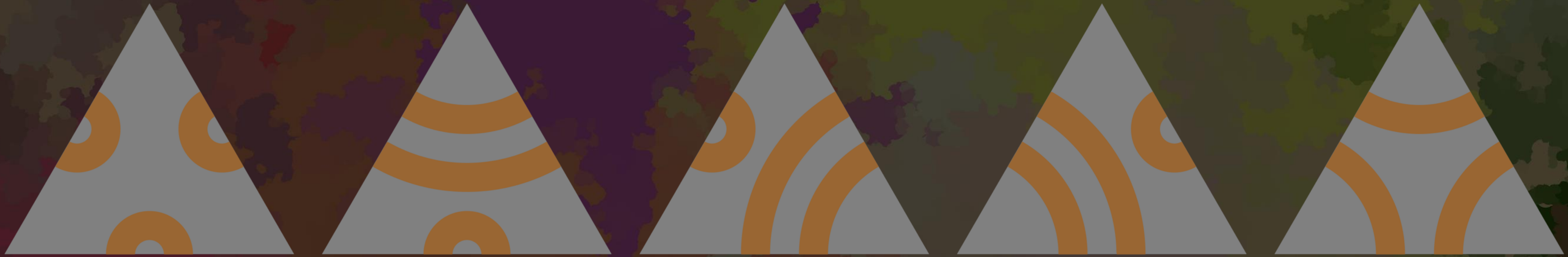
Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates an edge by $120°$. What is $|X^g|$?

$$|X^g| = 2 \times 5$$

$$= 10$$

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$



Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates an edge by 180°. What is $|X^g|$?

$$|X/G| = \frac{1}{|G|}\sum_{g \in G} |X^g|$$

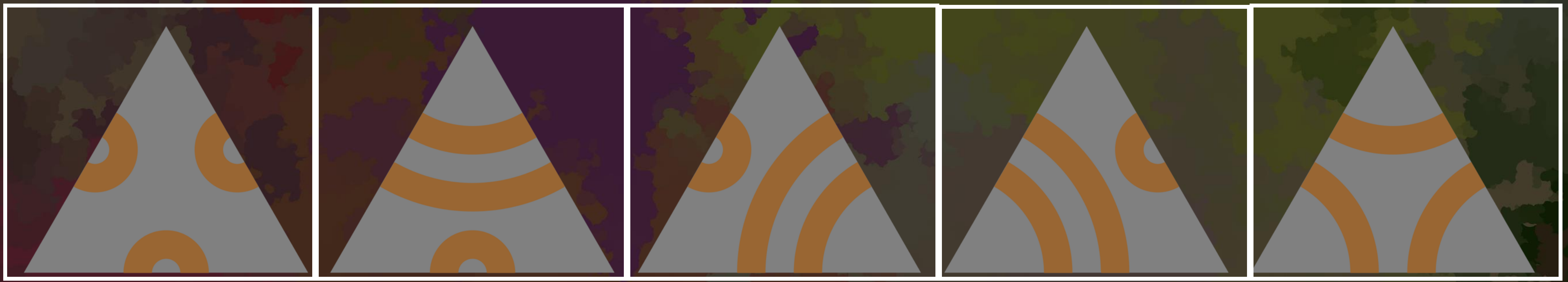
Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates an edge by $180°$. What is $|X^g|$?

What choices do we have for the front/back faces? The sides?

All **five** designs work, but the front/back must match and the sides must match.

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$



Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be a symmetry that rotates an edge by 180°. What is $|X^g|$?

$$|X^g| = 5 \times 5$$
$$= 25$$

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be the identity (do nothing) symmetry. What is $|X^g|$?

$$|X/G| = \frac{1}{|G|}\sum_{g \in G} |X^g|$$

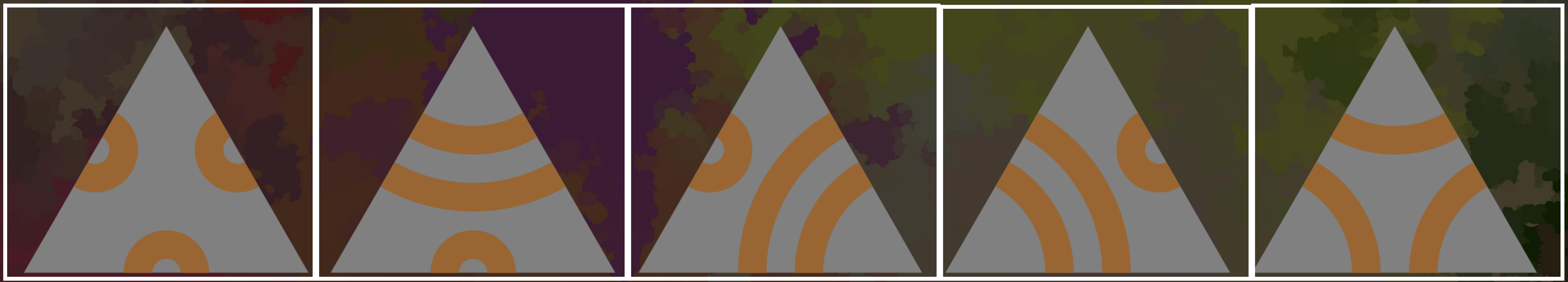
Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

Let g be the identity (do nothing) symmetry. What is $|X^g|$?

We can choose any of the **five** designs for any of the four faces.

$$|X^g| = 5^4 = 625$$

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$



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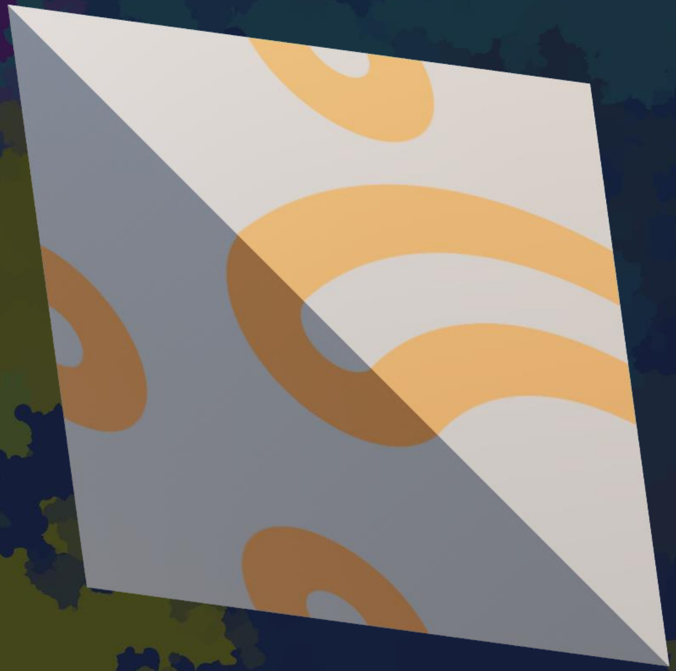


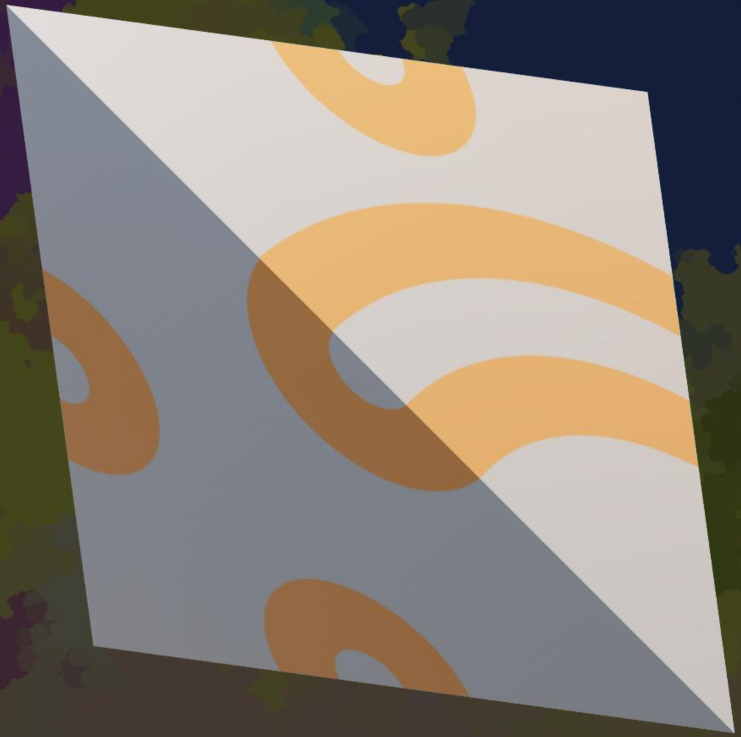
$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Let X be the tilings of a (fixed) tetrahedron with these five tile designs.

$$\left|\frac{X}{G}\right| = \frac{1}{12}(8 \times 10 + 3 \times 25 + 625)$$

$$= 65$$

There are 65 possible Zuckerian tetrahedra (up to rotation).



How do we systematically count tilings for more complex shapes?

Truchet Tile Ball by Jon-Paul Wheatley

A matrix representation of the full icosahedral group I (order 120).

$$\underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\varphi & -\frac{1}{2\varphi} \\ -\frac{1}{2}\varphi & \frac{1}{2\varphi} & -\frac{1}{2} \\ \frac{1}{2\varphi} & -\frac{1}{2} & \frac{1}{2}\varphi \end{bmatrix}}_{\text{Order 3 (120° rotation)}} \quad \underbrace{\begin{bmatrix} \frac{1}{2\varphi} & -\frac{1}{2} & -\frac{1}{2}\varphi \\ \frac{1}{2} & -\frac{1}{2}\varphi & \frac{1}{2\varphi} \\ -\frac{1}{2}\varphi & \frac{1}{2\varphi} & \frac{1}{2} \end{bmatrix}}_{\text{Order 5 (72° rotation)}} \quad \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\text{Order 2 (Inversion)}}$$

Order 3
(120° rotation)

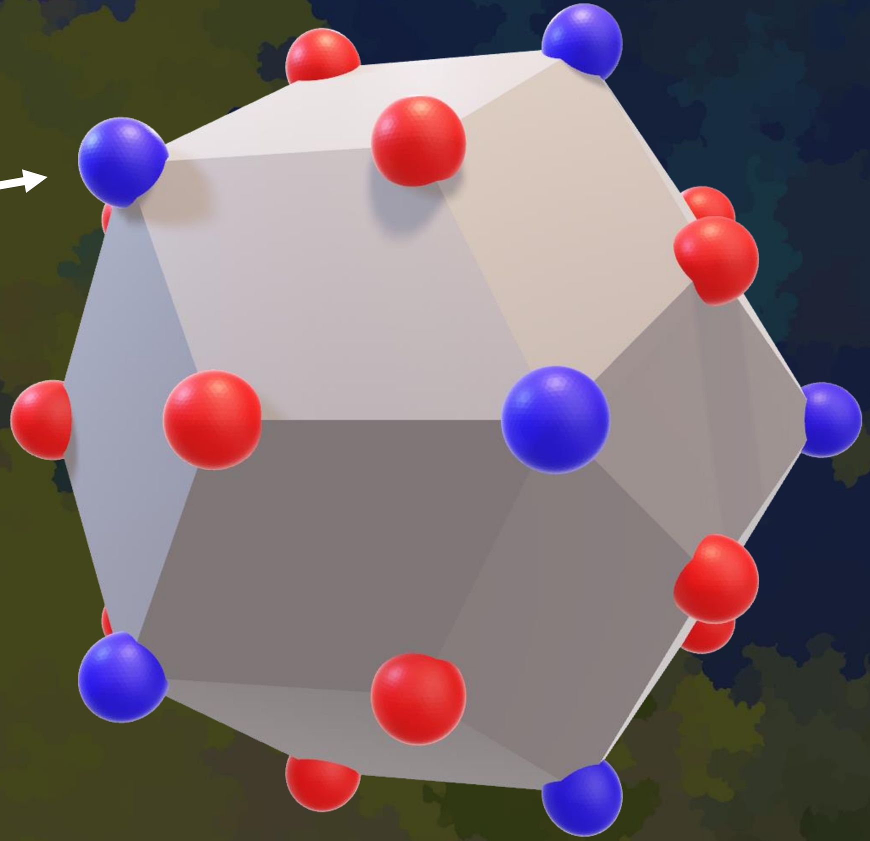
Order 5
(72° rotation)

Order 2
(Inversion)

Orient the rhombic triacontahedron so its symmetries are I.

Orbits of $(-1,-1,1)$

Orbits of $(0,-\varphi,1)$

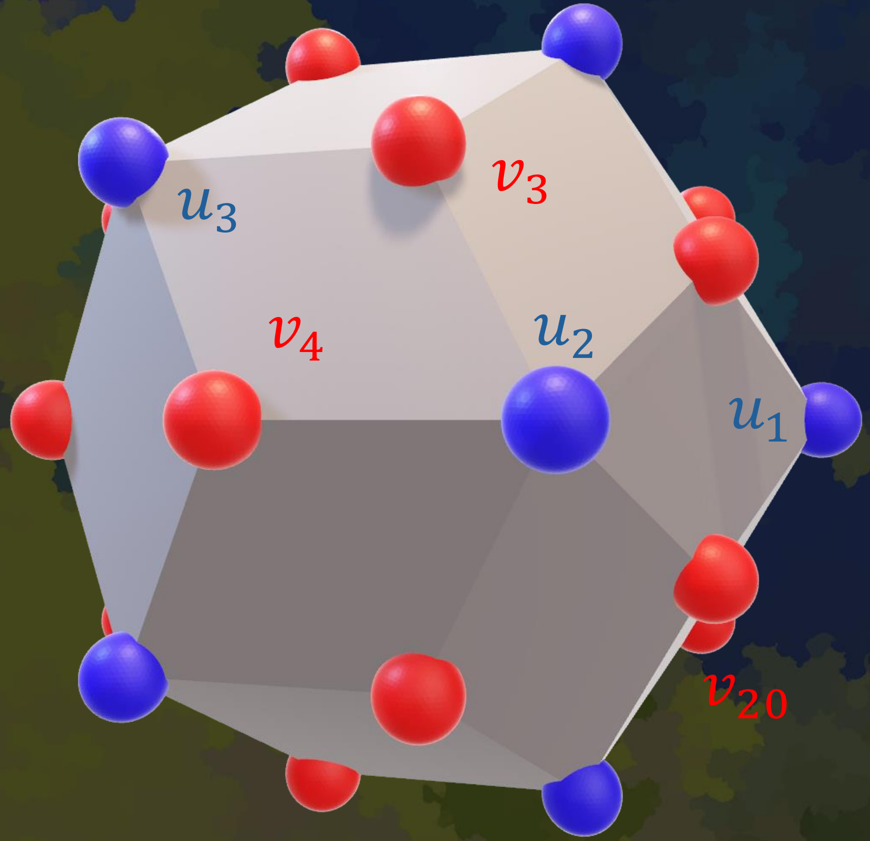


Choose representatives for each orbit of vertices.

Label each vertex.

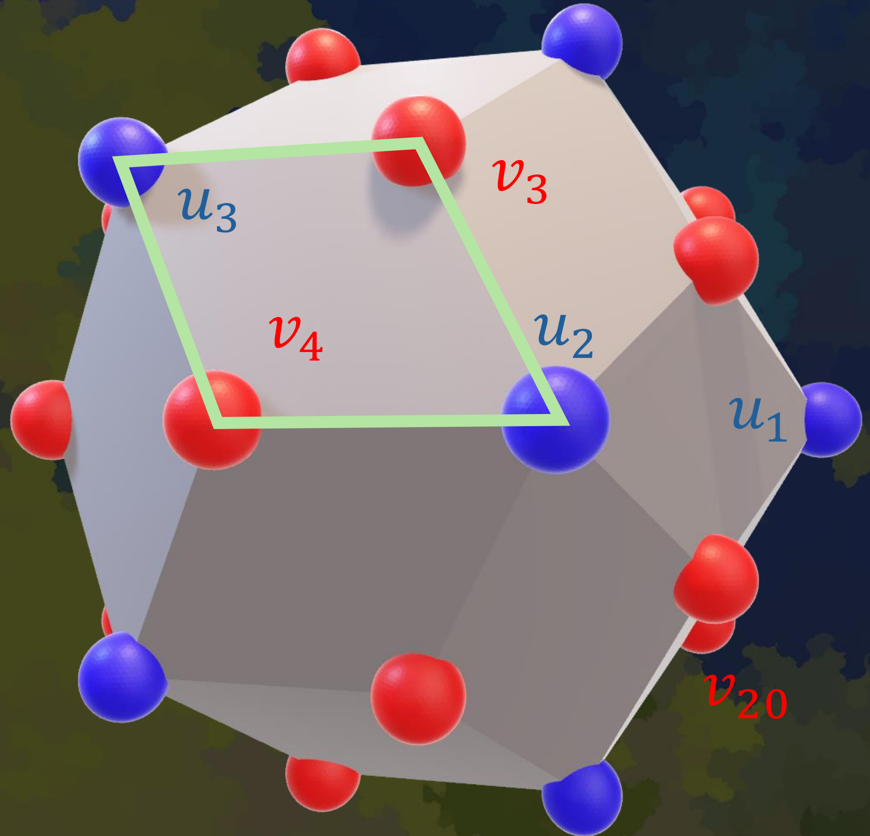
$u_1, u_2, \ldots, u_{12}$

$v_1, v_2, \ldots, v_{20}$



Assign each face an ordered tuple of vertices.

$$f_1 = (u_3, v_4, u_2, v_3)$$



Choose some $g \in G$. (e.g. the order-6 symmetry which is a 30° rotation followed by a reflection) Look at the image of f_1 under repeated actions of g.

$$g = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$f_1 = (u_3, v_4, u_2, v_3)$

$g \cdot f_1 = (u_{10}, v_{11}, u_9, v_{13})$

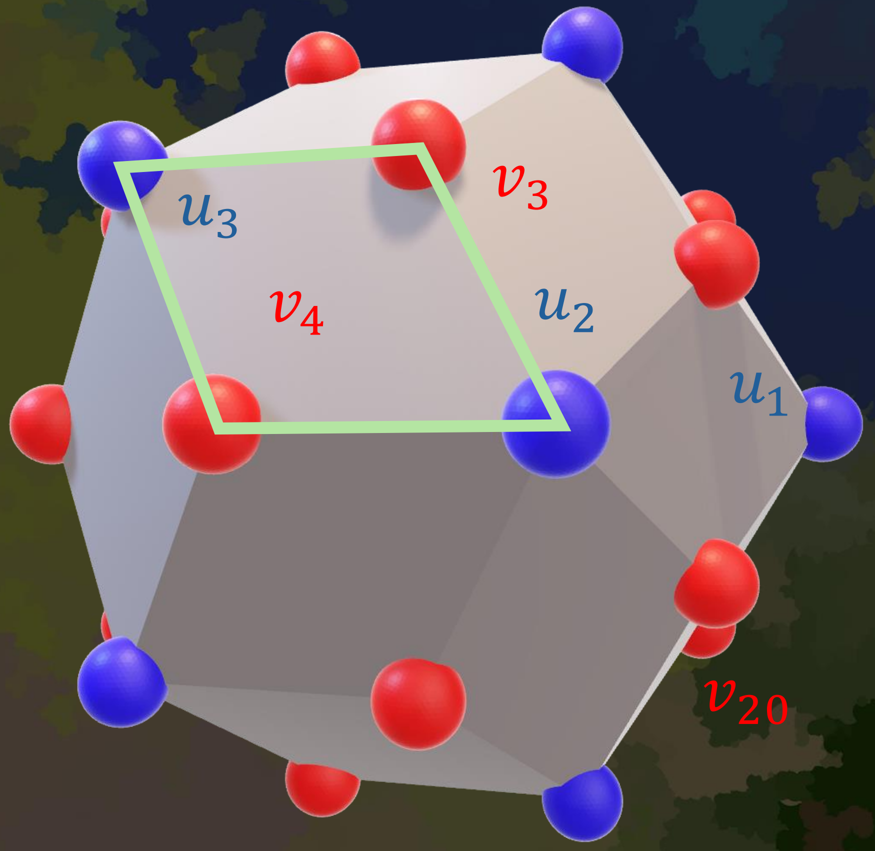
$g^2 \cdot f_1 = (u_2, v_3, u_4, v_1)$

$g^3 \cdot f_1 = (u_9, v_{11}, u_8, v_9)$

$g^4 \cdot f_1 = (u_4, v_3, u_3, v_5)$

$g^5 \cdot f_1 = (u_8, v_{11}, u_{10}, v_{18})$

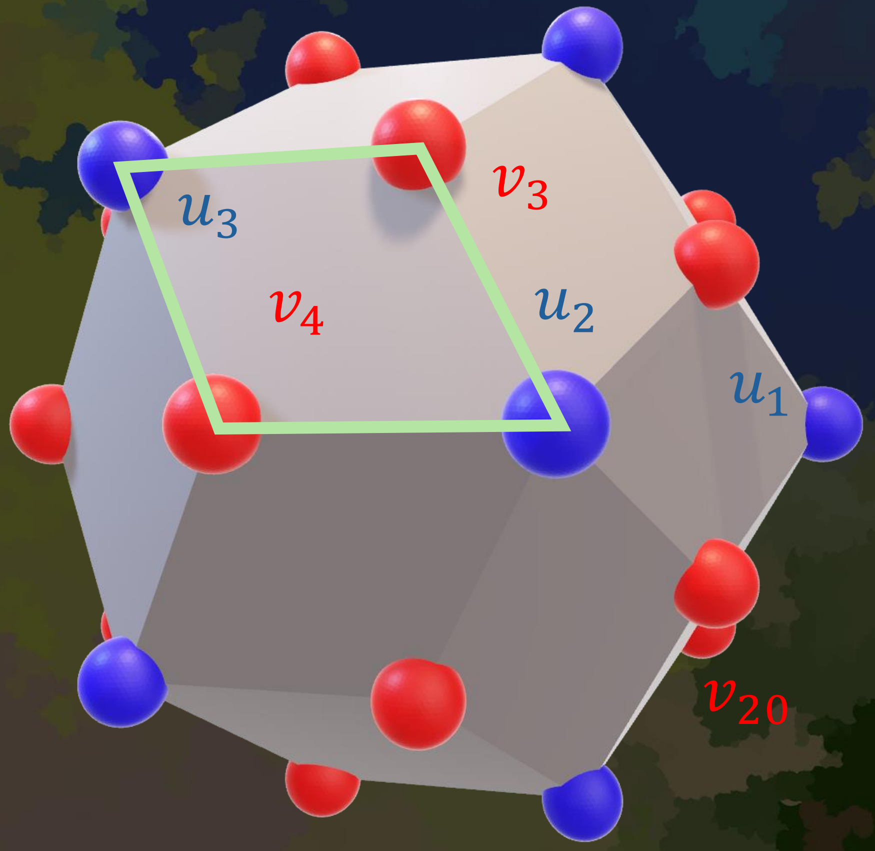
$g^6 \cdot f_1 = (u_3, v_4, u_2, v_3)$



Let $g \in G$, be the order-6 symmetry which is a $30°$ rotation followed by a reflection. Look at the image of f_1 under repeated actions of g.

$$g = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The orbit of f_1 under the cyclic group $\langle g \rangle$ has $|g| = 6$ elements, so we are free to choose any design for f_1.



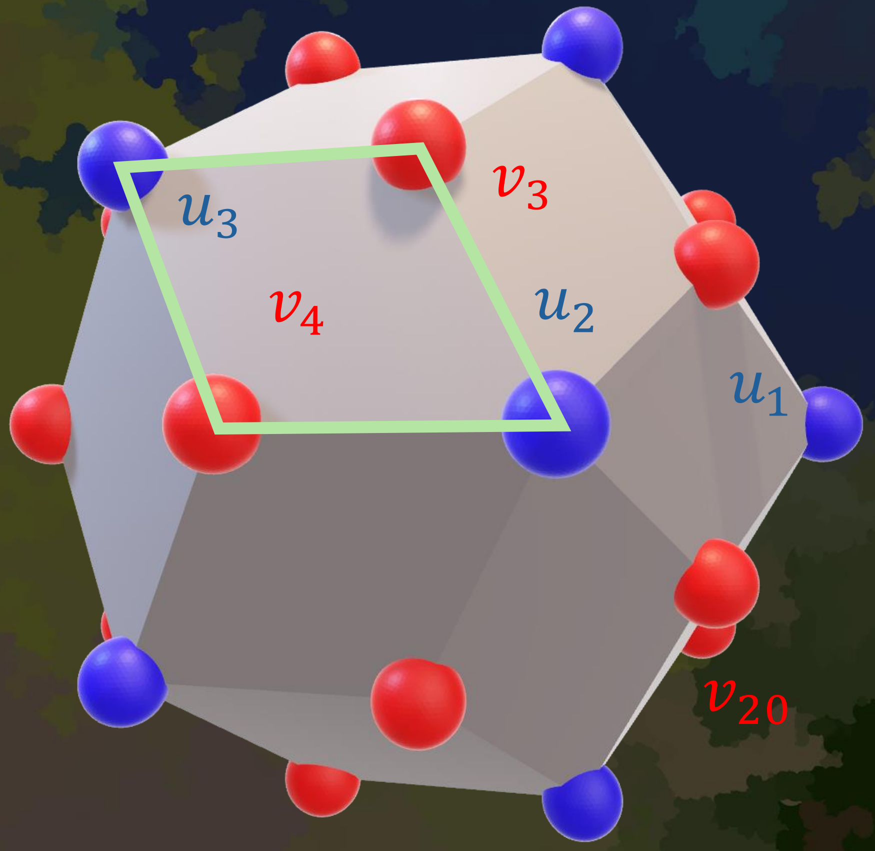
In this case, each of the 30 faces appears in an orbit of size 6, there are five orbits of size six that can each be given any tile design.

Let $g \in G$, be the order-6 symmetry which is a 30° rotation followed by a reflection. Look at the image of f_1 under repeated actions of g.

$$g = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

In this case, each of the 30 faces appears in an orbit of size 6, there are five orbits of size six that can each be given any tile design.

If there are t possible tile designs, then $|X^g| = t^5$.

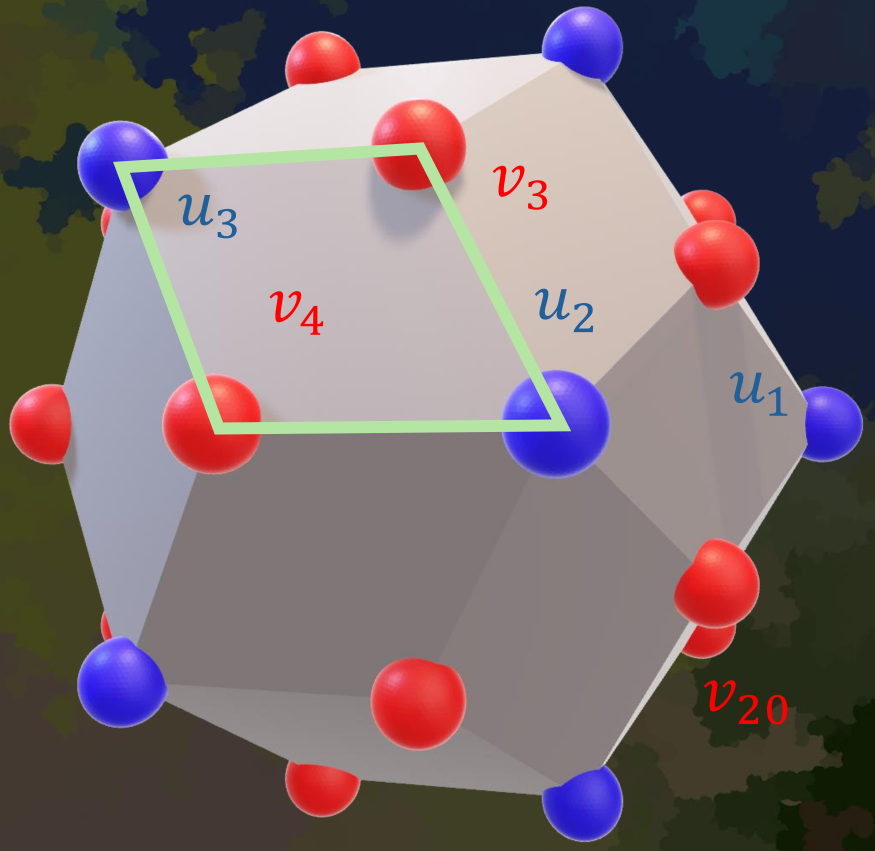


Let $g \in G$, be the order-2 reflectional symmetry. Look at the image of f_1 under repeated actions of g.

$$g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

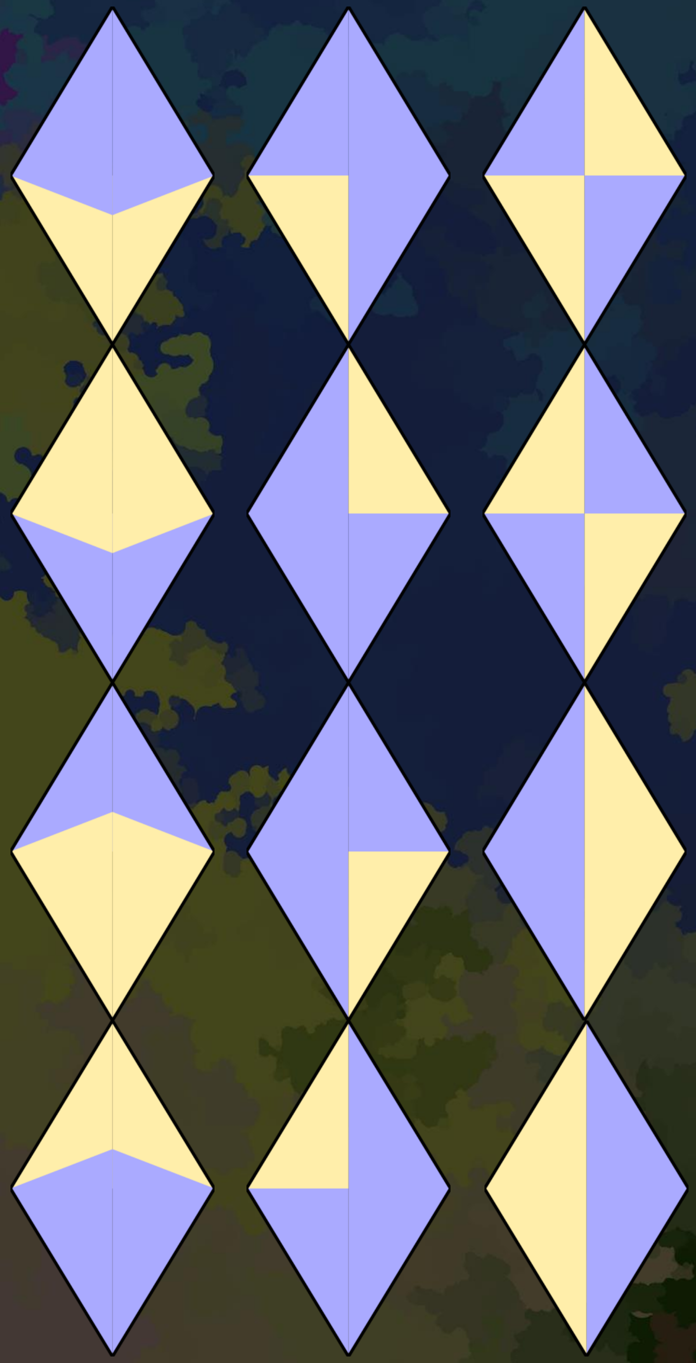
$$f_1 = (u_3, v_4, u_2, v_3)$$
$$g \cdot f_1 = (u_2, v_4, u_3, v_3)$$
$$g^2 \cdot f_1 = (u_3, v_4, u_2, v_3)$$

In this case, g maps a reflected version of f_1 onto itself, so if the tiling is fixed under g, f_1 must be tiled with a design that has reflectional symmetry.



Let's count the number of ways to tile the rhombic triacontahedron with the following twelve tile designs, which we classify by their symmetries in

$$D_2 = \{r, f \mid r^2 = f^2 = (rf)^2 = id\}$$



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$$D_2 = \{r, f \mid r^2 = f^2 = (rf)^2 = id\}$$

How many designs are fixed under id?

$t_{id} = 12$

How many designs are fixed under f? (left-to-right reflection)

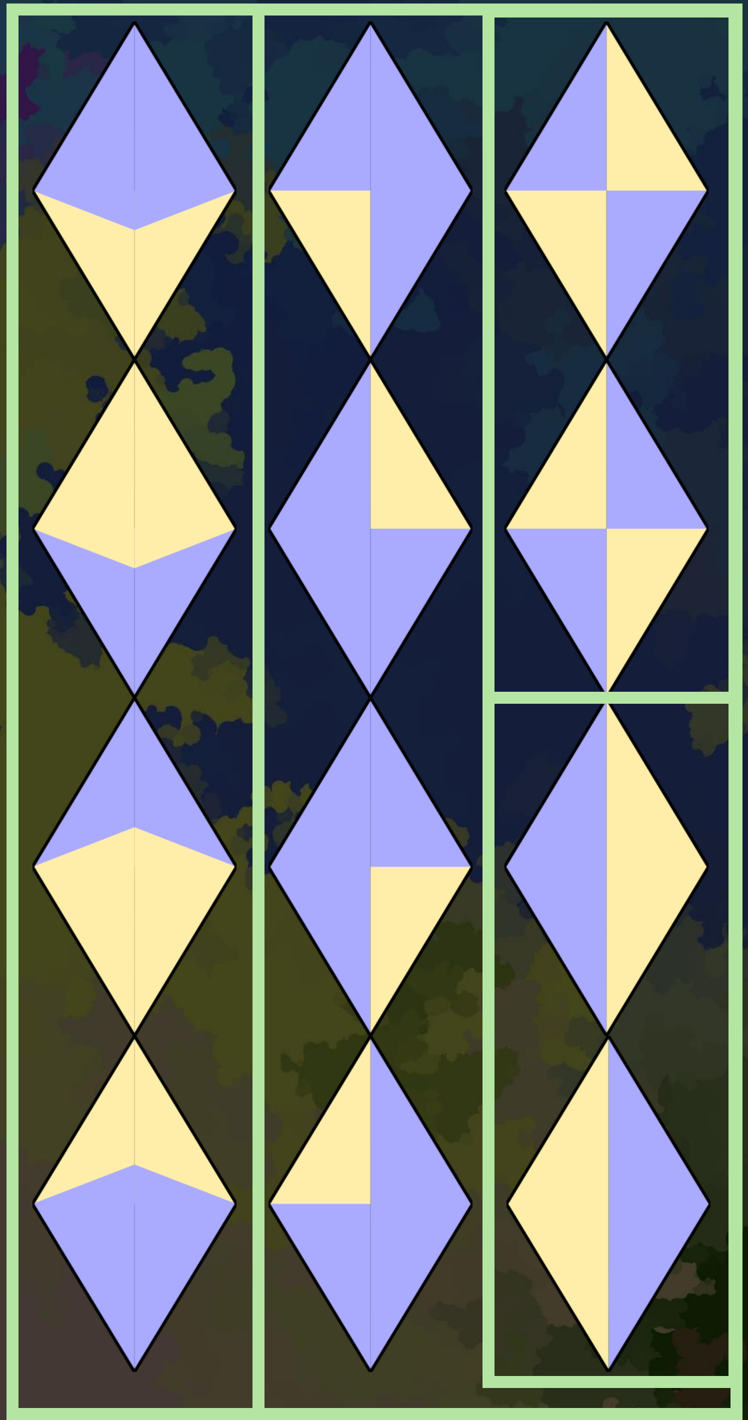
$t_f = 4$

How many designs are fixed under r? (180° rotation)

$t_r = 2$

How many designs are fixed under rf? (top-to-bottom reflection)

$t_{rf} = 2$

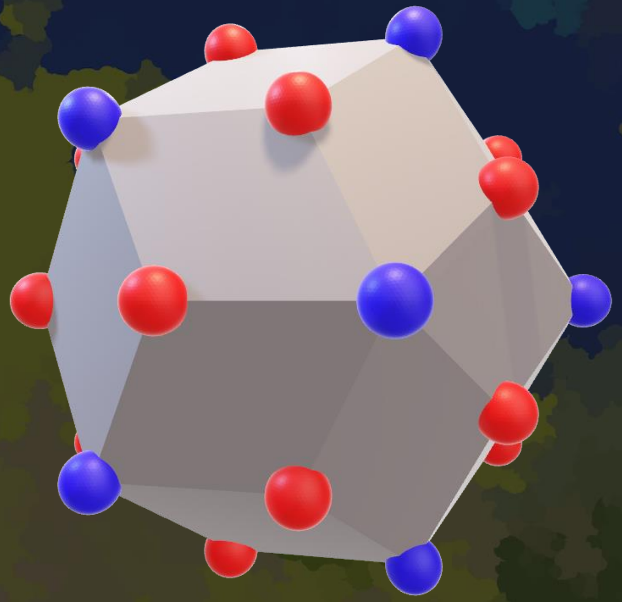


Icosahedral group conjugacy classes

Identity	1
Rotation by 72°	12
Rotation by 144°	12
Rotation by 120°	20
Rotation by 180°	15

Inversion	1
Rotation by 36° + reflection	12
Rotation by 108° + reflection	12
Rotation by 60° + reflection	20
Reflection	15

$$|X^g| = {t_{id}}^{30} = 12^{30}$$



$t_{id} = 12 \quad t_f = 4$

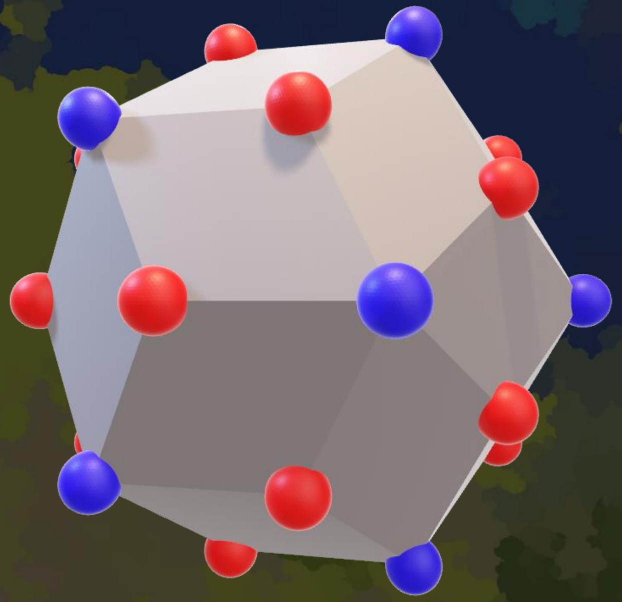
$t_r = 2 \quad t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1
Rotation by $72°$	12
Rotation by $144°$	12
Rotation by $120°$	20
Rotation by $180°$	15

Inversion	1
Rotation by $36°$ + reflection	12
Rotation by $108°$ + reflection	12
Rotation by $60°$ + reflection	20
Reflection	15

$$|X^g| = {t_{id}}^6 = 12^6$$



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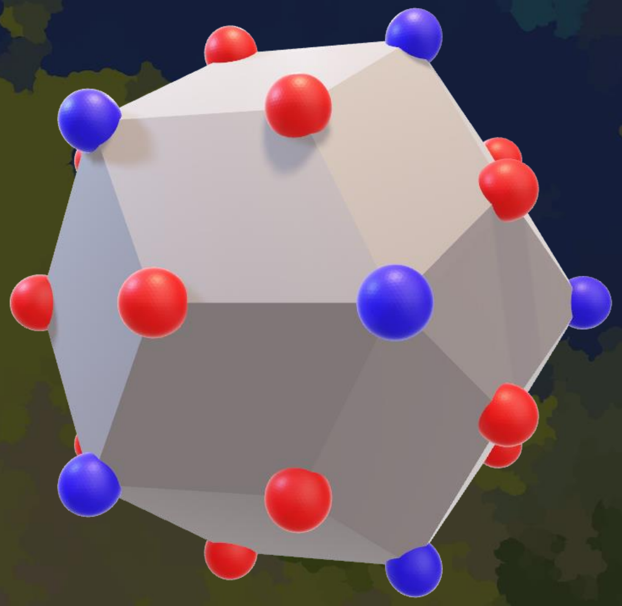
Reflection	15

$$|X^g| = {t_{id}}^6 = 12^6$$



$t_{id} = 12 \quad t_f = 4$

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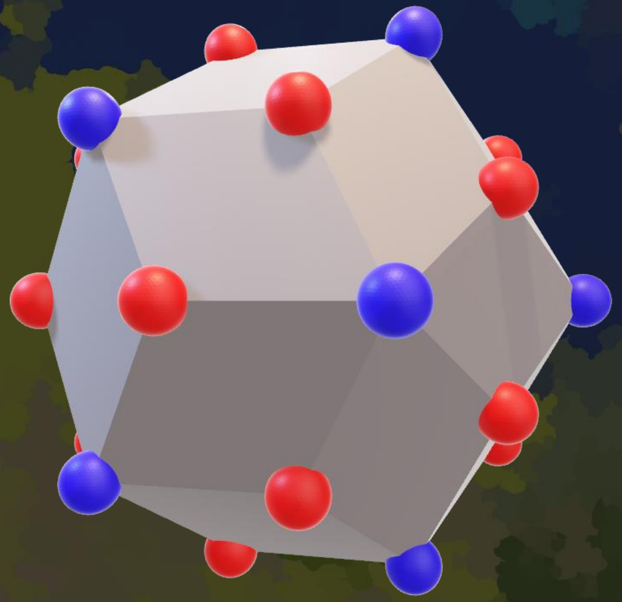
Icosahedral group conjugacy classes

Identity	1	Inversion	1
Rotation by 72°	12	Rotation by 36° + reflection	12
Rotation by 144°	12	Rotation by 108° + reflection	12
Rotation by 120°	20	Rotation by 60° + reflection	20
Rotation by 180°	15	Reflection	15



$$|X^g| = {t_{id}}^{10} = 12^{10}$$

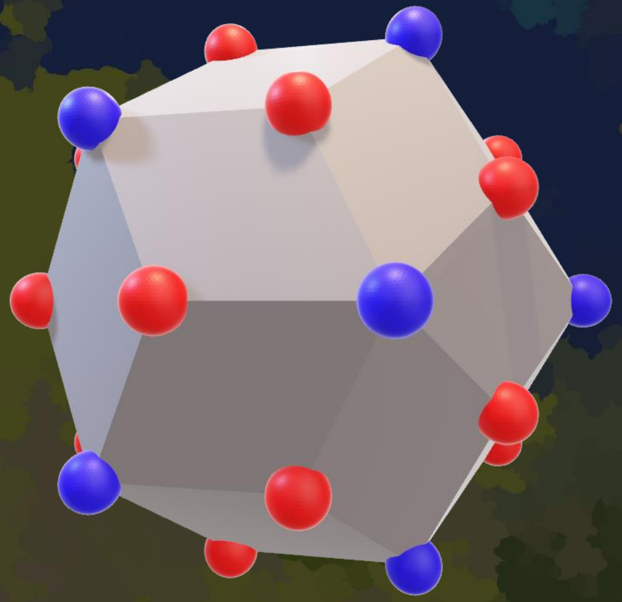
$t_{id} = 12 \quad t_f = 4$

$t_r = 2 \quad t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1
Rotation by 72°	12
Rotation by 144°	12
Rotation by 120°	20
Rotation by 180°	15

Inversion	1
Rotation by 36° + reflection	12
Rotation by 108° + reflection	12
Rotation by 60° + reflection	20
Reflection	15



$$|X^g| = {t_{id}}^{14}{t_r}^{2} = 12^{14}2^2$$

$t_{id} = 12 \quad t_f = 4$

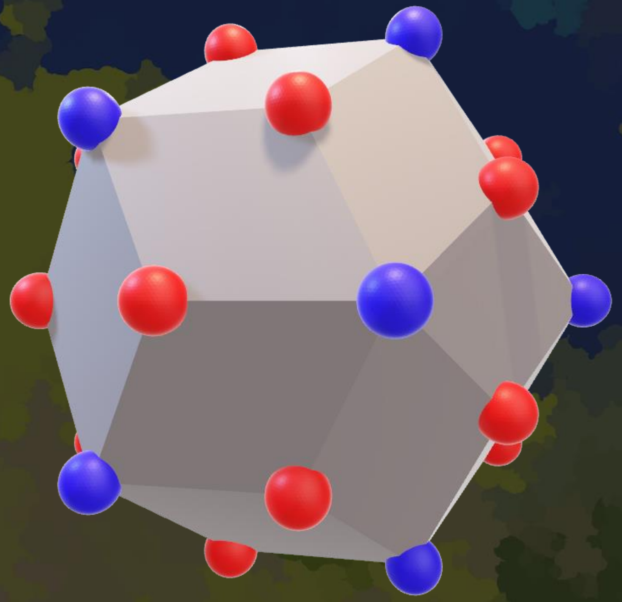
$t_r = 2 \quad t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1
Rotation by 72°	12
Rotation by 144°	12
Rotation by 120°	20
Rotation by 180°	15

Inversion	1
Rotation by 36° + reflection	12
Rotation by 108° + reflection	12
Rotation by 60° + reflection	20
Reflection	15

$$|X^g| = {t_{id}}^{15} = 12^{15}$$



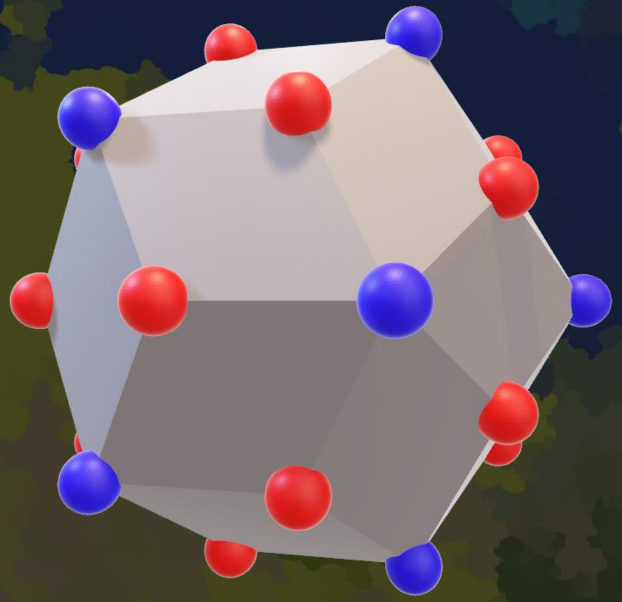
$t_{id} = 12$ $t_f = 4$

$t_r = 2$ $t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1	Inversion	1
Rotation by 72°	12	Rotation by 36° + reflection	12
Rotation by 144°	12	Rotation by 108° + reflection	12
Rotation by 120°	20	Rotation by 60° + reflection	20
Rotation by 180°	15	Reflection	15

$$|X^g| = {t_{id}}^3 = 12^3$$



$t_{id} = 12$ $t_f = 4$

$t_r = 2$ $t_{rf} = 2$

Icosahedral group conjugacy classes

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Rotation by 72°	12
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Rotation by 120°	20
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Inversion	1
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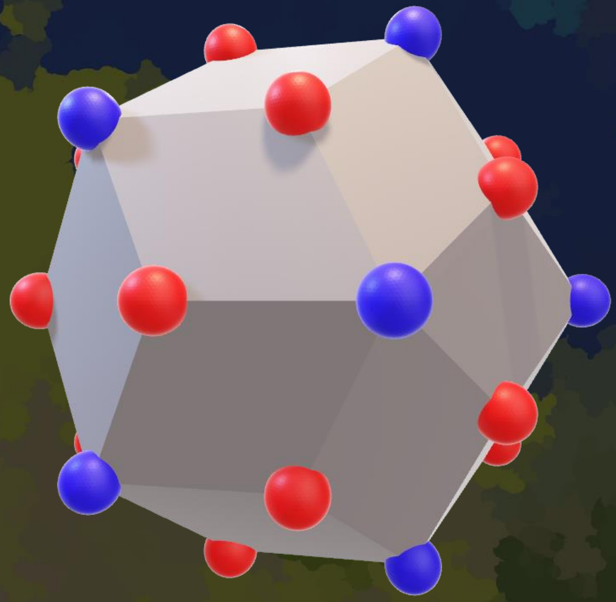

$$|X^g| = {t_{id}}^3 = 12^3$$

$t_{id} = 12 \quad t_f = 4$

$t_r = 2 \quad t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1
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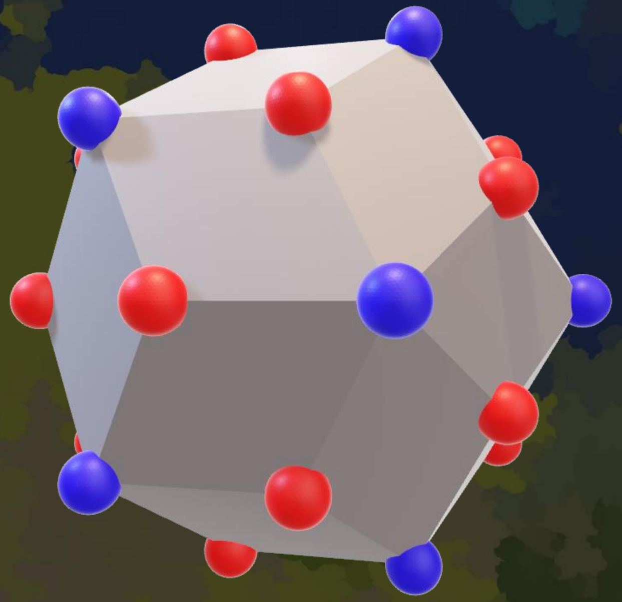


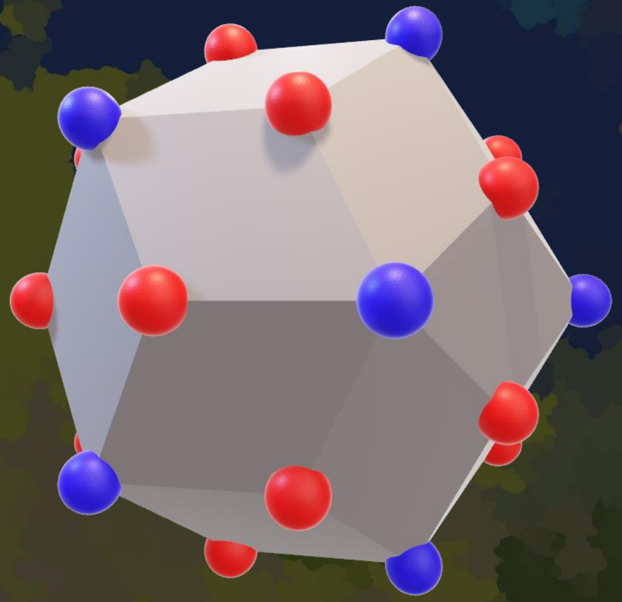
$$|X^g| = {t_{id}}^5 = 12^5$$

$$t_{id} = 12 \quad t_f = 4$$
$$t_r = 2 \quad t_{rf} = 2$$

Icosahedral group conjugacy classes

Identity	1
Rotation by 72°	12
Rotation by 144°	12
Rotation by 120°	20
Rotation by 180°	15

Inversion	1
Rotation by 36° + reflection	12
Rotation by 108° + reflection	12
Rotation by 60° + reflection	20
Reflection	15



$$|X^g| = {t_{id}}^{13}{t_f}^2{t_{rf}}^2$$
$$= 12^{13}(4^2)(2^2)$$

$t_{id} = 12 \quad t_f = 4$

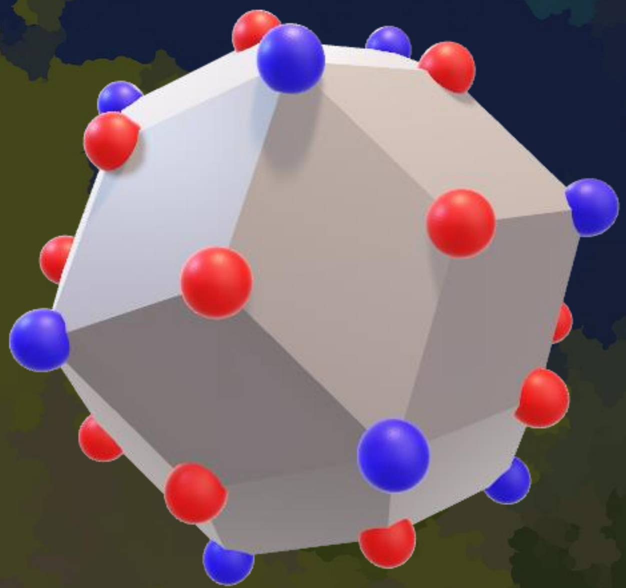
$t_r = 2 \quad t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1
Rotation by 72°	12
Rotation by 144°	12
Rotation by 120°	20
Rotation by 180°	15

Inversion	1
Rotation by 36° + reflection	12
Rotation by 108° + reflection	12
Rotation by 60° + reflection	20
Reflection	15

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$



$t_{id} = 12 \quad t_f = 4$

$t_r = 2 \quad t_{rf} = 2$

Icosahedral group conjugacy classes

Identity	1
Rotation by 72°	12
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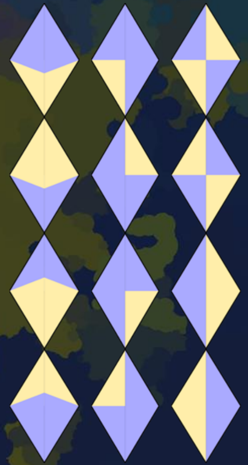
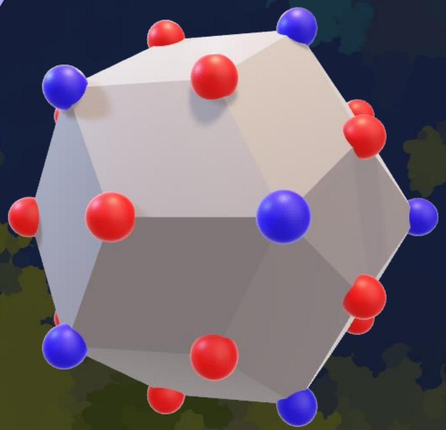


$$t_{id} = 12$$
$$t_r = 2$$
$$t_f = 4$$
$$t_{rf} = 2$$

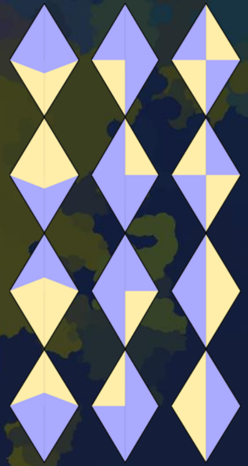


$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

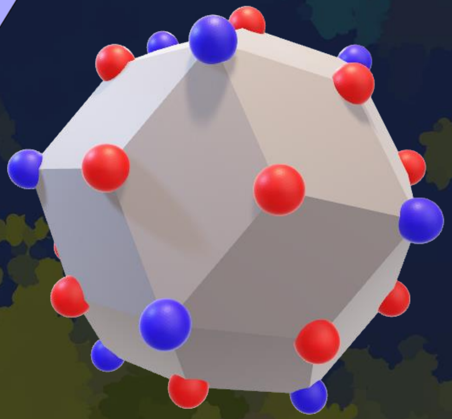
$$= \frac{1}{120}\left(t_{id}{}^{30} + 12t_{id}{}^{6} + 12t_{id}{}^{6} + 20t_{id}{}^{10} + 15t_{id}{}^{14}r^{2} + t_{id}{}^{15} + 12t_{id}{}^{3} + 12t_{id}{}^{3} + 20t_{id}{}^{5} + 15t_{id}{}^{13}t_f{}^{2}t_{rf}{}^{2}\right)$$

Icosahedral group conjugacy classes

Identity	1	Inversion	1
Rotation by 72°	12	Rotation by 36° + reflection	12
Rotation by 144°	12	Rotation by 108° + reflection	12
Rotation by 120°	20	Rotation by 60° + reflection	20
Rotation by 180°	15	Reflection	15



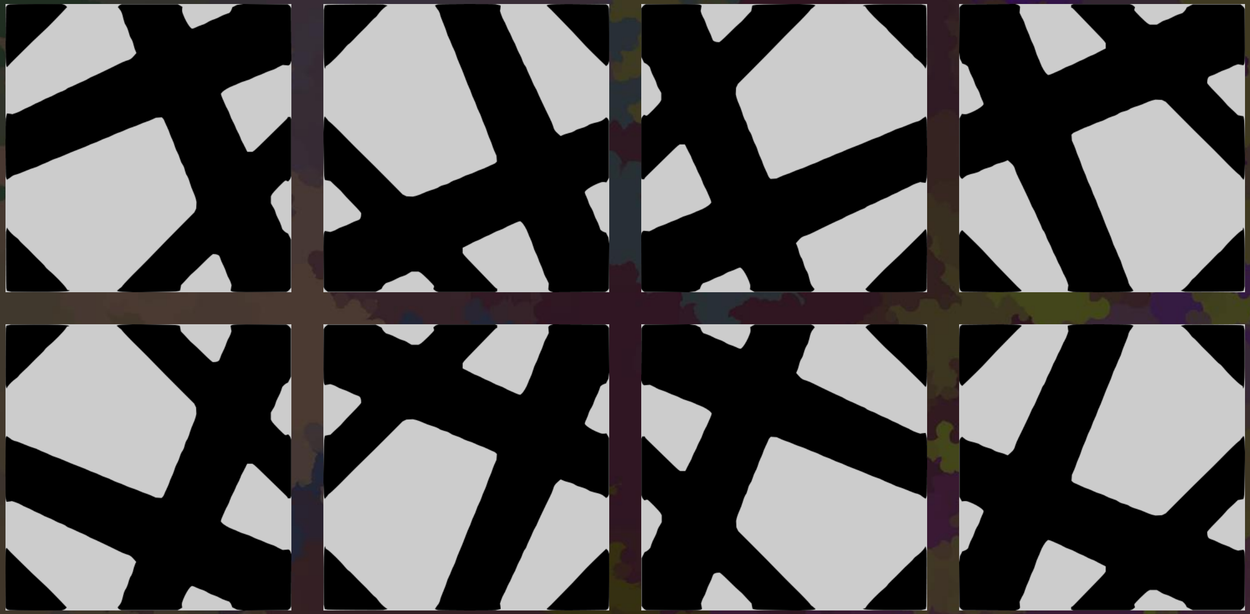
$t_{id} = 12$
$t_r = 2$
$t_f = 4$
$t_{rf} = 2$



$$|X/G| = \frac{1}{120}\left(t_{id}{}^{30} + 12t_{id}{}^{6} + 12t_{id}{}^{6} + 20t_{id}{}^{10} + 15t_{id}{}^{14}r^{2} + t_{id}{}^{15} + 12t_{id}{}^{3} + 12t_{id}{}^{3} + 20t_{id}{}^{5} + 15t_{id}{}^{13}t_f{}^{2}t_{rf}{}^{2}\right)$$

$$\approx 1.978 \times 10^{30}$$

Escher's cubes (up to rotation and reflection)

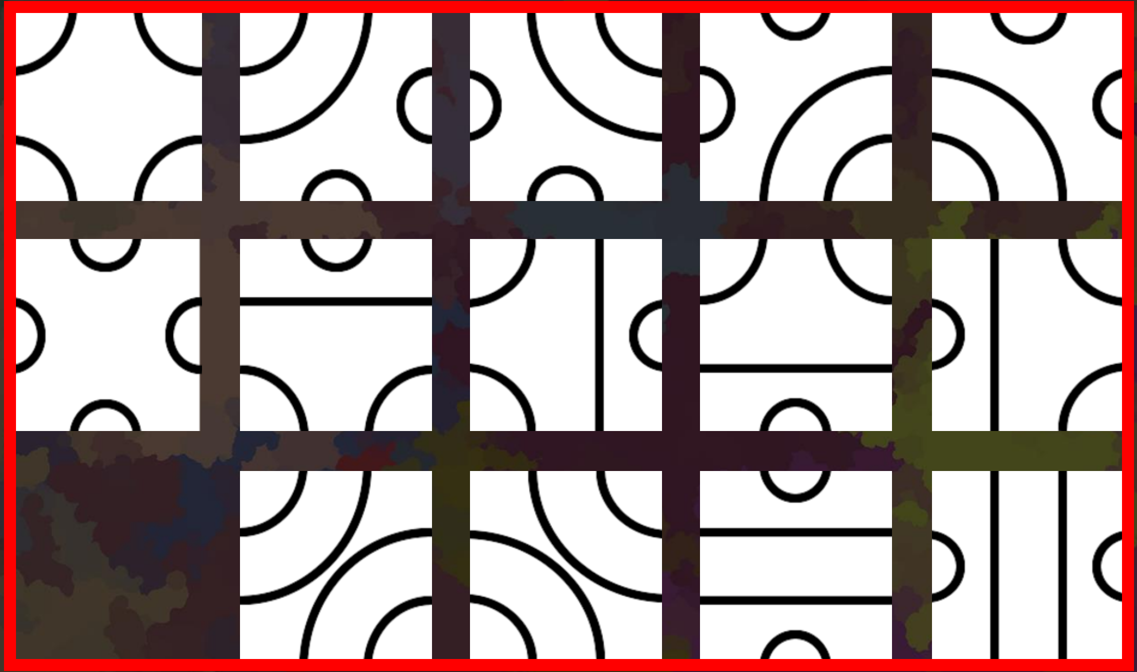


$$\frac{1}{48}\left({t_{id}}^6 + 7{t_{id}}^3 + 8{t_{id}}^2 + 8t_{id} + 6t_{id}{t_r}^2 + 6t_{id}t_{r^2} + 3{t_{id}}^2{t_{r^2}}^2 + 3t_{id}{t_f}^4 + 6{t_{id}}^2{t_{rf}}^2\right)$$

$$t_{id} = 8 \qquad t_r = t_f = t_{rf} = t_{r^2} = 0$$

$$|X/G| = 5548$$

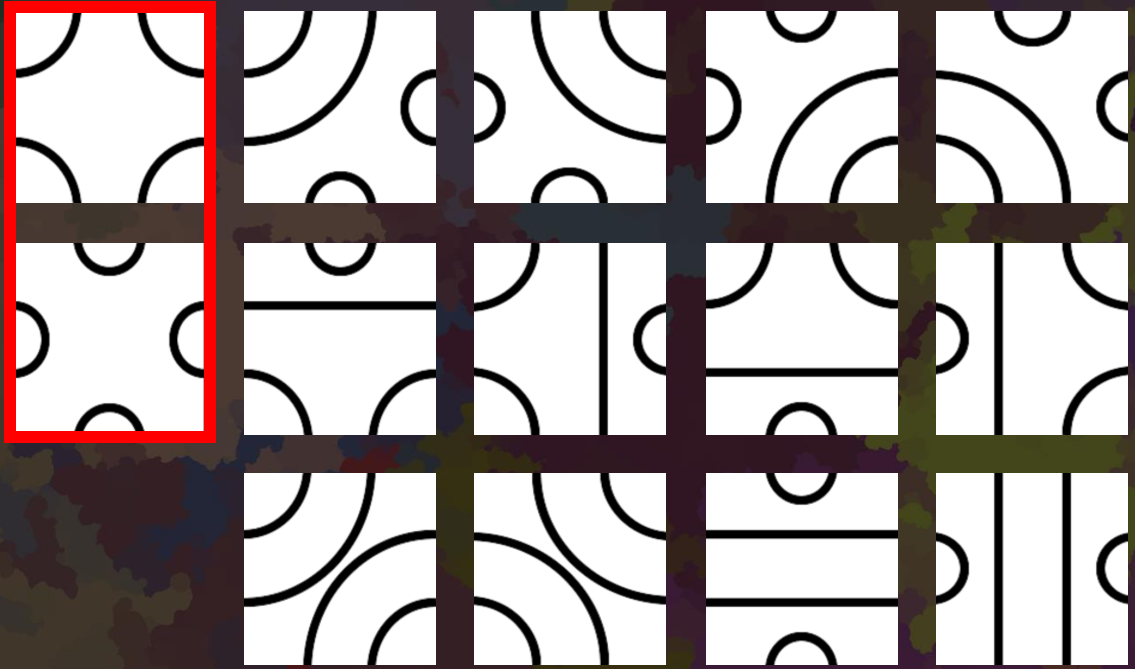
Reimann's cubes (up to rotation and reflection)



$t_{id} = 14$

$$\frac{1}{48}\left(t_{id}{}^{6} + 7t_{id}{}^{3} + 8t_{id}{}^{2} + 8t_{id} + 6t_{id}t_{r}{}^{2} + 6t_{id}t_{r^2} + 3t_{id}{}^{2}t_{r^2}{}^{2} + 3t_{id}t_{f}{}^{4} + 6t_{id}{}^{2}t_{rf}{}^{2}\right)$$

Reimann's cubes (up to rotation and reflection)

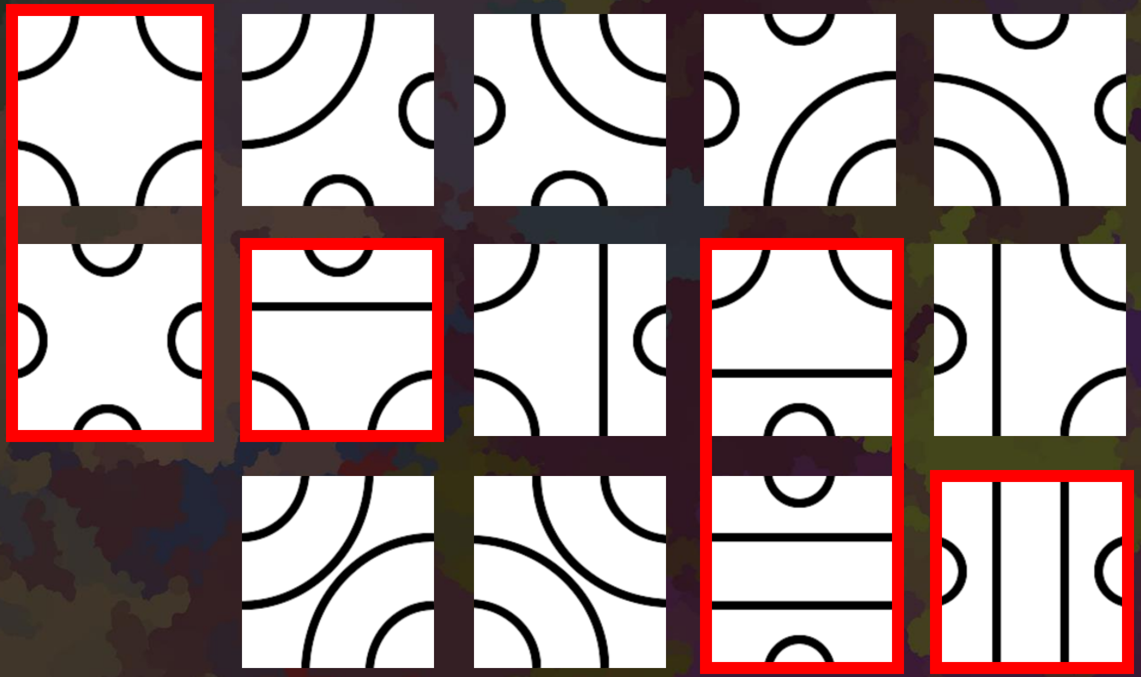


$t_{id} = 14$

$t_r = 2$

$$\frac{1}{48}\left(t_{id}{}^6 + 7t_{id}{}^3 + 8t_{id}{}^2 + 8t_{id} + 6t_{id}t_r{}^2 + 6t_{id}t_{r^2} + 3t_{id}{}^2t_{r^2}{}^2 + 3t_{id}t_f{}^4 + 6t_{id}{}^2t_{rf}{}^2\right)$$

Reimann's cubes (up to rotation and reflection)



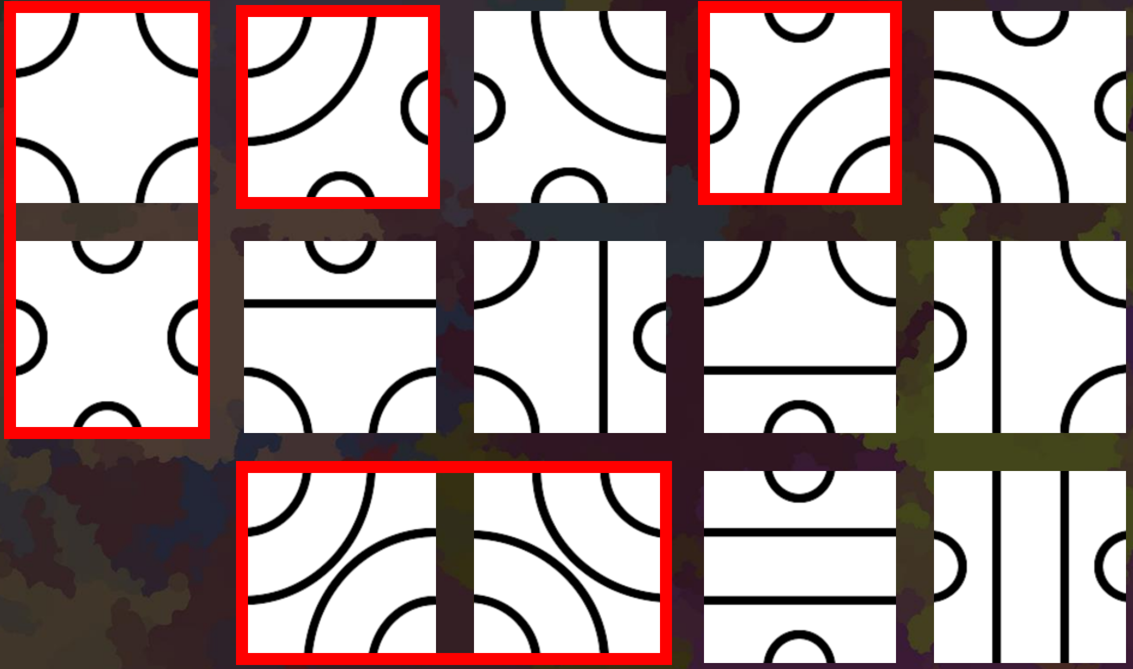
$t_{id} = 14$

$t_r = 2$

$t_f = 6$

$$\frac{1}{48}\left(t_{id}{}^{6} + 7t_{id}{}^{3} + 8t_{id}{}^{2} + 8t_{id} + 6t_{id}t_{r}{}^{2} + 6t_{id}t_{r^2} + 3t_{id}{}^{2}t_{r^2}{}^{2} + 3t_{id}t_{f}{}^{4} + 6t_{id}{}^{2}t_{rf}{}^{2}\right)$$

Reimann's cubes (up to rotation and reflection)



$t_{id} = 14$

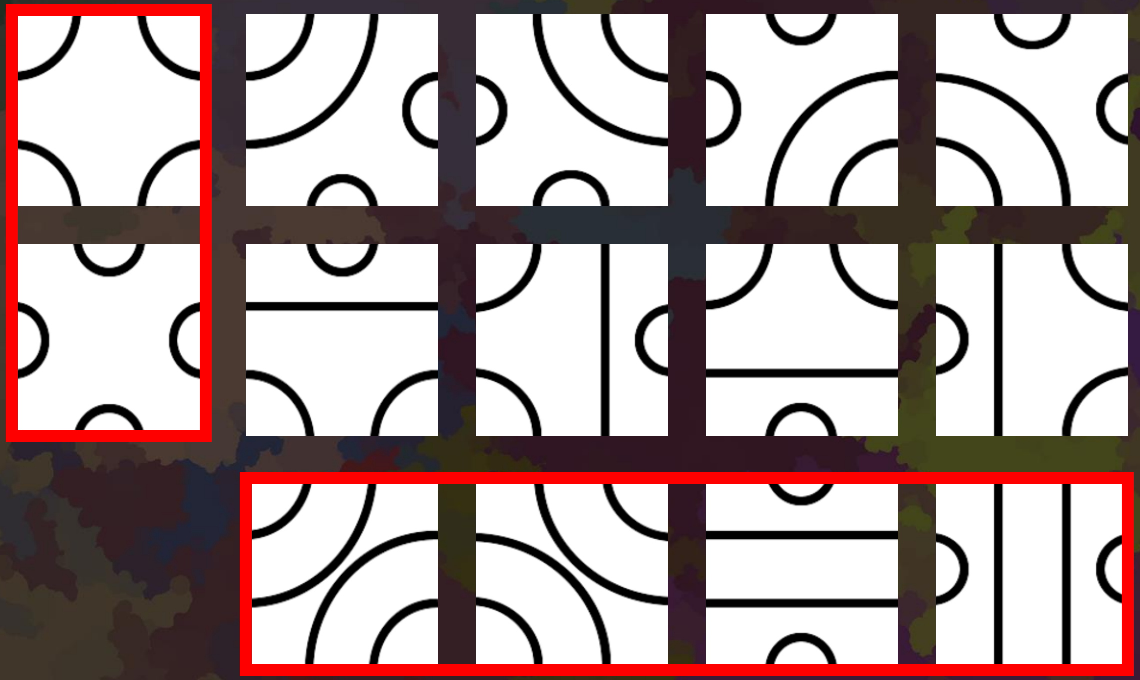
$t_r = 2$

$t_f = 6$

$t_{rf} = 6$

$$\frac{1}{48}\left(t_{id}{}^6 + 7t_{id}{}^3 + 8t_{id}{}^2 + 8t_{id} + 6t_{id}t_r{}^2 + 6t_{id}t_{r^2} + 3t_{id}{}^2t_{r^2}{}^2 + 3t_{id}t_f{}^4 + 6t_{id}{}^2t_{rf}{}^2\right)$$

Reimann's cubes (up to rotation and reflection)



$t_{id} = 14$

$t_r = 2$

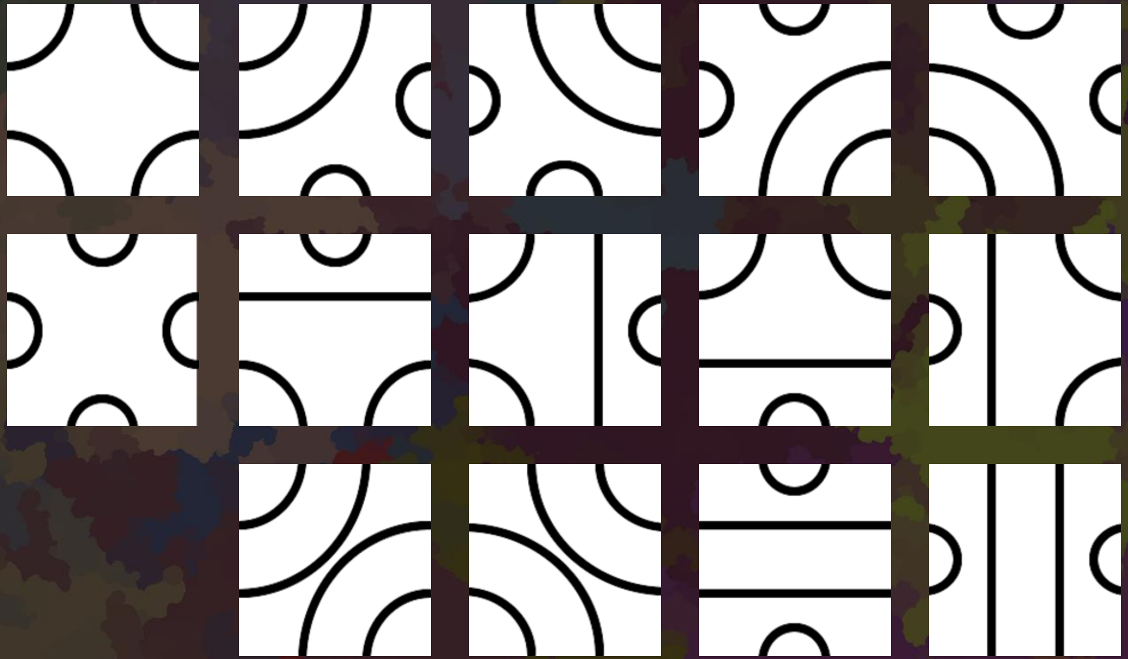
$t_f = 6$

$t_{rf} = 6$

$t_{r^2} = 6$

$$\frac{1}{48}\left({t_{id}}^6 + 7{t_{id}}^3 + 8{t_{id}}^2 + 8t_{id} + 6t_{id}{t_r}^2 + 6t_{id}t_{r^2} + 3{t_{id}}^2{t_{r^2}}^2 + 3t_{id}{t_f}^4 + 6{t_{id}}^2{t_{rf}}^2\right)$$

Reimann's cubes (up to rotation and reflection)



$t_{id} = 14$

$t_r = 2$

$t_f = 6$

$t_{rf} = 6$

$t_{r^2} = 6$

$$\frac{1}{48}\left({t_{id}}^6 + 7{t_{id}}^3 + 8{t_{id}}^2 + 8t_{id} + 6t_{id}{t_r}^2 + 6t_{id}t_{r^2} + 3{t_{id}}^2{t_{r^2}}^2 + 3t_{id}{t_f}^4 + 6{t_{id}}^2{t_{rf}}^2\right)$$

$$|X/G| = 159{,}775$$

Truchet tile ball (up to rotation)

$t_{6,id} = 3$

$t_{6,r} = 0$

$t_{5,id} = 1$

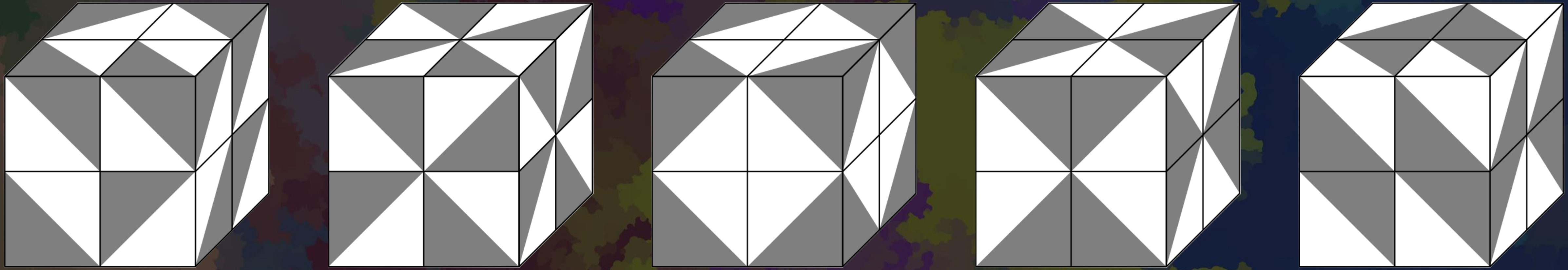
$t_{5,r} = 1$

$$|X/G| = \frac{1}{60}\left(t_{6,id}{}^{20} t_{5,id}{}^{10} + 24 t_{6,id}{}^{4} t_{5,id}{}^{2} t_{5,r}{}^{2} + 20 t_{6,id}{}^{6} t_{6,r}{}^{2} t_{5,id}{}^{4} + 15 t_{6,id}{}^{10} t_{5,id}{}^{6} \right)$$

$$|X/G| = 58{,}127{,}868$$

Truchet Tile Ball
by Jon-Paul Wheatley

The motivating question!



$$\frac{1}{24}\left({t_{id}}^6 + 7{t_{id}}^3 + 8{t_{id}}^2 + 8t_{id} + 6t_{id}{t_r}^2 + 6t_{id}t_{r^2} + 3{t_{id}}^2{t_{r^2}}^2 + 3t_{id}{t_f}^4 + 6{t_{id}}^2{t_{rf}}^2\right)$$

$t_{id} = 4^4$ $t_{r^2} = 4^2$ $t_{rf} = 4(2^2)$ $t_r = 4$ $t_f = 4^2$

$$|X/G| = 5{,}864{,}068{,}667{,}776$$

Thank you!

Peter Kagey

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