Dyadic shifts, sparse domination, and commutators in the non-doubling setting

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December 12, 2024, Harmonic Analysis and Convexity at ICERM

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[1]Supported in part by National Science Foundation grant DMS-2203272

Plan for this lecture

A Dyadic Operator

Definition (Haar Functions)

Fix a dyadic grid $\mathcal{D}$ on $\mathbb{R}$. Let $I \in \mathcal{D}$. Define the Haar function associated to I

$$h_I(x) = \frac{1}{\sqrt{|I|}} \left(\mathbb{1}_{I_-}(x) - \mathbb{1}_{I_+}(x) \right)$$

where I_- denotes the left "child" of I and I_+ denotes the right child. Recall that $\{h_I\}_{I \in \mathcal{D}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

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Definition (Dyadic Hilbert Transform)

The dyadic Hilbert transform is the operator

$$\text{Ш}_{\mathcal{D}} f(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle (h_{I_-}(x) - h_{I_+}(x)), \quad f \in L^2(\mathbb{R}), x \in \mathbb{R}.$$

Motivation

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Theorem (S. Petermichl)

There exists a constant $c_0 \neq 0$ so that

$$\frac{c_0}{t-x} = \lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x)\, d\alpha\, \frac{dr}{r}$$

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Recall that (continuous) Hilbert transform $H : L^p(w) \to L^p(w)$ iff w satisfies the Muckenhoupt $A_p(\mathbb{R})$ condition

$$[w]_{A_p(\mathbb{R})} := \sup_{I\subset\mathbb{R}} \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}(x)\,dx\right)^{p-1} < \infty$$

Determining the sharp dependence of $\|H\|_{L^p(w)}$ (and more generally for generic CZO) on the weight characteristic was a major problem in harmonic analysis with important connections to PDE (Beltrami equation)

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$$\|Hf\|_{L^2(w)} \leq C[w]_{A_2(\mathbb{R})}\|f\|_{L^2(w)}.$$

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• Hytönen proved the analogous result for general CZOs on $\mathbb{R}^n$. His argument again relied on showing T is a suitable average of dyadic shift operators (which can be more complicated than $\text{Ш}_{\mathcal{D}}$ in general).
• Natural question: what happens if we change Lebesgue measure on $\mathbb{R}$ (or $\mathbb{R}^n$) to some arbitrary Borel measure μ?

Non-homogeneous measures

1. If μ is a Borel measure on $\mathbb{R}^n$ that satisfies the *doubling condition* $\mu(B(x, 2r)) \lesssim \mu(B(x, r))$, the theory for continuous CZOs and dyadic shifts is basically the same.

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3. Lopez-Sanchez, Martell, and Parcet [LSMP] considered an appropriate variant of the *dyadic Hilbert transform* :

$$H_{\mathcal{D}}^{\mu} f(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \big(h_{I_-}(x) - h_{I_+}(x) \big).$$

with Haar functions defined to be **orthonormal in** $L^2(\mu)$ (I will often drop the μ later). They characterized the measures for which $H_{\mathcal{D}}$ is bounded on $L^p(\mu)$ ($L^2(\mu)$ comes for free).

LSMP Results

Definition (Haar Function)

Given $I \in \mathcal{D}$, define the Haar function associated to I:

$$h_I(x) = \sqrt{m(I)} \left(\frac{\mathbb{1}_{I_-}(x)}{\mu(I_-)} - \frac{\mathbb{1}_{I_+}(x)}{\mu(I_+)} \right).$$

where

$$m(I) = \frac{\mu(I_-)\mu(I_+)}{\mu(I)} \sim \min\{\mu(I_-), \mu(I_+)\}.$$

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Definition

We say that μ is *balanced* (m-equilibrated in original article) if there exists an independent constant $C > 0$ so

$$\frac{1}{C} m(I) \leq m(\widehat{I}) \leq Cm(I)$$

for all $I \in \mathcal{D}$, where $\widehat{I}$ denotes the dyadic parent of I.

LSMP (continued)

Theorem (LSMP)

The operator $H_{\mathcal{D}}^{\mu}$, and its adjoint, map continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$ (or $L^p(\mu)$ to itself for all $1 < p < \infty$) if and only if μ is balanced.

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Additional assumptions on the measure in $\mathbb{R}^n$: μ is atomless, finite and nonzero on each cube, and infinite on each "quadrant". (Today: $n = 1$.)

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Definition (Dyadic Sparse Collection)

Let $\eta \in (0,1)$. A collection of dyadic intervals $\mathcal{S} \subsetneq \mathcal{D}$ is said to be η-sparse, if for each $I \in \mathcal{S}$, there exists a measurable subset E_I satisfying $\mu(E_I) \geq \eta\mu(I)$ and moreover the collection $\{E_J\}_{J\in\mathcal{S}}$ is pairwise disjoint . Equivalently (as long as μ is atomless), there exists a constant $\Lambda > 0$ so that for all $I \in \mathcal{D}$:

$$\sum_{\substack{J\subseteq I:\\ J\in\mathcal{S}}} \mu(J) \leq \Lambda\mu(I).$$

Picture of Sparse Collection

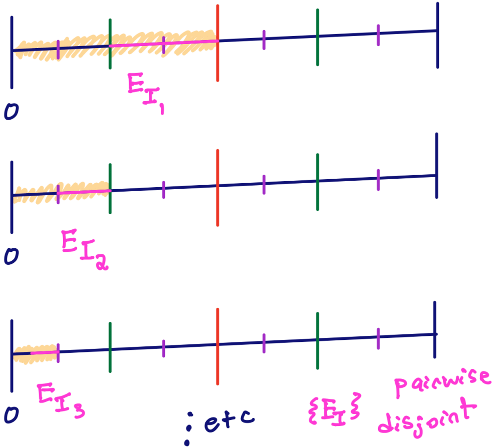

Fig 2.
An example of a sparse collection contained in $[0,1]$

Dyadic Shifts

Definition (Haar shift of complexity (s, t))

For integers $s, t > 0$, define

$$T^{s,t,\alpha} f(x) = \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}_s(I)} \sum_{K \in \mathcal{D}_t(I)} \alpha^I_{J,K} \langle f, h_J \rangle h_K.$$

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Pairing $\langle\cdot,\cdot\rangle$ (and integral averages $\langle\cdot\rangle_I$) are taken with respect to μ:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)d\mu(x), \quad \langle f\rangle_I = \frac{1}{\mu(I)}\int_I f(x)d\mu(x).$$

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- Question:

$$\langle | T^{s,t,\alpha} f, g \rangle | \lesssim \sum_{I \in \mathcal{S}} \langle f \rangle_I \langle g \rangle_I \, \mu(I) =: \mathcal{A}_{\mathcal{S}}(f,g)? \qquad \text{(Sparse)}$$

Here, $\mathcal{S}$ is sparse family of dyadic intervals and f, g non negative. This result is true in the Lebesgue (or doubling) measure case (Culiuic, Di Plinio, Ou)

Counterexamples

Theorem (Counterexample, CAPW)

There exists a balanced measure μ on $[0,1]$ s.t. $H_{\mathcal{D}}$ fails (Sparse).

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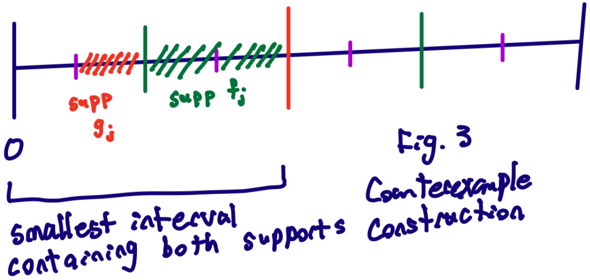
•[CAPW] also gives positive results: modified sparse domination result, and a suitably modified A_p class that is both necessary and sufficient. The modified sparse bound is in some sense "as useful" because it still allows for a proof of L^p and weak-type estimates (re-proving [LSMP] results).

Picture of Counterexample



Modification for a generic Haar shift

- Let $\alpha = \{\alpha^I_{J,K}\}_{I,J,K\in\mathcal{D}}$ be a bounded sequence, i.e., assume $\|\alpha\|_{\ell^\infty} \leq 1$. Recall, for $s, t \in \mathbb{N}$, a *dyadic shift of complexity* (s, t) is:

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- For a sparse $\mathcal{S} \subsetneq \mathcal{D}$ and $f, g \in L^2(d\mu)$ non-negative, set $\mathfrak{A}_{\mathcal{S}}(f,g) := \sum_{I \in \mathcal{S}} \langle f \rangle_I \langle g \rangle_I \mu(I)$.

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- We need new sparse forms associated to the collection $\mathcal{S}$ that reflect the complexity of the operator.

Definition (Dyadic Distance)

Define the *dyadic distance* between two intervals $I, J \in \mathcal{D}$ that share a common ancestor:

$$\operatorname{dist}_{\mathcal{D}}(I, J) := \min_{(s,t): I^{(s)} = J^{(t)}} (s + t).$$

Note that $\operatorname{dist}_{\mathcal{D}}(I, J) = 0$ if and only if $I = J$.

Definition (Sparse forms associated to higher complexities)

Given $\mathcal{S}$, $N \in \mathbb{N}$, and $f_1, f_2 \in L^2(d\mu)$, define:

$$\mathfrak{C}_{\mathcal{S}}^{N}(f_1, f_2) := \sum_{\substack{J,K \in \mathcal{S}: \\ 2 < \mathrm{dist}_{\mathcal{D}}(J,K) \leq N+2, \\ J \cap K = \emptyset}} \langle f_1 \rangle_J \langle f_2 \rangle_K \sqrt{m(J)}\sqrt{m(K)}.$$

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Theorem (CAPW)

Let μ be balanced, $\|\alpha\|_{\ell^\infty} \leq 1$ and $N \in \mathbb{Z}^+$. There exist constants $C > 0$ and $\eta \in (0,1)$ (independent of α) so that for each pair of compactly supported, bounded nonnegative functions f_1, f_2 there exists an η-sparse collection $\mathcal{S} \subset \mathcal{D}$ such that for any integers s, t satisfying $s + t \leq N$, we have the estimate

$$\left|\langle T^{s,t,\alpha} f_1, f_2\rangle\right| \leq C \left(\mathfrak{A}_{\mathcal{S}}(f_1, f_2) + \mathfrak{C}_{\mathcal{S}}^N(f_1, f_2)\right).$$

Hilbert Transform and Adjoint

Let μ be balanced. There exists $C > 0$ and $\eta \in (0, 1)$ so that for f_1, f_2, compactly supported, bounded nonnegative functions there exists an η-sparse collection $\mathcal{S} \subset \mathcal{D}$:

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$$|\langle H_{\mathcal{D}} f_1, f_2 \rangle| + |\langle H_{\mathcal{D}}^* f_1, f_2 \rangle| \leq C \Bigg(\sum_{I \in \mathcal{S}} \langle f_1 \rangle_I \langle f_2 \rangle_I \mu(I) + \sum_{\substack{I \in \mathcal{S}: \\ I_-^s \in \mathcal{S}}} \langle f_1 \rangle_I \langle f_2 \rangle_{I_-^s} m(I)$$

$$+ \sum_{\substack{I \in \mathcal{S}: \\ I_+^s \in \mathcal{S}}} \langle f_1 \rangle_I \langle f_2 \rangle_{I_+^s} m(I) + \sum_{\substack{I \in \mathcal{S}: \\ I_-^s \in \mathcal{S}}} \langle f_2 \rangle_I \langle f_1 \rangle_{I_-^s} m(I)$$

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Definition (Dyadic BMO)

b belongs to $\text{BMO}_\mathcal{D}$ (dyadic BMO) if

$$\|f\|_{\text{BMO}_\mathcal{D}} := \sup_{I \in D} \frac{1}{\mu(I)} \int_I |f - \langle f \rangle_{\widehat{I}}| \, d\mu < \infty.$$

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This is the same definition of BMO that arises from martingales.

Construction of Sparse Set

Let f_1, f_2 be bounded nonnegative functions supported on I_0. For $I \subset I_0$ dyadic, let $\mathcal{B}(I)$ denote the selected intervals in the CZ decomposition applied to $f_1 \mathbb{1}_I$ and $f_2 \mathbb{1}_I$ at heights $\lambda_1 = 16\langle f_1 \rangle_I$ and $\lambda_2 = 16\langle f_2 \rangle_I$.

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Set $\mathcal{S}_1 = \mathcal{B}(I_0) = \{I_k^0\}$. Repeat this process on each of the disjoint I_k^0. Set $\mathcal{S}_2 = \bigcup_{I \in \mathcal{S}_1} \mathcal{B}(I)$. In general,

$$\mathcal{S}_j = \bigcup_{I \in \mathcal{S}_{j-1}} \mathcal{B}(I) \text{ and set } \quad \mathcal{S} = \{I_0\} \cup \bigcup_{j=1}^{\infty} \mathcal{S}_j$$

Construction of Sparse Set

Let f_1, f_2 be bounded nonnegative functions supported on I_0. For $I \subset I_0$ dyadic, let $\mathcal{B}(I)$ denote the selected intervals in the CZ decomposition applied to $f_1 \mathbb{1}_I$ and $f_2 \mathbb{1}_I$ at heights $\lambda_1 = 16\langle f_1 \rangle_I$ and $\lambda_2 = 16\langle f_2 \rangle_I$.

Set $\mathcal{S}_1 = \mathcal{B}(I_0) = \{I_k^0\}$. Repeat this process on each of the disjoint I_k^0. Set $\mathcal{S}_2 = \bigcup_{I \in \mathcal{S}_1} \mathcal{B}(I)$. In general,

$$\mathcal{S}_j = \bigcup_{I \in \mathcal{S}_{j-1}} \mathcal{B}(I) \text{ and set } \quad \mathcal{S} = \{I_0\} \cup \bigcup_{j=1}^{\infty} \mathcal{S}_j$$

To prove sparse, using the fact that the intervals in the sum are pairwise disjoint:

$$\sum_{\substack{J \in \mathcal{S}_{j+1}: \\ J \subset I}} \mu(J) \leq \frac{\mu(I)}{16} \sum_{\substack{J \in \mathcal{S}_{j+1}: \\ J \subset I}} \left(\frac{\int_J f_1 \, d\mu}{\int_I f_1 \, d\mu} + \frac{\int_J f_2 \, d\mu}{\int_I f_2 \, d\mu} \right) \leq \frac{\mu(I)}{8}.$$

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- Recall the notation $\mathfrak{C}_{\mathcal{S}}^N(f_1, f_2)$ for the "extra terms" in our modified sparse form. We begin by recovering the main result in [LSMP]. For a balanced μ, we have:

Theorem (CAPW)

Given a balanced μ, a sparse collection $\mathcal{S}$, and $N \in \mathbb{N}$, there exists C_p depending only on p, the parameter η associated to $\mathcal{S}$, and the μ, s.t. for $f_1 \geq 0 \in L^p(d\mu)$ and $f_2 \geq 0 \in L^{p'}(d\mu)$:

$$\mathfrak{A}_{\mathcal{S}}(f_1, f_2) + \mathfrak{C}_{\mathcal{S},N}(f_1, f_2) \leq C_p \|f_1\|_{L^p(d\mu)} \|f_2\|_{L^{p'}(d\mu)}.$$

Proof of L^p Estimate

Reduction: To begin the proof, label the dyadic intervals $J \in \mathcal{D}(I, N+2)$, for which $J \cap I = \emptyset$ and $2 < \text{dist}_{\mathcal{D}}(I, J) \leq N+2$ as $c_1(I), c_2(I), \cdots, c_{N'}(I)$, and we need to show:

$$\sum_{\substack{I \in \mathcal{S}: \\ c_j(I) \in \mathcal{S}}} \langle f_1 \rangle_I \langle f_2 \rangle_{c_j(I)} \sqrt{m(I) m(c_j(I))} \leq C_p \|f_1\|_{L^p(d\mu)} \|f_2\|_{L^{p'}(d\mu)}.$$

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Because μ is balanced: $\sqrt{m(I) m(c_j(I))} \lesssim m(I)^{1/p} m(c_j(I))^{1/p'}$

and now proceed, using Hölder, and the dyadic maximal function, as in the usual sparse form.

Weights and modified maximal functions

To prove the weighted estimates, we need to define a new maximal function.

To that end, define (upper bounded) constants for I, J dyadic intervals and $1 \leq p < \infty$.

$$C_p(I,J) = \begin{cases} 1 & I = J \\ \left(\frac{m(I)^{p/2} m(J)^{p/2}}{\mu(J)\mu(I)^{p-1}}\right) & \text{otherwise.} \end{cases}$$

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Observe that

$$C_1(I,J) = \frac{\sqrt{m(I)m(J)}}{\mu(J)}$$

A Modified Dyadic Maximal Function

Definition (New Dyadic Maximal)

Given $N \in \mathbb{N}$, define the following maximal dyadic operator for $f \in L^1(d\mu)$:

$$M_{\mathcal{D}}^N f(x) := \sup_{\substack{I,J \in \mathcal{D}: \\ \operatorname{dist}_{\mathcal{D}}(I,J) \leq N+2}} C_1(I,J) \langle |f| \rangle_I \mathbb{1}_J(x)$$

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- $\frac{\sqrt{m(I)m(J)}}{\mu(J)} \lesssim \frac{m(J)}{\mu(J)} \lesssim 1$
- $M_{\mathcal{D}}^N$ is bounded on $L^p(d\mu)$ for $1 < p \leq \infty$ and is weak-type $(1,1)$.
- $M_{\mathcal{D}}^N$ admits a (modified) sparse domination.
- Can use sparse domination result plus weak-type bound for maximal operator to provide a new proof for the weak-type estimate for a dyadic shift wrt balanced measure .

Weighted estimates

- Let $w(x)\, d\mu(x)$ be an absolutely continuous, locally finite, positive measure w.r.t. μ.

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Definition

Let w be a weight on $\mathbb{R}$ w.r.t. μ, $p \in (1,\infty)$ and $N \in \mathbb{Z}^+$. We say $w \in A_p^N$ if

$$[w]_{A_p^N} := \sup_{\substack{I,J\in\mathcal{D}:\\ 0\le \mathrm{dist}_{\mathcal{D}}(J,K)\le N+2}} C_p(I,J)\left(\langle w\rangle_I \langle w^{1-p'}\rangle_J^{p-1}\right) < \infty.$$

Properties of Weights

- If $w \in A_p^N$, $[w]_{A_p} \leq [w]_{A_p^N}$ for any $N \in \mathbb{N}$, so that $A_p^N \subset A_p$. The containment is, in general, strict for $N \geq 1$ (recall previous example).

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- $w \in A_p^N$ if and only if $w^{1-p'} \in A_{p'}^N$.
- Rubia de Francia extrapolation does hold for these weight classes.
- There is also a natural endpoint class of weights when $p = 1$.

Main Weighted Theorem

Theorem (CAPW)

Let $p \in (1, \infty)$ and $N \in \mathbb{N}$. There exists a constant C such that all operators

$$T \in \bigcup_{\substack{s,t \in \mathbb{N}: \\ s+t \leq N}} HS(s,t)$$

satisfy

$$\|T\|_{L^p(w d\mu) \to L^p(w d\mu)} \leq C$$

if and only if $w \in A_p^b(\mu)$.

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if and only if $w \in A_p^b(\mu)$.

Also have weighted estimates for new maximal function and $p = 1$.

A New Operator and Measure Condition

- Domelevo, Kakaroumpas, Soler i Gibert, and Petermichl considered the following new dyadic analog of the Hilbert transform:

$$\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathcal{H}(h_I) = h_{I^s} \cdot \mathrm{sign}(I)$$

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- Unlike Ш, $\mathcal{H}$ satisfies $\mathcal{H}^2 = -I$.
- In the biparameter (but still Lebesgue) setting, showed that this operator can be used to characterize little BMO.
- Motivation: Can this operator be used to characterize dyadic BMO in the non-homogeneous setting?
- First: a new (slightly weaker) condition on the measure called *sibling balanced* actually characterizes $L^p(\mu)$ bounds for $\mathcal{H}$:

$$m(I) \sim m(I^s), \quad I \in \mathcal{D}.$$

Commutator Bounds

Somewhat surprisingly and subtly, the commutator characterization of dyadic BMO fails in the non-homogeneous setting!

Theorem (Borges, Conde Alonso, Pipher, W. 2024)

Let μ be sibling balanced, $b \in BMO_{\mathcal{D}}$ and $1 < p < \infty$.

- **Upper estimate:**

$$\|[\mathcal{H}, b]\|_{L^p(\mu) \to L^p(\mu)} \lesssim \|b\|_{BMO_{\mathcal{D}}}.$$

- **Lower estimate:**

$$\|b\|_{BMO_{\mathcal{D}}} \lesssim \|[\mathcal{H}, b]\|_{L^p(\mu) \to L^p(\mu)} + \sup_{k \in \mathbb{Z}} \|[\mathcal{H}, \mathsf{E}_k b]\|_{L^p(\mu) \to L^p(\mu)}.$$

- **Failure of lower estimate in general:** *The estimate*

$$\|b\|_{BMO_{\mathcal{D}}} \lesssim \|[\mathcal{H}, b]\|_{L^2(\mu) \to L^2(\mu)}$$

fails in general, even if we allow the implicit constant to depend on μ.

Weighted Estimates for Commutators with Dyadic Shifts

Key Point: The identification of a non-homogeneous weight class, which we call $\widehat{A}_p(\mu)$ that satisfies a reverse Hölder inequality and characterizes dyadic BMO.

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Theorem (Borges, Conde Alonso, Pipher, W. 2024)

Let μ be a locally finite Borel measure on $\mathbb{R}$ satisfying $\mu(I) > 0$ for all $I \in \mathcal{D}$.

(a) *Suppose $w \in \widehat{A}_2(\mu)$. Then the function $\log w \in BMO_{\mathcal{D}}$. Conversely, suppose $b \in BMO_{\mathcal{D}}$. Then for sufficiently small $\delta > 0$, the weight function $e^{\delta b}$ belongs to $\widehat{A}_2(\mu)$.*

(b) *Let $1 < p < \infty$ and $w \in \widehat{A}_p(\mu)$. Then there exists $\gamma > 1$, depending only on p and $[w]_{\widehat{A}_p(\mu)}$, so that for all $I \in \mathcal{D}$,*

$$\left(\frac{1}{\mu(I)} \int_I w^\gamma \, d\mu\right)^{1/\gamma} \lesssim_w \left(\frac{1}{\mu(I)} \int_I w \, d\mu\right).$$

Weighted Commutator Theorem

Theorem (Borges, Conde Alonso, Pipher, W. 2024)

Let μ be sibling balanced and atomless. Let $1 < p < \infty$, $b \in BMO_{\mathcal{D}}$, and $w \in \widehat{A}_p(\mu)$. Then there holds

$$\|[\mathcal{H}, b]\|_{L^p(\omega) \to L^p(w)} \lesssim_{[w]_{\widehat{A}_p(\mu)}, p} \|b\|_{BMO_{\mathcal{D}}}.$$

Thanks

- Many thanks to the organizers, specifically, Sergei Treil, for the kind invitation.
- Thanks to all of you for your attention!

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New CZD

Lemma

Let $f_1, f_2 \in L^1(d\mu)$ be nonnegative; supported on I and $\lambda_1, \lambda_2 > 0$. Then there exist g_j, b_j s.t. $f_j = g_j + b_j$ for $j = 1, 2$ and satisfying:

1. *There exist pairwise disjoint, dyadic intervals $\{I_k\} \subset \mathcal{D}(I)$ s.t. $b_j = \sum_{k=1}^{\infty} b_{j,k}$ where each $b_{j,k}$ is supported on $\widehat{I_k}$, $\int_{\widehat{I_k}} b_{j,k} = 0$, and $\|b_{j,k}\|_{L^1(d\mu)} \lesssim \int_{I_k} |f_j| \, d\mu$ for $j = 1, 2$.*
   So, for $j = 1, 2$ we have
   $$b_{j,k} = f_j \mathbb{1}_{I_k} - \langle f_j \mathbb{1}_{I_k} \rangle_{\widehat{I_k}} \mathbb{1}_{\widehat{I_k}}.$$
2. *For $j = 1, 2$, the function $g_j \in L^p(d\mu)$ for all $1 \leq p < \infty$ and satisfies $\|g_j\|_{L^p(d\mu)}^p \leq C_p \lambda_j^{p-1} \|f_j\|_{L^1(\mu)}$.*
3. *For $j = 1, 2$ the function $g_j \in BMO_{\mathcal{D}}$ and $\|g_j\|_{BMO_{\mathcal{D}}} \leq \lambda_j$.*

Iteration Lemma

Lemma

Suppose μ is balanced, $I_0 \in \mathcal{D}$, and that $T^{m,n}_{s,t,\alpha}$ as before. Set $N = s + t$. Let f_1, f_2 be bounded nonnegative functions supported on I_0. For any $I \subset I_0$ dyadic, let $\mathcal{B}(I)$ denote the selected intervals in the Calderón-Zygmund decomposition applied to $f_1 \mathbb{1}_I$ and $f_2 \mathbb{1}_I$ at heights $\lambda_1 = 16\langle f_1 \rangle_I$ and $\lambda_2 = 16\langle f_2 \rangle_I$, and let

$$\mathcal{G}(I) := \{J \in \mathcal{D}(I) : J \not\subset K \text{ for any } K \in \mathcal{B}(I)\}.$$

Then, the following estimate holds:

$$\left| \sum_{J \in \mathcal{G}(I)} \alpha^J_{J^n_s, J^m_t} \langle f_1, h_{J^m_s} \rangle \langle h_{J^n_t}, f_2 \rangle \right| \lesssim \left(\langle f_1 \rangle_I \langle f_2 \rangle_I \mu(I) + \sum_{\substack{S \in \mathcal{B}^s(I),\, T \in \mathcal{B}^t(I): \\ \mathrm{dist}_{\mathcal{D}}(S,T) \le N+2}} \langle f_1 \rangle_S \langle f_2 \rangle_T \sqrt{m(S)} \sqrt{m(T)} \right).$$