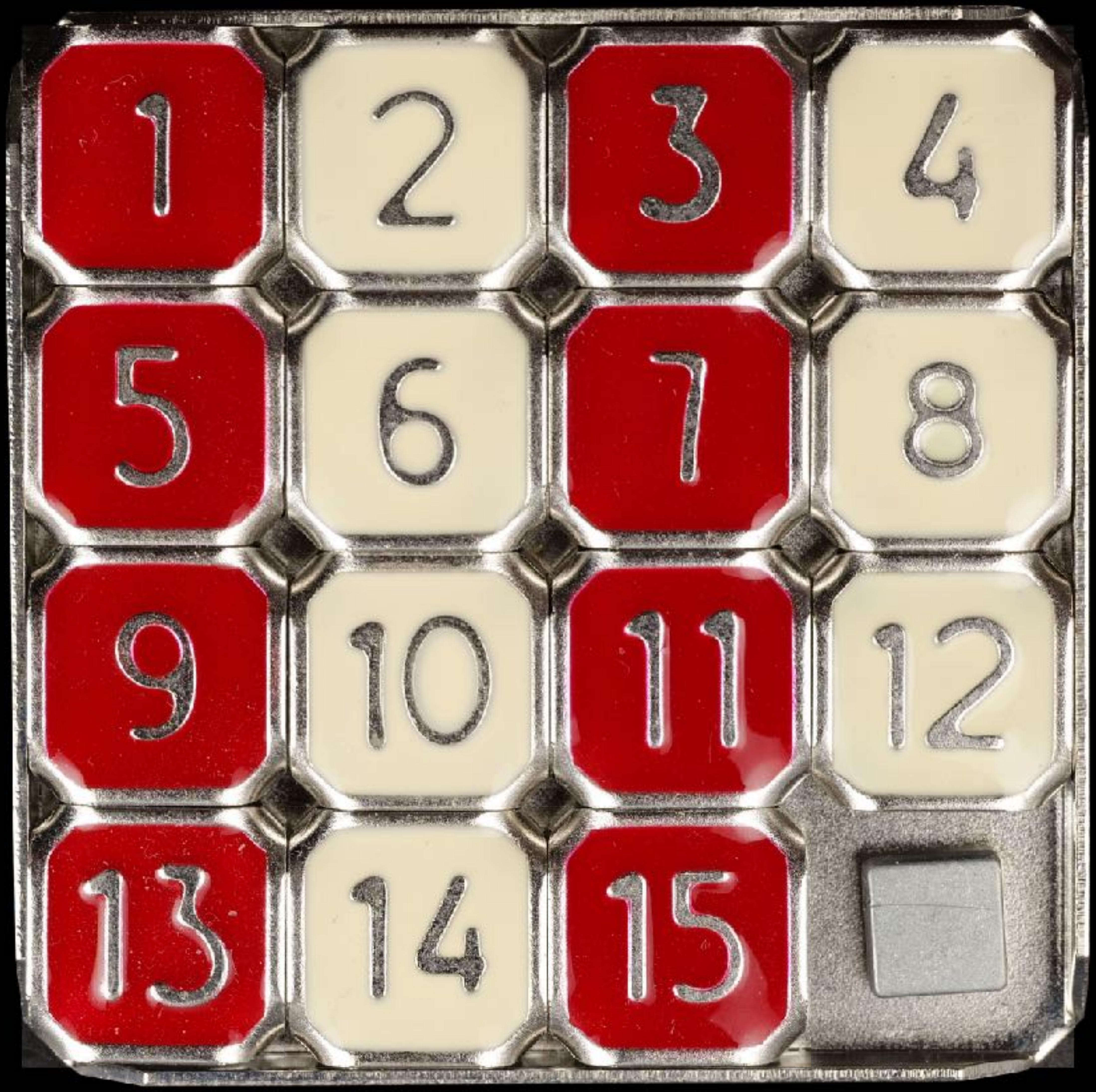
Variants of the 15-puzzle and the effects of holonomy

Henry Segerman (he/him)
Oklahoma State University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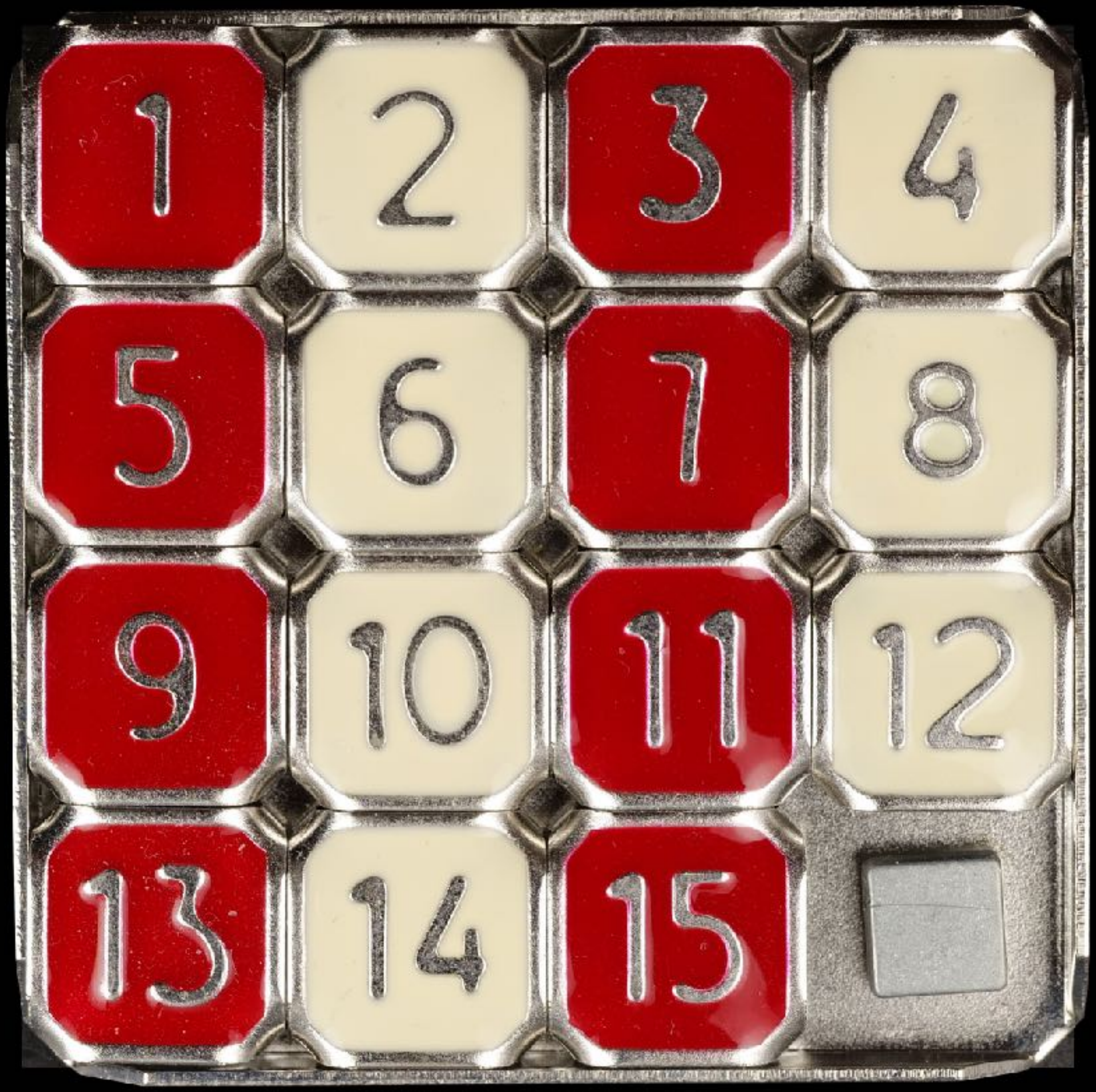
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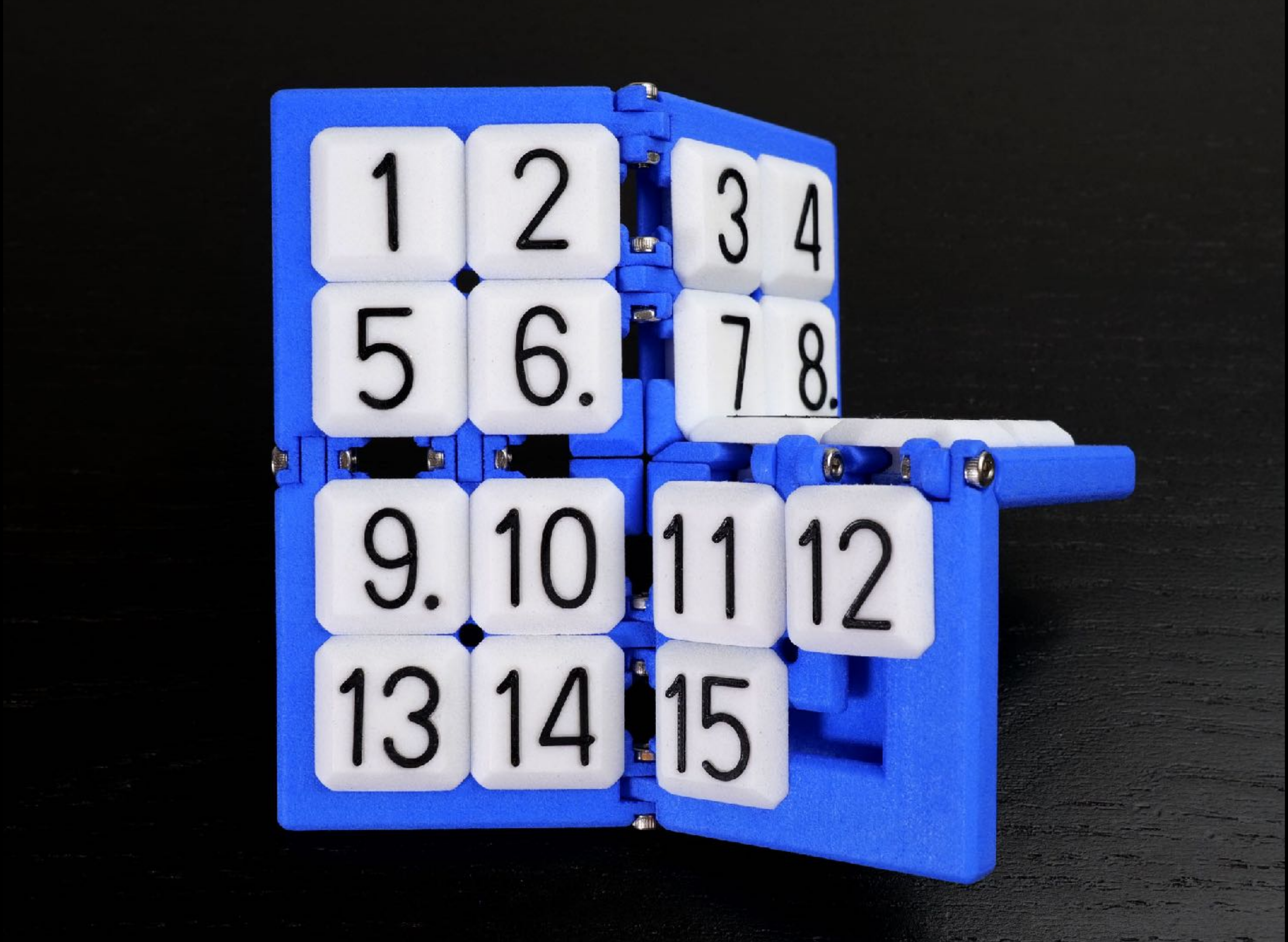
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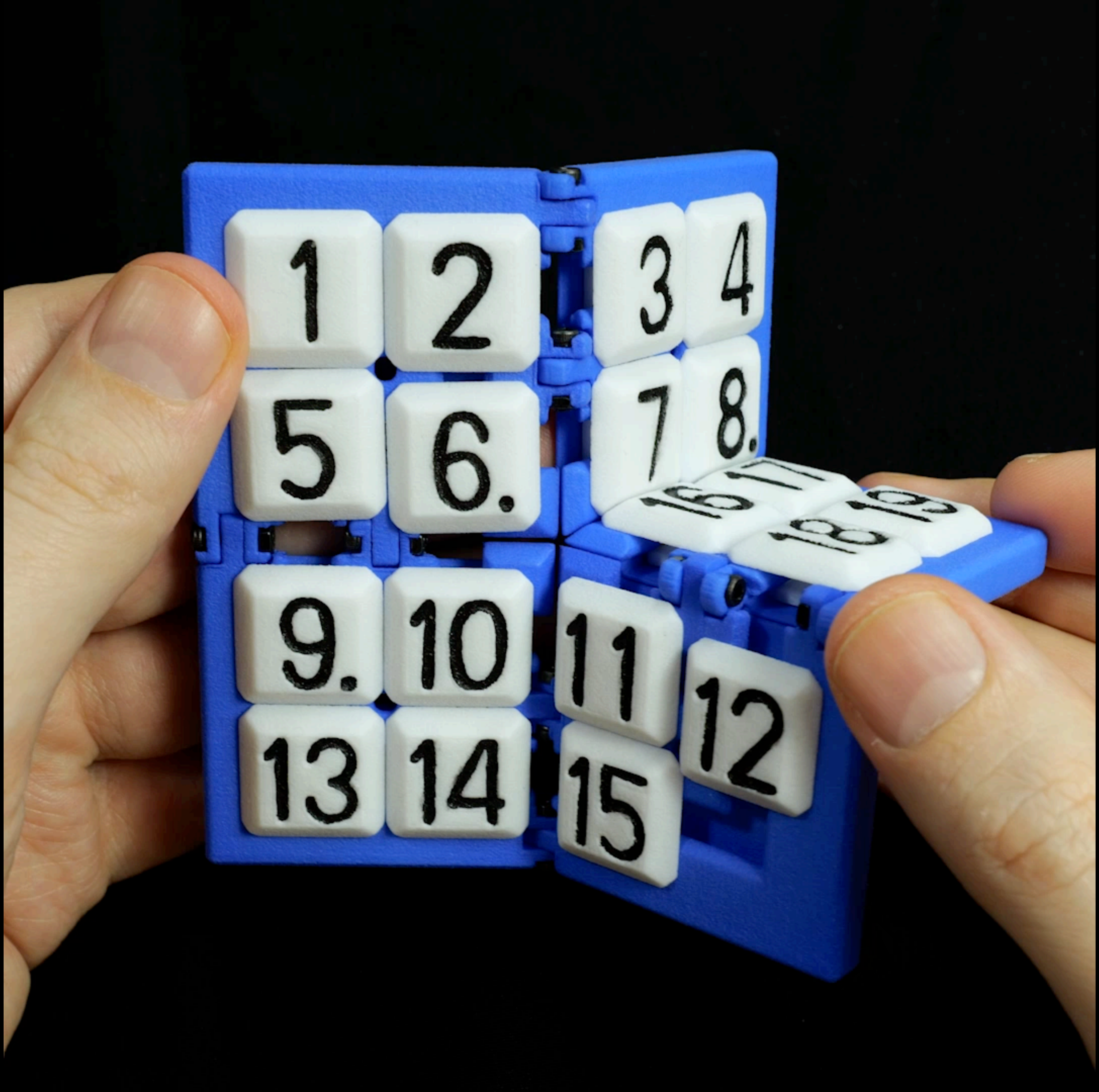
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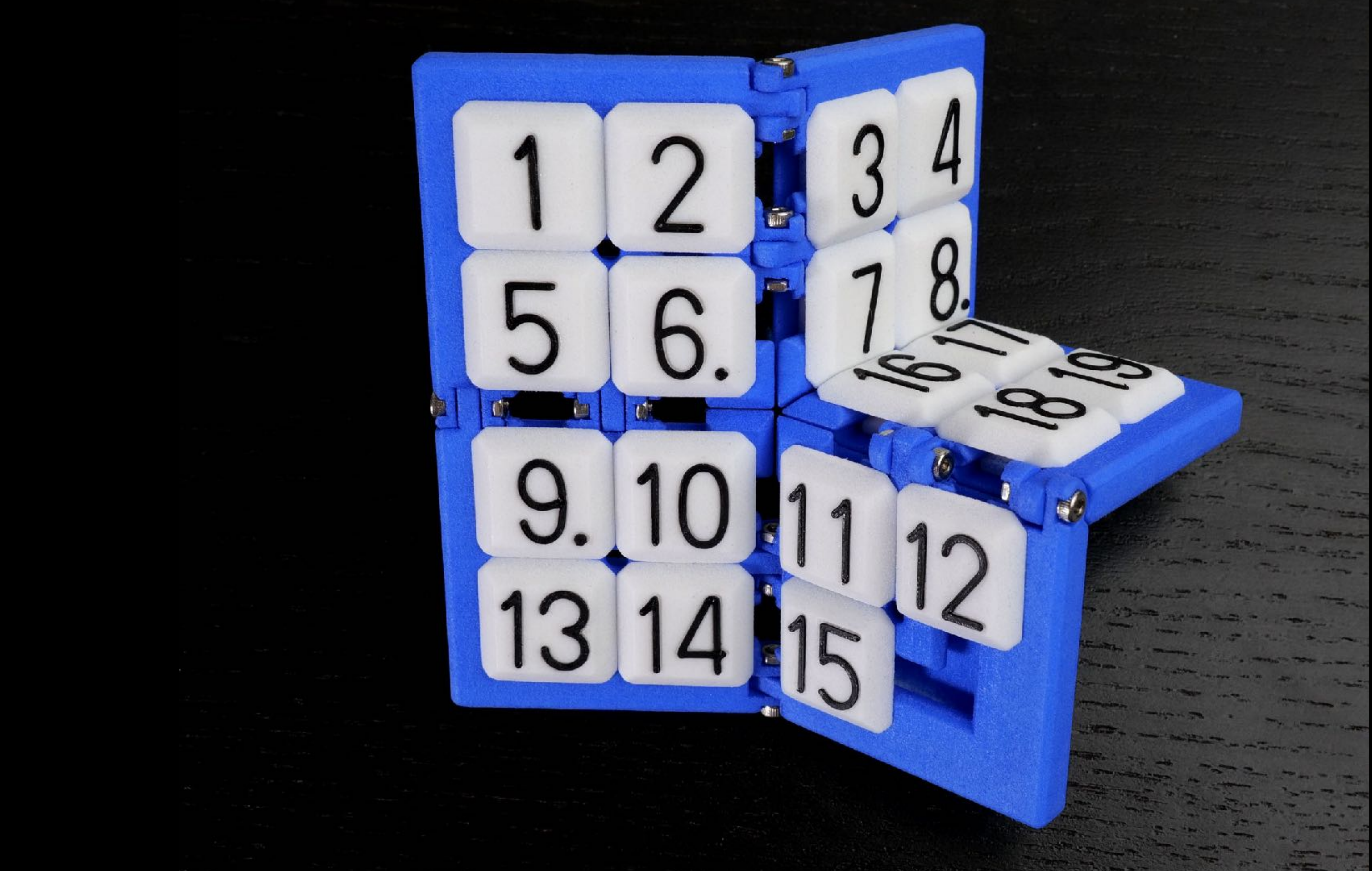
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DISCRETE MATHEMATICS

Sliding piece puzzles with oriented tiles[1]

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Abstract

In this paper, we consider n identical tiles which are placed on the $n+1$ vertices of a graph and which move along the edges of the graph. The tiles come with an "orientation", an element of an arbitrary finite group H. Moving a tile along a given edge into the empty vertex changes the orientation of the tile in a prescribed way. We study the group of oriented positions of the tiles achievable from an initial position which fix the empty vertex. It may be thought of as a subgroup of the semidirect product $H^n \rtimes S_n$ or the wreath product $H \operatorname{wr} S_n$.

Theorem 3. *The group of a sliding piece puzzle with oriented tiles on a nonseparable graph $\mathscr{G}$ is isomorphic to one of the following.*

If $\mathscr{G}$ is a polygon,

(1) *the cyclic group of order $n|H_0|$.*

If $\mathscr{G}$ is bipartite,

(2) $H_0^n \rtimes A_n$, *or*

(3) $\{((h_1, \ldots, h_4), \sigma) \in H_0^4 \rtimes A_4\colon\ h_1 \cdots h_4 \in K_0 h_0^r$ and $\sigma \in K_4(1,2,3)^r,\ r = 0,\ 1,$ *or* $2\}$, *for some normal subgroup K_0 of H_0 of index 3 and some $h_0 \in H_0 - K_0$.*

If $\mathscr{G}$ is isomorphic to θ_0,

(4) $H_0^6 \rtimes \langle (1,2,3,4), (1,4,5,6)\rangle$, *or*

(5) $\{((h_1, \ldots, h_n), \sigma) \in H_0^6 \rtimes \langle (1,2,3,4), (1,4,5,6)\rangle\colon\ h_1 \cdots h_n \in K_0$ *iff* $\sigma \in \langle (1,2,3,4), (1,4,5,6)\rangle \cap A_6\}$, *for some normal subgroup K_0 of H_0 of index 2.*

If $\mathscr{G}$ is not a polygon, bipartite, or isomorphic to θ_0,

(6) $H_0^n \rtimes S_n$, *or*

(7) $\{((h_1, \ldots, h_n), \sigma) \in H_0^n \rtimes S_n\colon\ h_1 \cdots h_n \in K_0$ *iff* $\sigma \in A_n\}$, *for some normal subgroup K_0 of H_0 of index 2.*

In particular, except for the cyclic case, the index of G in $H_0^n \rtimes S_n$ is 1, 2, 6, or 12, with 6 possible only for the nonseparable, bipartite graph on 5 vertices and for θ_0, and 12 possible only for θ_0.


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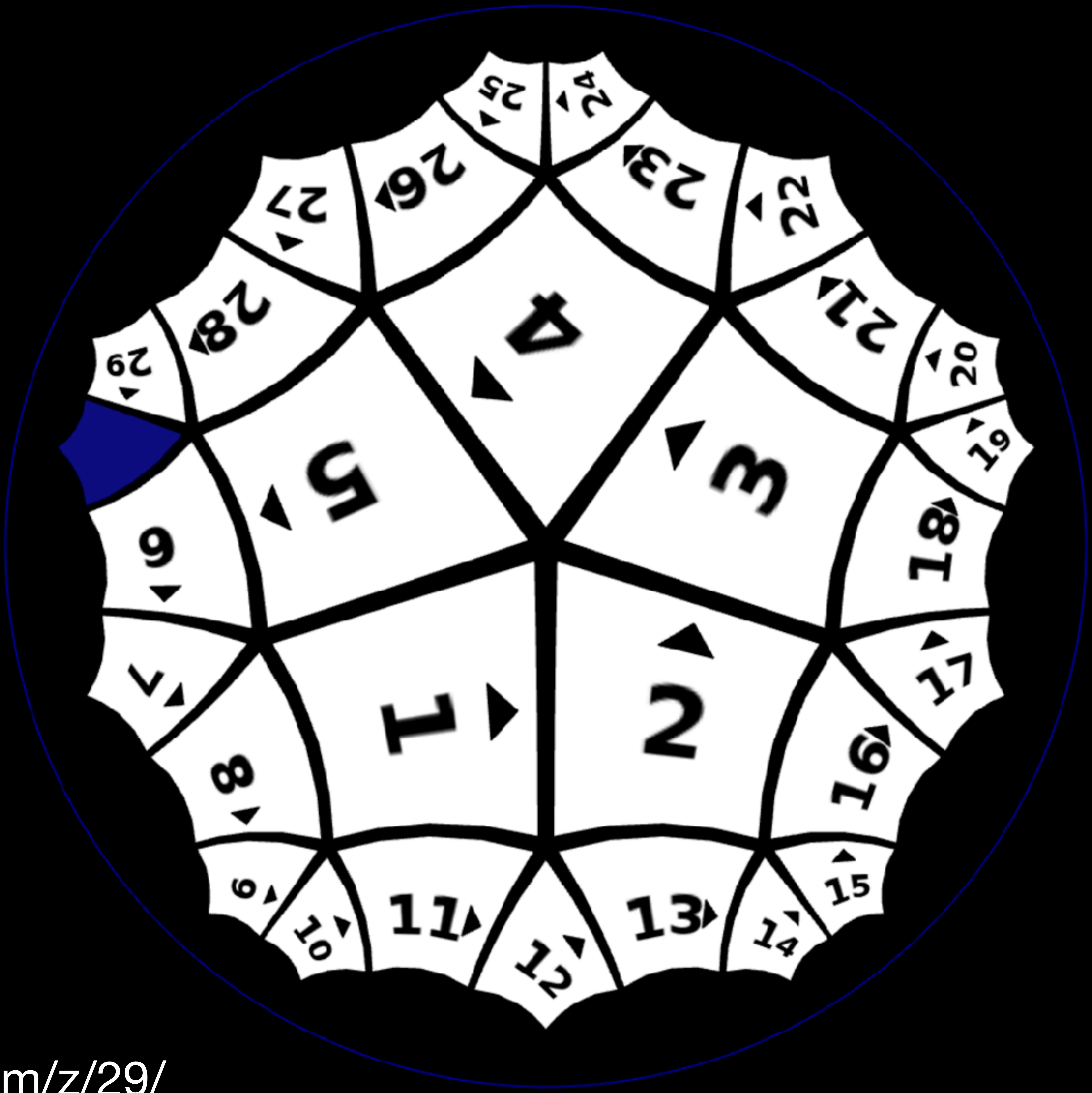
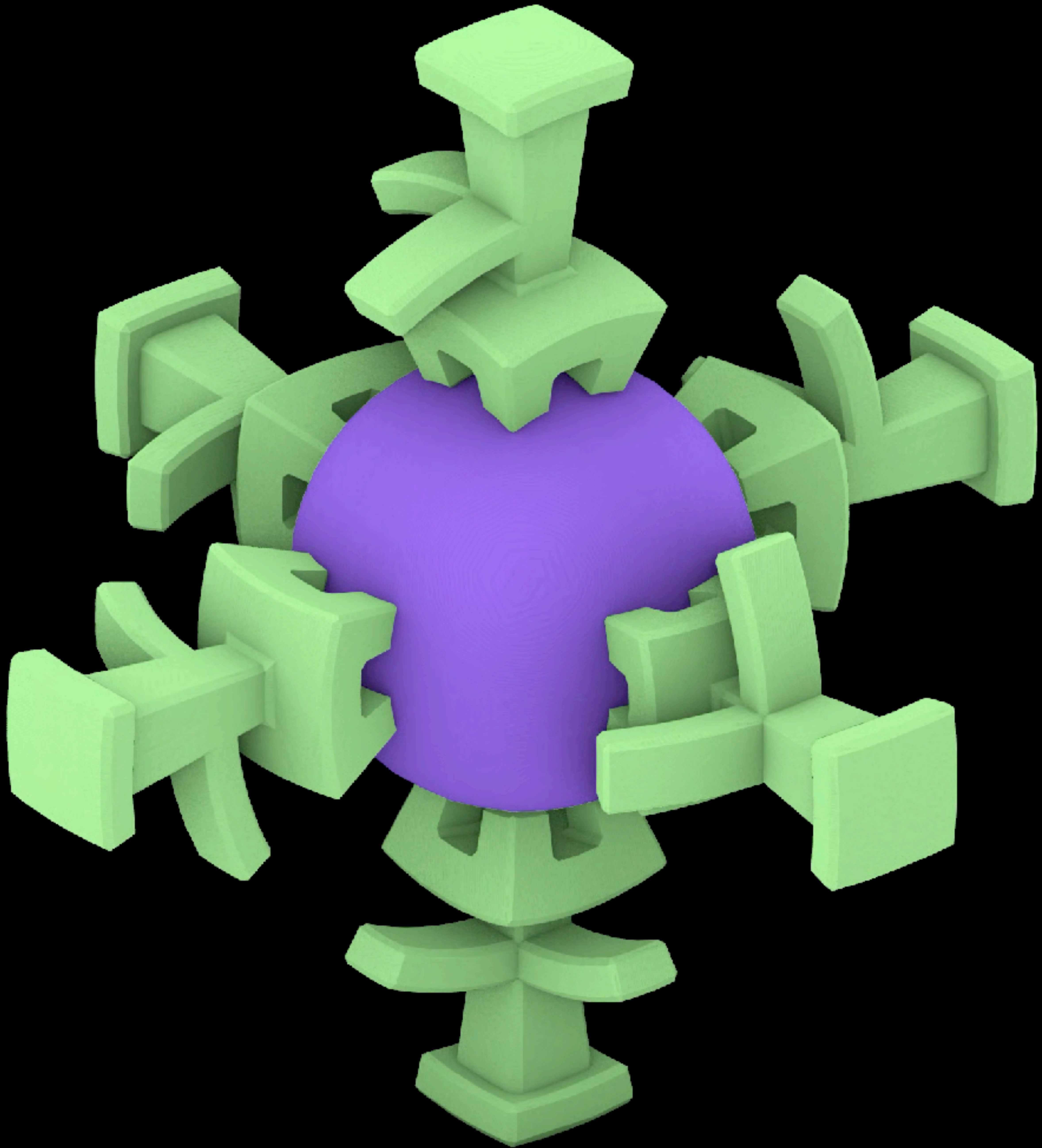
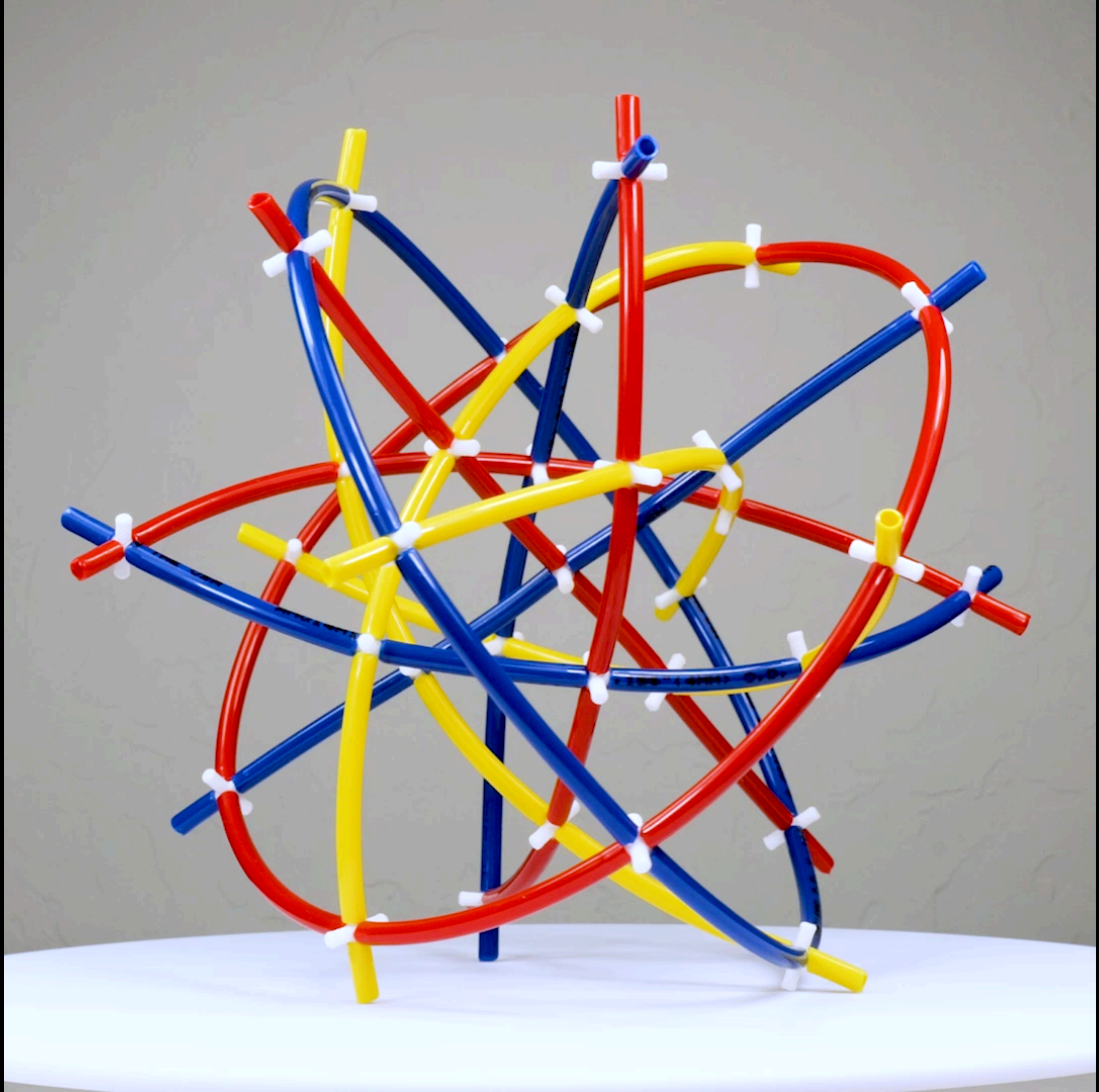


http://roguetemple.com/z/29/

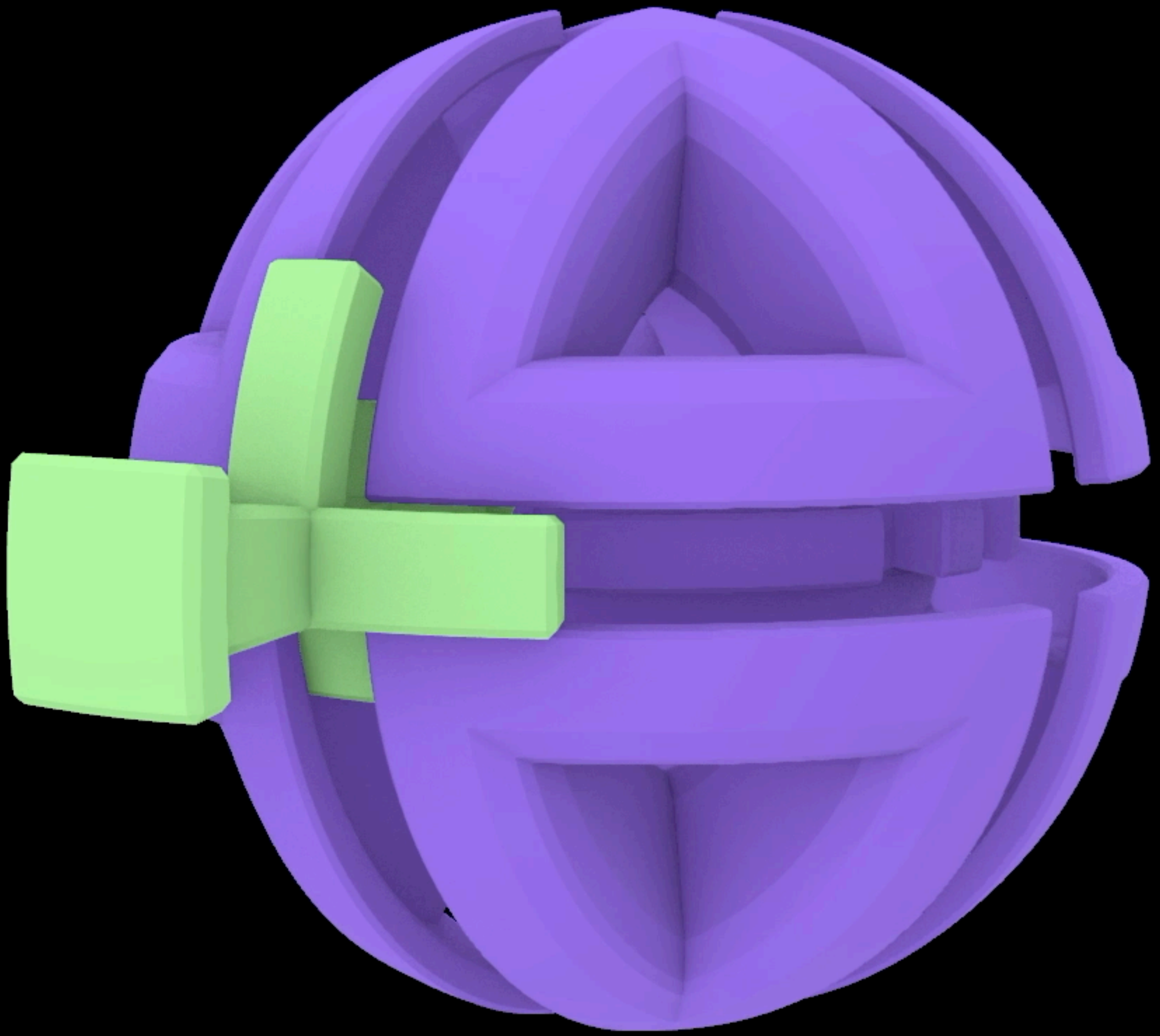
(v) menu

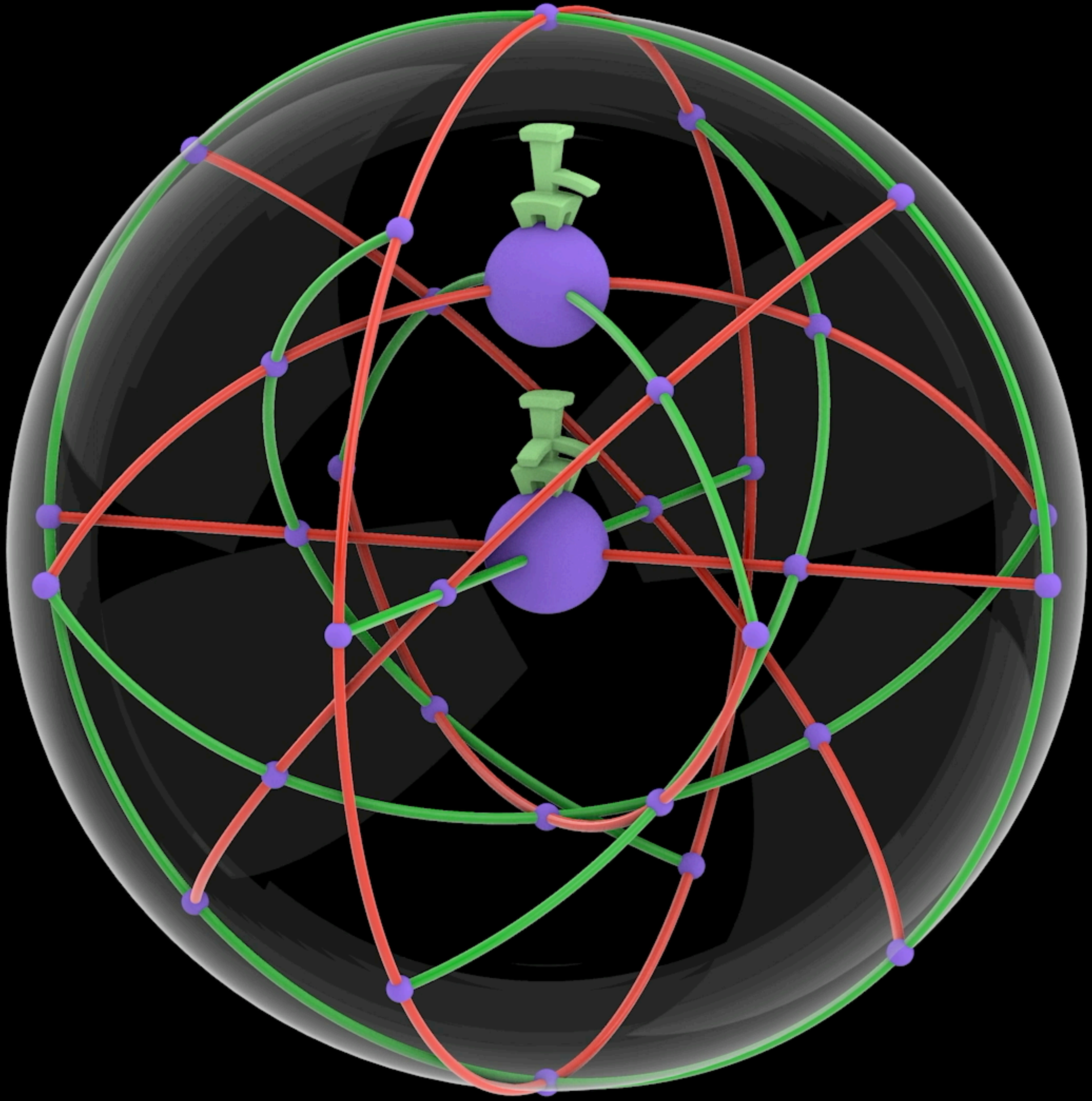
How should we choose pegs to make a good maze?





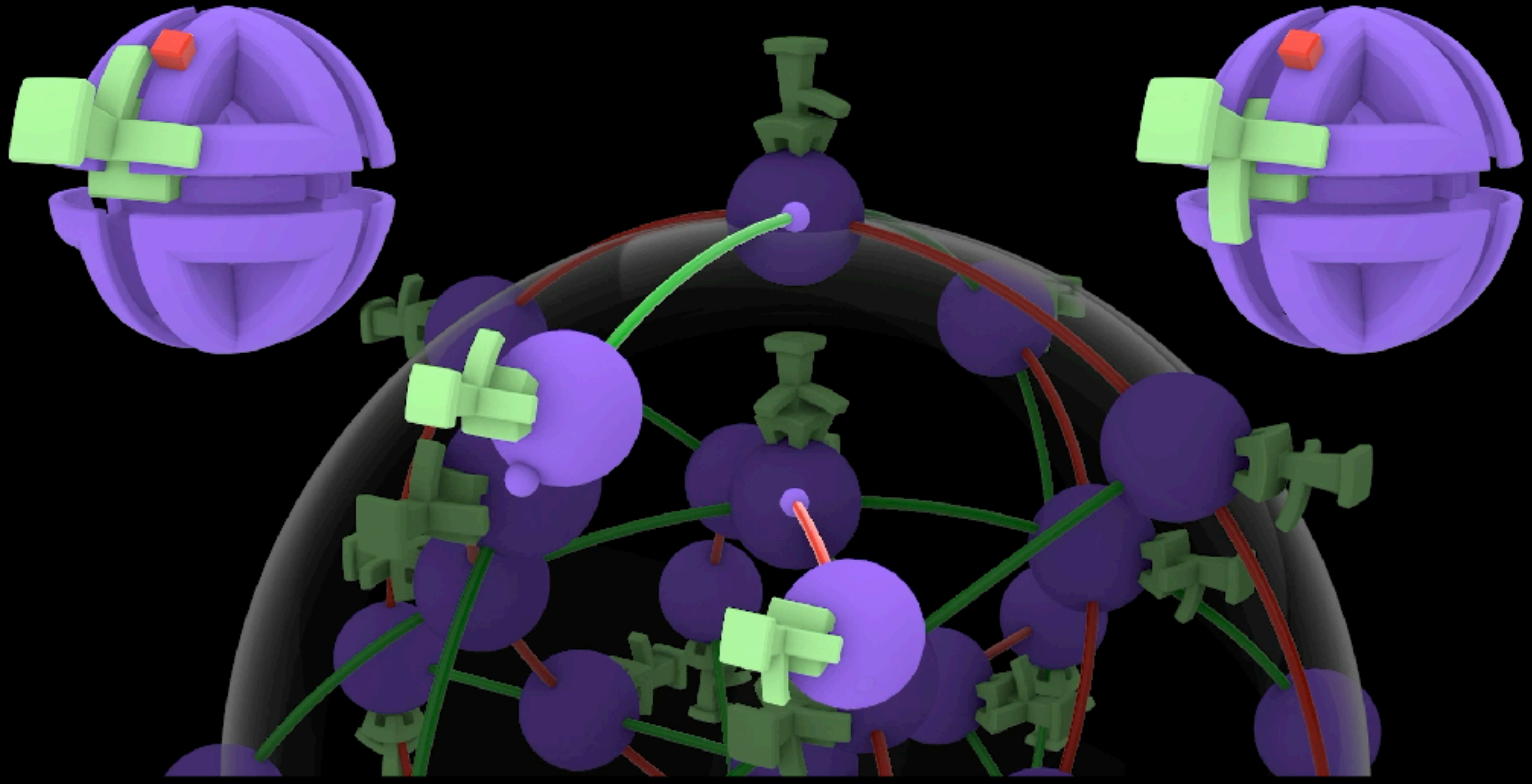
Sculpture by Chaim Goodman-Strauss





Use a quaternion (modulo sign) to record a position in $\mathbb{R}P^3$. We have:

- A quaternion for each node of the graph
- A pair of quaternions for each edge of the graph
- A quaternion for each peg position



The map from pegs to pairs of blocked edges is equivariant under quaternion composition.

This lets us build the graph and test peg positions algorithmically.

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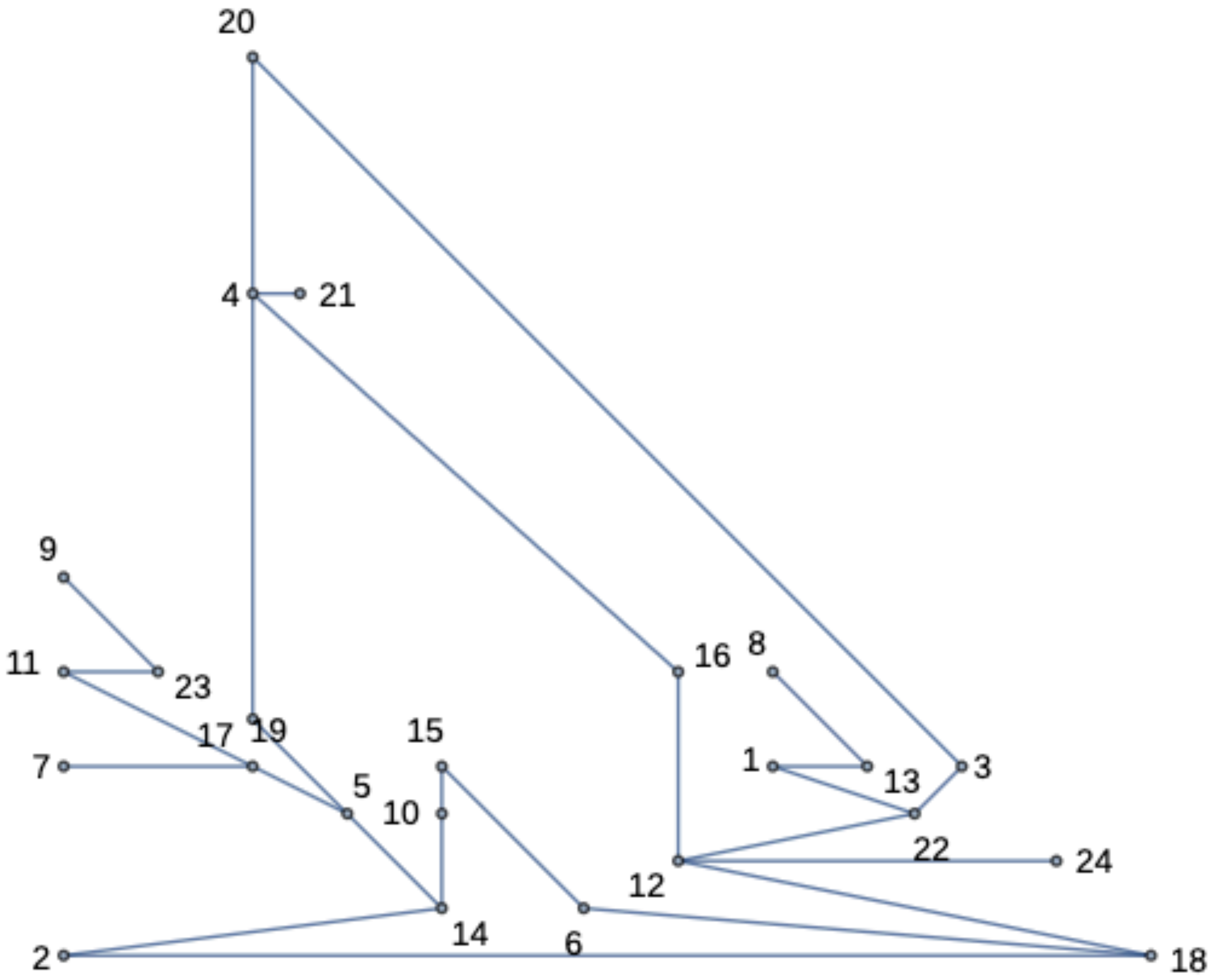

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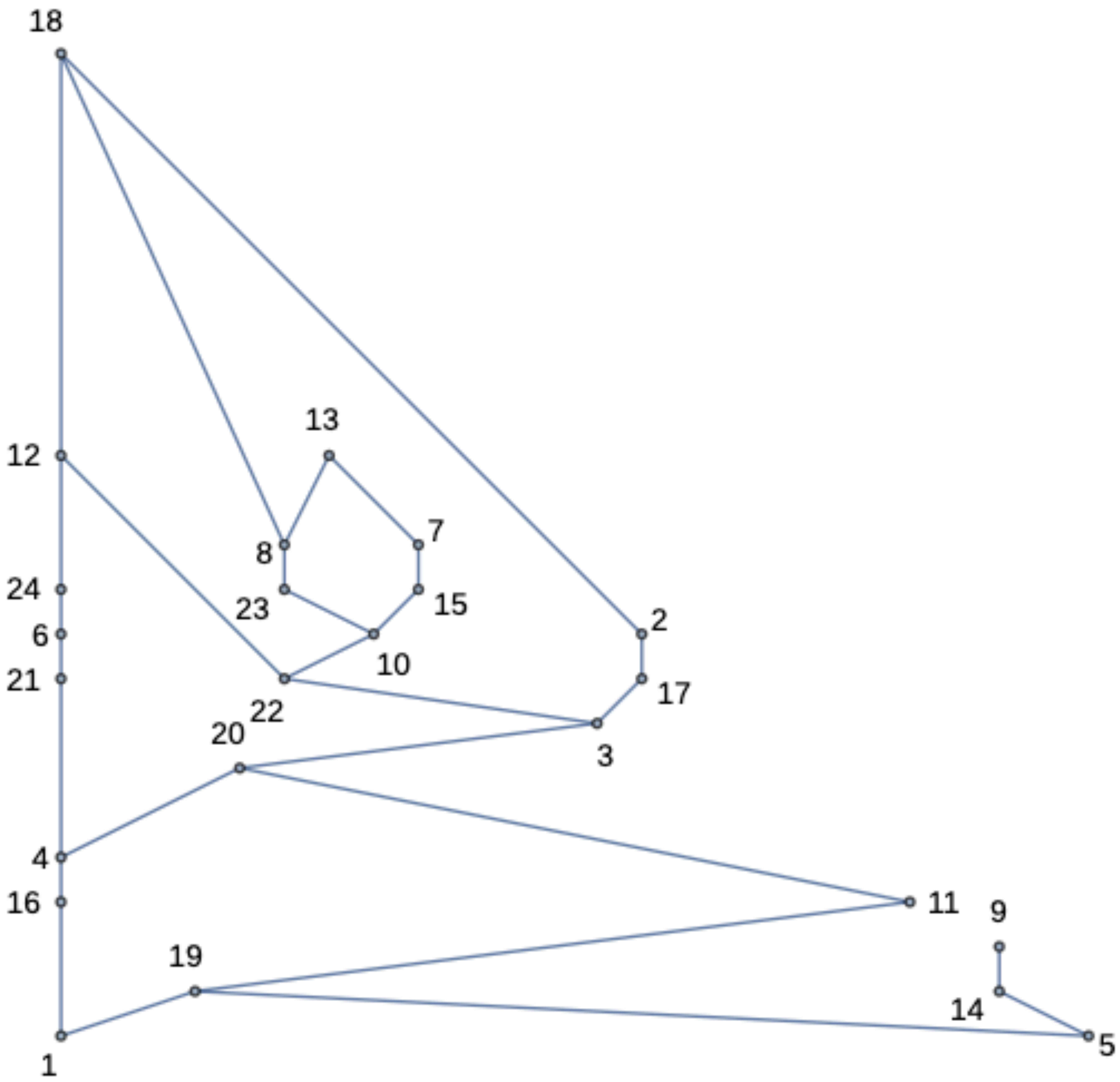

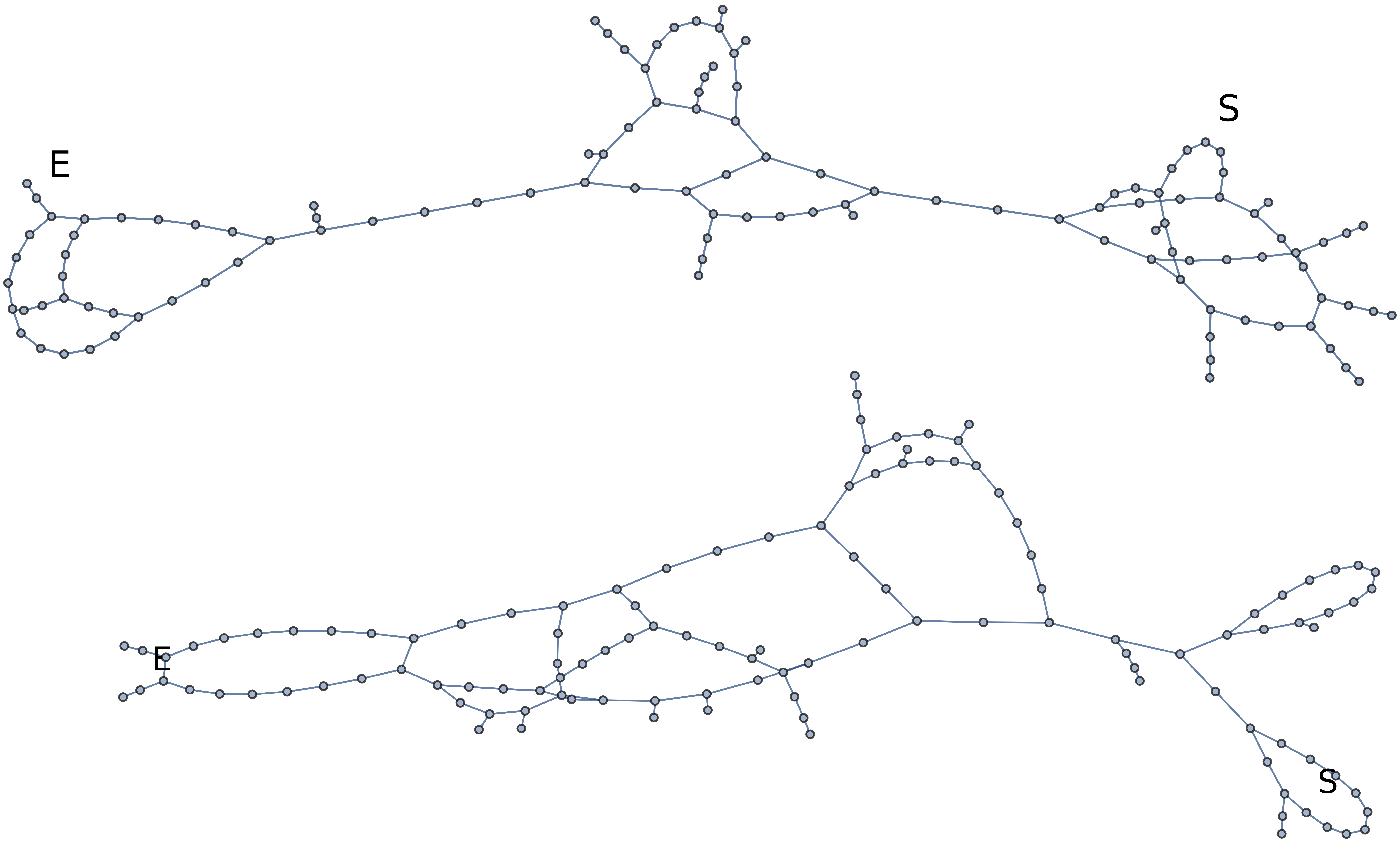
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Joint work with Edmund Harriss

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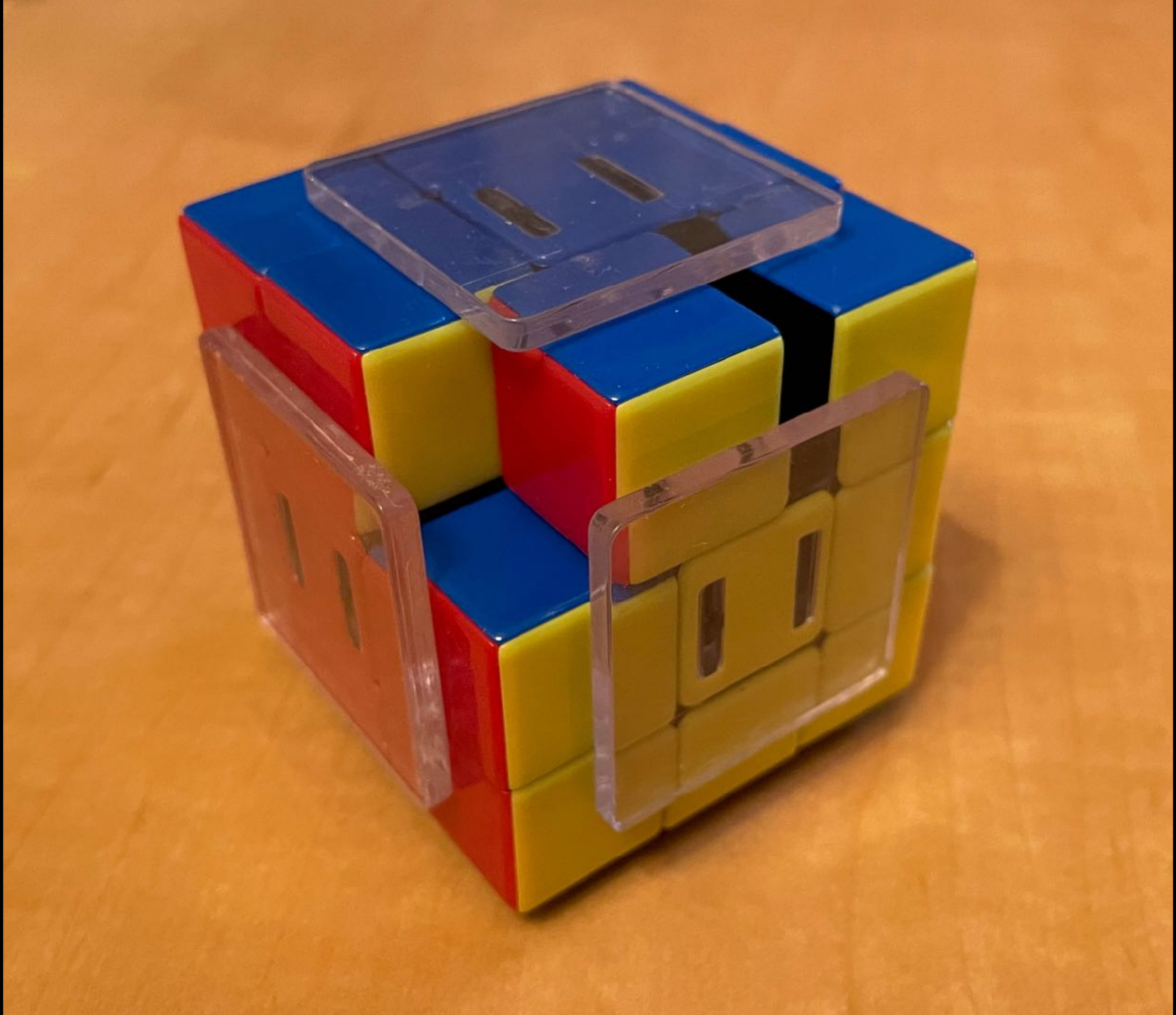
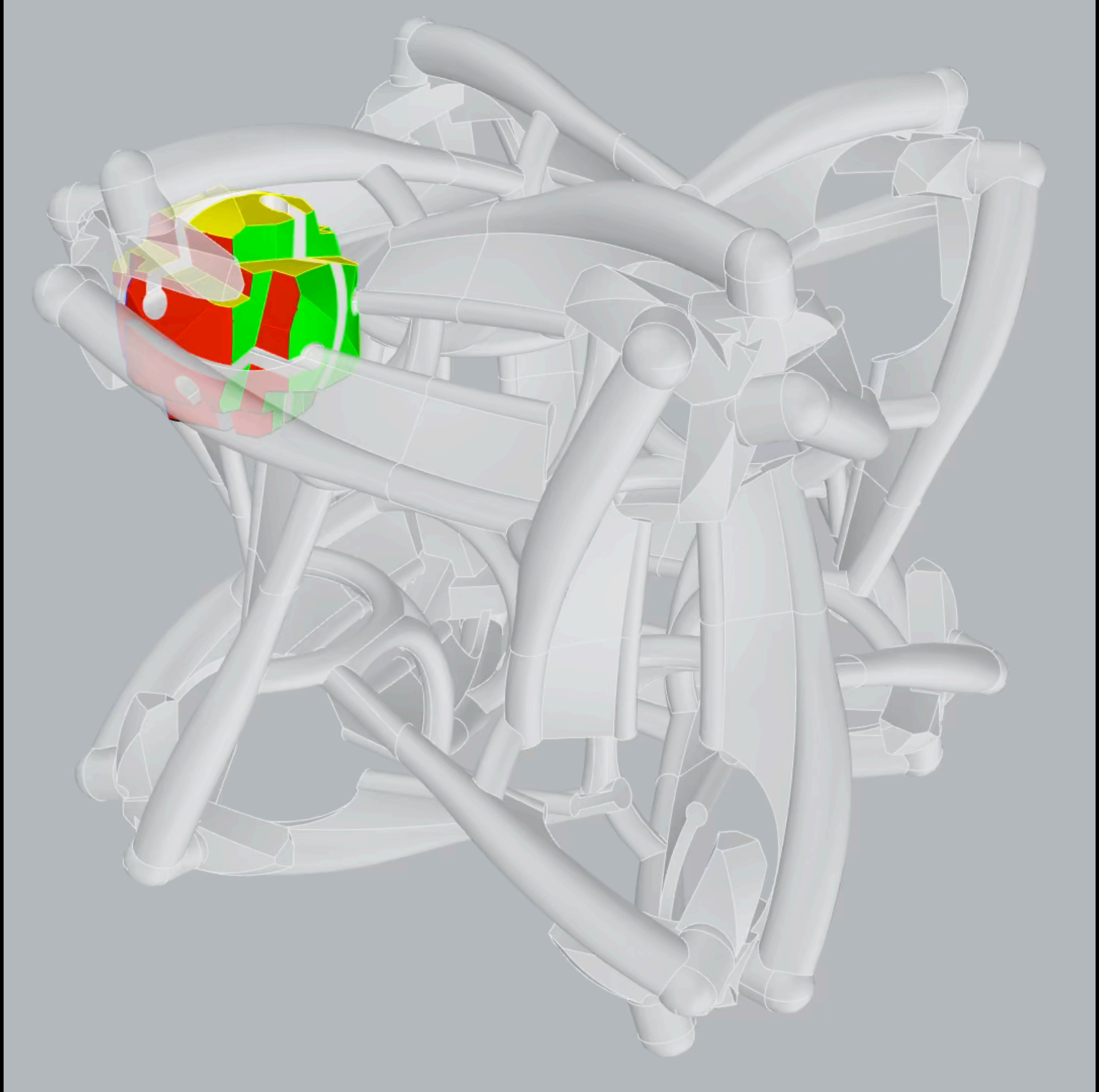
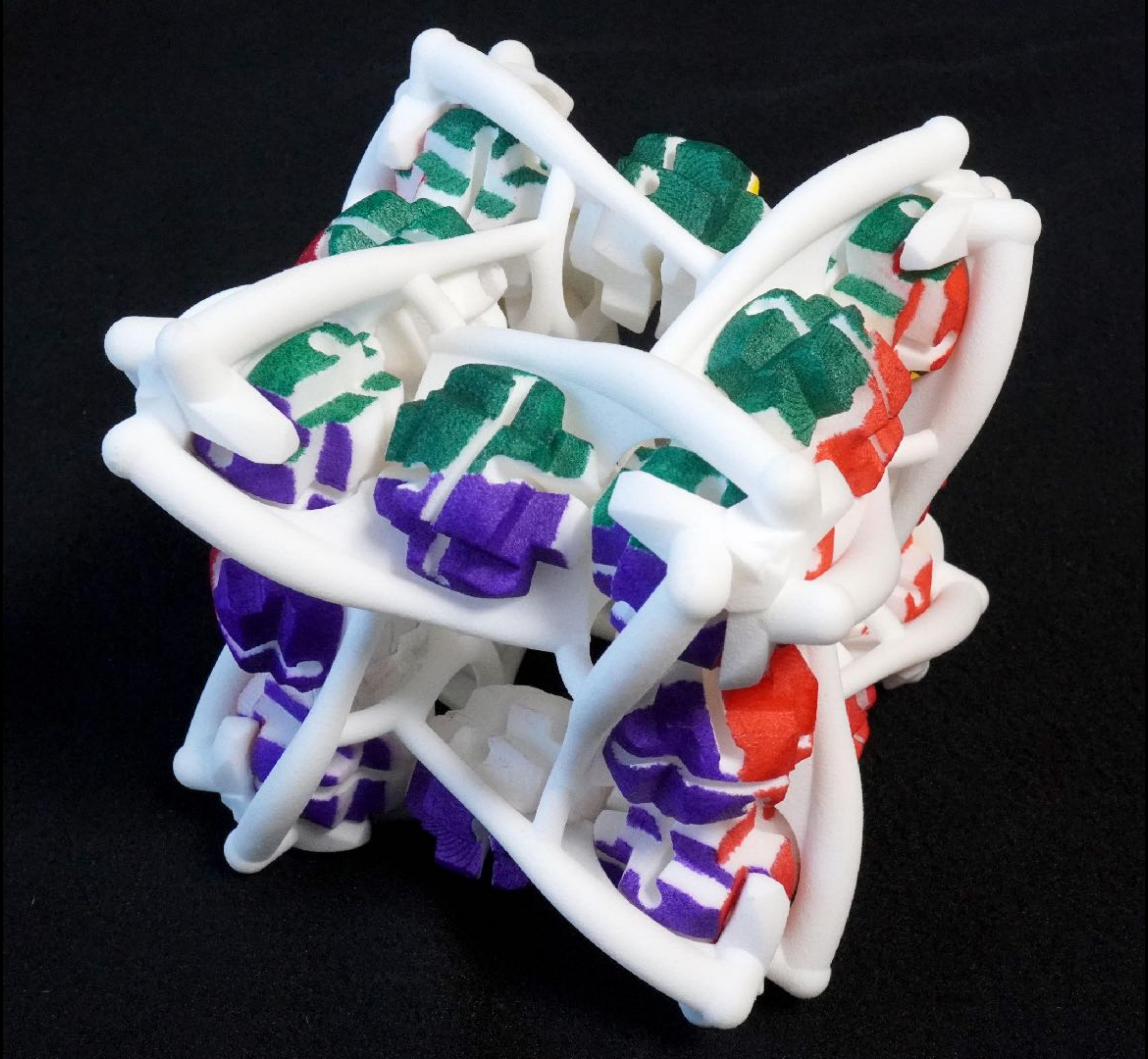
Photo by Alice Zhang








Thanks!
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