Empirical Methods for Dual Mixed Volumes

with G. Paouris and P. Pivovarov

Recall for a convex body K in $\mathbb{R}^n$, its m-th quermassintegral is defined by

$$V_{n-m}(K) = c_{n,m} \int_{G(n,n-m)} |P_H K| \, dH$$

Correspondingly, define the m-th dual quermassintegral of K by

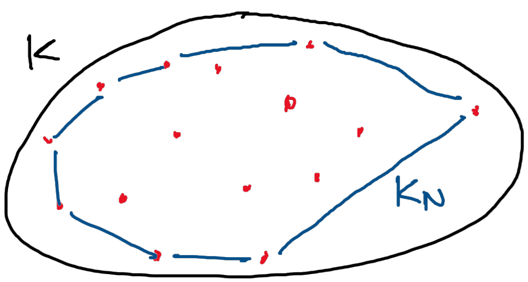
$$\tilde{V}_{n-m}(K) = c_{n,m} \int_{G(n,n-m)} |K \cap H| \, dH = c_{n,m} \int_{S^{n-1}} \rho_K^{n-m}(\theta) \, d\theta$$

These quantities come up in the context of Steiner formula:

Brunn - Minkowski Theory	Dual Brunn-Minkowski Theory
Let K be a convex body	Let K be a star body
Steiner formula:	Dual Steiner formula
$$V(K + tB_2^n) = \sum_{m=0}^{n} \binom{n}{m} V_{n-m}(K)\, t^m$$	$$V(K \tilde{+} tB_2^n) = \sum_{m=0}^{n} \binom{n}{n-m} \omega_m \tilde{V}_{n-m}(K)\, t^m$$
Minkowski sum	radial sum
$$V_{n-m}(K) = C_{n,m} \int_{G(n,n-m)} \lvert P_H K \rvert \, dH$$	$$\tilde{V}_{n-m}(K) = \tilde{C}_{n,m} \int_{G(n,n-m)} \lvert K \cap H \rvert \, dH$$
projection	section

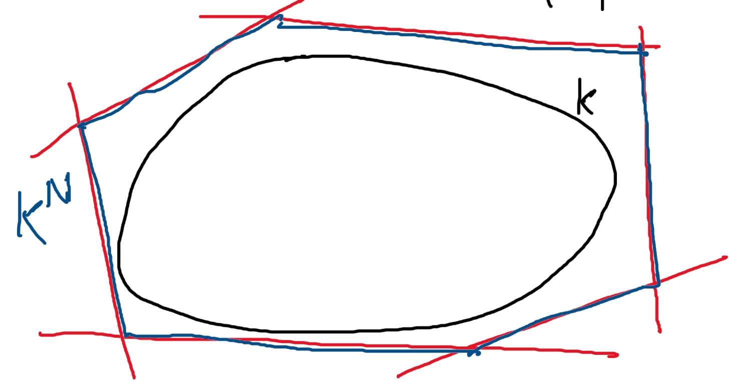
Many constructions in Brunn-Minkowski theory can be seen through a random convex approximations.

Examples: Convex hulls of randomly sampled points



Volumetric inequality in Busemann Random Simplex Inequality '53. See survey by Paouris-Pivovarov '14

Intersection of random half-spaces



Studying mean width in Böröczky-Schneider '10.

If K is a star body, can we have a random construction for $\tilde{V}_{n-m}(K)$?

We can interpret the quermassintegrals through Gaussian lens.

Let $G = [g_1 \cdots g_m]$, $g_1, \ldots, g_m$ be i.i.d Gaussians in $\mathbb{R}^n$.

Tsirelson '86 observed that since $\langle g_1, \ldots, g_m \rangle^\perp$ is uniformly distributed in $G(n, n-m)$,

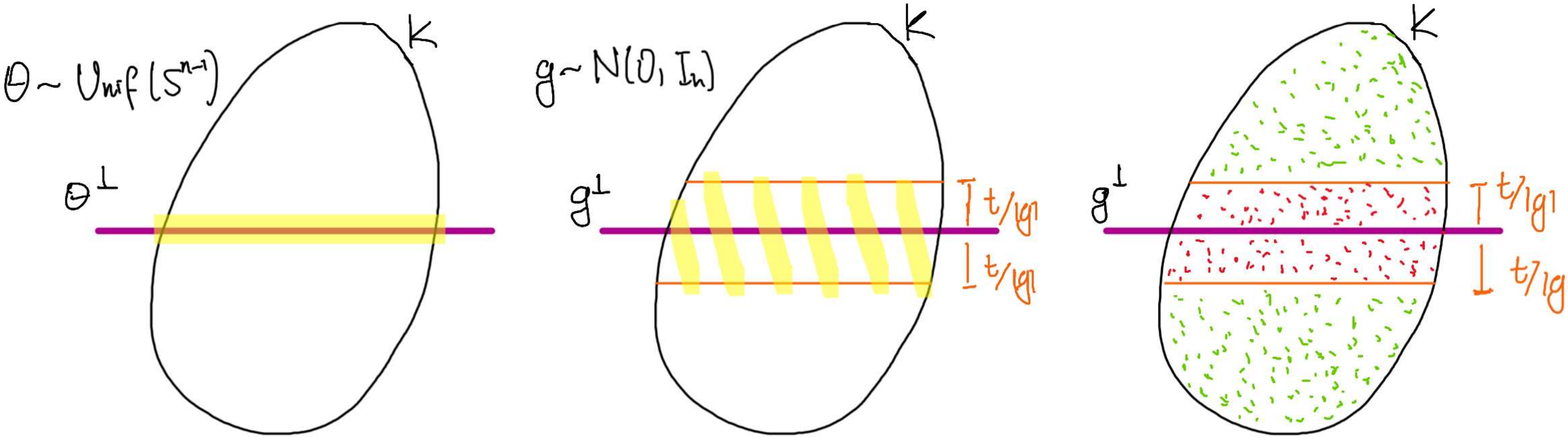
$$V_{n-m}(K) = \mathbb{E}\,|\det(G^T G)|^{1/2} \cdot \mathbb{E}\,|P_{(\mathrm{Im}\,G)^\perp} K|$$

By analogy, $\tilde{V}_{n-m}(K) = \mathbb{E}\,|\det(G^T G)|^{-1/2} \cdot \mathbb{E}\,|K \cap \ker G^T|$

Here $K \cap \ker G^T = \{ x \in K \mid \langle x, g_i \rangle = 0 \text{ for } i = 1, \ldots, m \}$

But how to approximate this section?

Sketch of Approximation for Codimension 1 Section:



Wanted: $|K \cap \theta^\perp|$

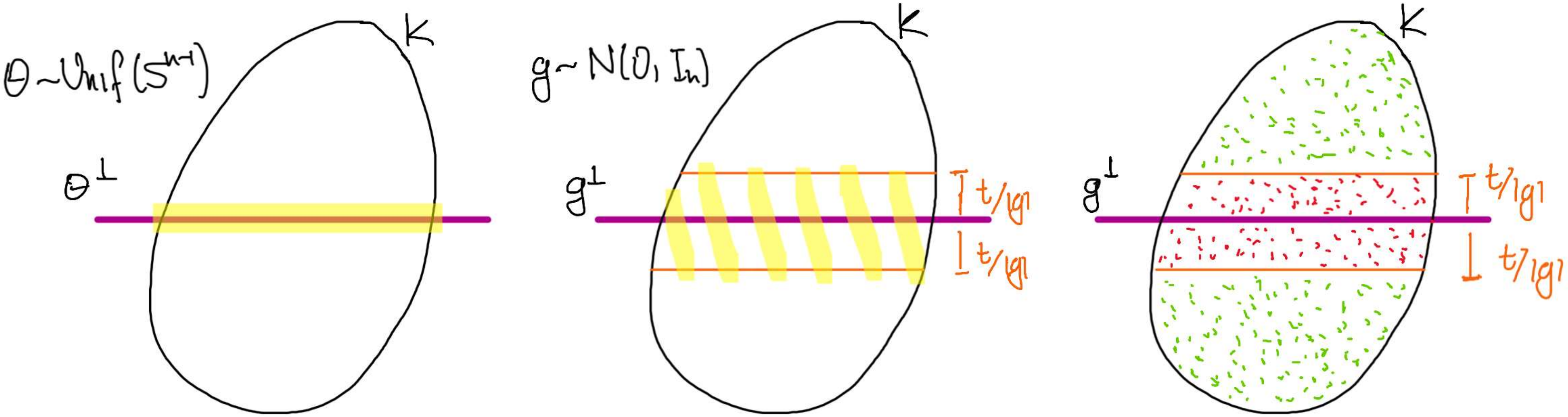
Thicken: $|K \cap H_{g,t}|$

Fact: $\frac{g}{|g|}$ is uniformly distributed on S^{n-1}.

Approximate:

$$\bar{\rho}_{t,N}(g) = \frac{|K|}{N} \sum_{i=1}^{N} \chi_{[-t,t]}(\langle X_i, g \rangle)$$

$X_i \sim Unif(K)$



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$X_i \sim \mathrm{Unif}(K)$

This thickening procedure was used by Anttilla-Ball-Perissinaki '00 to show CLT for sections of convex body.

Define the empirical m-th dual intrinsic volume of K

$$\tilde{V}_{n-m,t,N}(K) = \frac{|K|}{(2t)^m \, \mathbb{E}|\det(G^T G)|^{-1/2}} \, \mathbb{E}_g \left(\frac{1}{N} \sum_{i=1}^{N} \prod_{j=1}^{m} \chi_{[-t,t]}(\langle X_i, g_j \rangle) \right)$$

where $X_i \sim \mathrm{Unif}(K)$ independent samples

In more general view, we can replace K by a (bounded) probability distribution, so we consider $\tilde{V}_{n-m,t,N}(f)$.

This probabilistic view of geometric construction was started by Paouris '06 and has been applied to show sharp isoperimetric inequalities (e.g. Paouris - Pivovarov '13, Adamczak - Paouris - Pivovarov - S. '24)

Note that $\tilde{V}_{n-m,t,N}(K)$ is a random quantity, so we cannot compare different bodies with inequalities.

To do this, we need a notion of stochastic dominance:

a random variable ξ is stochastically dominated by a random variable η if $\mathbb{P}(\xi \geq t) \leq \mathbb{P}(\eta \geq t)$ for all t.

Then denote it by $\xi \prec \eta$

Immediate consequence: by layer cake representation,

$$\mathbb{E}|\xi|^p \leq \mathbb{E}|\eta|^p \text{ for all } p>0 \text{ (if the moment exists).}$$

Theorem 1: Let f be a bounded probability distribution on $\mathbb{R}^n$.

Then for each $t>0$ and N, $\tilde{V}_{n-m,t,N}(f) < \tilde{V}_{n-m,t,N}(f^*)$

Here f^* is the symmetric decreasing rearrangement of f: the unique distribution on $\mathbb{R}^n$ so that $\{f^* > s\}$ is an Euclidean ball with same measure as $\{f > s\}$.

Corollary: For a star body K in $\mathbb{R}^n$, $\tilde{V}_{n-m,t,N}(K) < \tilde{V}_{n-m,t,N}(K^*)$, where $K^* = rB_2^n$ so that $|K^*| = |K|$.

Note: actually $\tilde{V}_{n-m,t,N}(f)$ is (stochastically) monotone by Steiner symmetrization

By Law of Large Numbers,

$$\lim_{t \to 0} \lim_{N \to \infty} \tilde{V}_{n-m, t, N}(K) = \tilde{V}_{n-m}(K).$$

Thus Theorem 1 implies Lutwak's inequality for dual quermassintegrals

Theorem (Lutwak '75): let K be a star body in $\mathbb{R}^n$. For each $m \in [n]$,

$$\tilde{V}_{n-m}(K) \leq \tilde{V}_{n-m}(K^*)$$

These quantities admit isoperimetric-type inequalities:

Brunn - Minkowski Theory	Dual Brunn-Minkowski Theory
Let K be a convex body	Let K be a star body
Quermassintegral	Dual quermassintegral
$V_{n-m}(K) = c_{n,m} \int_{G(n,n-m)} \lvert P_H K \rvert \, dH$	$\tilde{V}_{n-m}(K) = \tilde{c}_{n,m} \int_{G(n,n-m)} \lvert K \cap H \rvert \, dH$
projection	section
(Consequence of) Alexandrov - Fenchel Inequality	Lutwak '75:
$V_{n-m}(K) \geq V_{n-m}(K^*)$	$\tilde{V}_{n-m}(K) \leq \tilde{V}_{n-m}(K^*)$
Proof by symmetrization	Proof by Hölder's inequality

Proof of Theorem 1:

Recall the definition

$$\tilde{V}_{n-m,t,N}(f) = \frac{|K|}{(2t)^m \, \mathbb{E}\,|\det(GG^T)|^{-1/2}} \underbrace{\mathbb{E}_g\left(\frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m} \chi_{[-t,t]}(\langle X_i, g_j\rangle)\right)}_{\text{we'll focus on this}},$$

$$X_i \sim f$$

Let $H_{X,t} = \{ y \in \mathbb{R}^n \mid |\langle y, X\rangle| \le t \} = \{ y \in \mathbb{R}^n \mid |\langle y, \frac{X}{|X|}\rangle| \le \frac{t}{|X|} \}$

Then

$$\mathbb{E}_g\left(\frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m} \chi_{[-t,t]}(\langle X_i, g_j\rangle)\right) = \sum_{i=1}^{N}\prod_{j=1}^{m} \gamma_n(H_{X_i,t}) = \sum_{i=1}^{N} \gamma_n^m(H_{X,t})$$

Gaussian measure in $\mathbb{R}^n$.

Lemma A: For each $t>0$, if $X_i \sim f$ and $X_i^* \sim f^*$,

then $\gamma_n(H_{X,t}) \leq \gamma_n(H_{X^*,t})$

Proof: Let $s>0$. By rotation invariance,

$$\mathbb{P}(\gamma_n(H_{X,t}) > s) = \mathbb{P}\left(\gamma_1\left(\left[-\frac{t}{|X|}, \frac{t}{|X|}\right]\right) > s\right)$$

But $s \leq \gamma_1\left(\left[-\frac{t}{|X|}, \frac{t}{|X|}\right]\right) = 2\Phi\left(\sqrt{2}\,\frac{t}{|X|}\right) - 1 \Rightarrow |X| \leq c_{s,t}$

So it's equivalent to showing $\mathbb{P}(X \in c_{s,t}B_2^n) \leq \mathbb{P}(X^* \in c_{s,t}B_2^n)$ for $s,t>0$.

Since $\mathbb{P}(X \in c_{s,t}B_2^n) = \int_{\mathbb{R}^n} \chi_{c_{s,t}B_2^n}(x) f(x)\,dx$, this inequality

follows from Riesz rearrangement inequality.

Lemma B: Suppose $\xi_i, \eta_i \geq 0$ and $\xi_i \prec \eta_i$ for $i = 1, \dots, n$. Then for $t_1, \dots, t_n \geq 0$,

$$\sum_{i=1}^{n} t_i \xi_i \prec \sum_{i=1}^{n} t_i \eta_i .$$

Pf'chen: By independence of the ξ_i's and η_i's, we can condition one at a time: for $s > 0$,

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} t_i \xi_i > s \right) &= \mathbb{E}_{\xi_2, \dots, \xi_n} \mathbb{P}\left(\xi_1 \geq s - t_2 \xi_2 - \cdots - t_n \xi_n \right) \\
&\leq \mathbb{E}_{\xi_2, \dots, \xi_n} \mathbb{P}\left(\eta_1 \geq s - t_2 \xi_2 - \cdots - t_n \xi_n \right) \\
&\leq \cdots \\
&\leq \mathbb{P}\left(\sum_{i=1}^{n} t_i \eta_i > s \right)
\end{aligned}$$

Now we can prove Theorem 1:

Recall $\tilde{V}_{n-m,t,N}(K) = \dfrac{|K|}{(2\pi)^m \, \mathbb{E}\,|\det(GG^T)|^{1/2}} \cdot \dfrac{1}{N} \sum_{i=1}^{N} \gamma_n^m (H_{X_{i,t}})$

By Lemma A, $\gamma_n^m(H_{X_{i,t}}) < \gamma_n^m(H_{X^*_{i,t}})$ $i = 1, \dots, N$

By Lemma B, we can pass the stochastic dominance to linear combination.

Reverse the identity to finish.

We can use Steiner formulas for other bodies

Brunn - Minkowski Theory

Let K, L be convex bodies in $\mathbb{R}^n$.

Steiner formula:

$$V(K + tL) = \sum_{m=0}^{n} \binom{n}{m} V_{n-m}(K,L)\, t^m$$

Minkowski sum

Mixed volume as coefficients, in some cases can be expressed in terms of projections

Dual Brunn-Minkowski Theory

Let K, L be star bodies

Dual Steiner formula

$$V(K \tilde{+} tL) = \sum_{m=0}^{n} \binom{n}{n-m} \omega_m \tilde{V}_{n-m}(K,L)\, t^m$$

radial sum

Dual mixed volume

$$\tilde{V}_{n-m}(K,L) = \tilde{c}_{n,m} \int_{S^{n-1}} \rho_K^{n-m}(\theta)\, \rho_L^{m}(\theta)\, d\theta$$

Next we'll see how to adapt the set-up and proof of Theorem 1 for dual mixed volumes.

Recall the definition: for $m \in [n]$ and K, L star bodies in $\mathbb{R}^n$,

$$\tilde{V}_{n-m}(K,L) = \int_{S^{n-1}} \rho_K^{n-m}(\theta)\, \rho_L^{m}(\theta)\, d\theta$$

For $t>0$ and N, define empirical m-th dual mixed volume of K & L by

$$\tilde{V}_{n-m,t,N}(K,L) = \frac{C_{K,L}}{(2t)^m\, \mathbb{E}|\det(GG^T)|^{1/2}}\, \mathbb{E}_g\left(\frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m} \chi_{[-t,t]}(\langle X_i, g_j\rangle)\right),$$

where $X_i \sim \chi_K(x)\cdot \rho_L^m\left(\frac{x}{|x|}\right)$ independent samples

Where does this definition come from?

rough iterating subspaces and the fact that $\langle g_1, \dots, g_m \rangle^{\perp}$ is uniform in $G(n, n-m)$,

$$\tilde{V}_{n-m}(K, L) = C_{n,m} \, \mathbb{E}_g \int_{\langle g_1, \dots, g_m \rangle^{\perp}} \rho_L^m \left(\frac{x}{|x|} \right) \chi_K(x) \, dx$$

Now we can apply the thickening procedure as before, with the samples now are weighted according to radial function of L.

Theorem 2: let K, L be star bodies in $\mathbb{R}^n$. Then

$$\tilde{V}_{n-m,t,N}(K,L) \leq \tilde{V}_{n-m,t,N}(K^*, L^*).$$

Since $\lim_{t\to\infty} \lim_{N\to\infty} \tilde{V}_{n-m,t,N}(K,L) = \tilde{V}_{n-m}(K,L)$, we recover another Lutwak's classical inequality.

Theorem (Lutwak '75): $\tilde{V}_{n-m}(K,L) \leq \tilde{V}_{n-m}(K^*, L^*)$.

As before, this inequality find its counterpart in (non-dual) Brunn-Minkowski theory with a different type of proof.

We should be able to adapt the proof of Theorem 1 to Theorem 2

The main difference in $\tilde{V}_{n-m,t,N}$ is the sampling:

we sampled from K weighted by the the radius of L.

Thus taking f in Theorem 1 to be $\rho_L\left(\frac{x}{|x|}\right)\chi_k(x)$ gives us

$$\tilde{V}_{n-m,t,N}\left(\rho_L\left(\frac{\cdot}{|\cdot|}\right)\chi_k(\cdot)\right) < \tilde{V}_{n-m,t,N}\left(\left(\rho_L\left(\frac{\cdot}{|\cdot|}\right)\chi_k(\cdot)\right)^*\right)$$

To end at Euclidean balls, we need an application of the Bathtub Principle

One application of Theorem 2 is to get an empirical version of dual Brunn-Minkowski theorem:

Theorem 3: Let K, L be star bodies in $\mathbb{R}^n$. Define

$$\tilde{V}_{t,N}(K \tilde{+} L) = \sum_{m=0}^{n} \frac{C_{K,L}^{(m)}}{(2t)^m \, \mathbb{E}\left|\det\left(G^{(m)}(G^{(m)})^T\right)\right|^{1/2}} \cdot \mathbb{E}_g\left(\frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m}\chi_{[-t,t]}\left(\langle X_i^{(m)}, g_j\rangle\right)\right)$$

where $G^{(m)} = [g_1 \cdots g_m]$, $X_i^{(m)} \sim \rho_L^m\left(\frac{x}{|x|}\right)\chi_K(x)$ independent samples

Then $\tilde{V}_{t,N}(K \tilde{+} L) \le \tilde{V}_{t,N}(K^* \tilde{+} L^*)$.

An interesting quantity is the n-th moment of the section functionals.

Define the dual m-th quermassintegral of K by

$$\Phi_m(K) = \left(\int_{G(n,n-m)} |K \cap H|^n \, dH \right)^{\frac{1}{m}}$$

If $m=1$, $\Phi_1(K) = \Phi(K)$ gives the volume of IK, the intersection body of K

It is known that IK admits an isoperimetric inequality

Theorem 4 (Busemann Intersection Inequality '53): for star body K in $\mathbb{R}^n$,

$$|IK| \leq |I(K^*)|$$

Busemann Intersection Inequality has only several known proofs.

- The original proof by Busemann using Busemann random simplex inequality (symmetrization on slices)
- Using a family of inequalities for random dual centroid bodies (Adamczak - Paouris - Pivovarov - S. '24).
- Through studying the action of continuous Steiner symmetrizations (Milman - Shabelman - Yehudayoff '24)

Here we will give a new proof.

Define $\Phi_{t,N}(K) = \left[\frac{|K|}{(2t)^n \, \mathbb{E}\left(|g|^{-n} \chi_{t<|g|<1/t}\right)} \mathbb{E}_g \underbrace{\left(\frac{1}{N} \sum_{i=1}^{N} \chi_{[-t,t]}(\langle X_i, g\rangle) \right)}_{\bar{\rho}_{t,N}(g)}^n \right]^{1/n}$,

$X_i \sim \mathrm{Unif}(K)$ independent samples

By Law of Large Numbers, $\lim_{N\to\infty} \bar{\rho}_{t,N}(g) = |K \cap H_{g,t}|$ ← slab $\{|\langle x, g\rangle| \le t\}$

Proposition: for $0<p\le n$,

$$\lim_{t\to 0} \mathbb{E}\left[\left(\frac{|K\cap H_{g,t}|}{2t} \right)^p \cdot \frac{1}{\mathbb{E}\left(|g|^{-p} \chi_{t<|g|<1/t}\right)} \right] = \mathbb{E}\, |K \cap g^\perp|^p$$

Note on proof: we used the Lipschitz continuity of ρ_K, but since Gaussian doesn't have the $(-n)$-th moment, we cannot get $p<-n$.

Last, we need to show

$$\mathbb{E}_{\underline{X}} \mathbb{E}_g \left(\bar{\rho}_{t,N}(g)\right)^n \le \mathbb{E}_{\underline{X}} \mathbb{E}_g \left(\bar{\rho}^{*}_{t,N}(g)\right)^n$$

i.e.

$$\boxed{\mathbb{E}_{\underline{X}} \mathbb{E}_g \left(\frac{1}{N} \sum_{i=1}^{N} \chi_{[-t,t]}(\langle X_i, g\rangle)\right)^n \le \mathbb{E}_{\underline{X}} \mathbb{E}_g \left(\frac{1}{N} \sum_{i=1}^{N} \chi_{[-t,t]}(\langle X_i^*, g\rangle)\right)^n}$$

with $X_i \sim \mathrm{Unif}(K)$, $X_i^* \sim \mathrm{Unif}(K^*)$

If we expand the LHS, a typical term looks like

$$\mathbb{E}_{\underline{X}} \mathbb{E}_g \prod_{i=1}^{k} \chi_{[-t,t]}(\langle X_i, g\rangle) = \mathbb{E}_{\underline{X}} \mathbb{E}_g \, \chi_{[-t,t]^k}\Big(\underbrace{[X_1 \cdots X_k]^T}_{\underline{X}^T} g\Big) \qquad k \in [n]$$

$$= \mathbb{E}_{\underline{X}} \gamma_n\left(\left(t^{-1} \underline{X} B_1^{k}\right)^{\circ}\right)$$

Gaussian measure in $\mathbb{R}^n$

A symmetrization theorem (Cordero-Erausquin, Fradelizi, Paouris, Pivovarov '15):

Let $X_1, \dots, X_N \sim f$ probability density

$X_1^*, \dots, X_N^* \sim f^*$ symmetric decreasing rearrangement of f

C · origin-symmetric convex body

μ radial measure w/ decreasing density

Then $\mathbb{E}_{\mathcal{X}}\, \mu\left(\left([X_1 \cdots X_N]C\right)^\circ\right) \leq \mathbb{E}_{\mathcal{X}}\, \mu\left(\left([X_1^* \cdots X_N^*]C\right)^\circ\right)$

This result is a part of a large family of symmetrization inequality called the Brascamp-Lieb-Luttinger inequalities. Probably the most known example is the Riesz-Sobolev/Riesz rearrangement inequality.

$$\boxed{\text{Thm (CEFPP '15): } \mathbb{E}_{\underline{X}}\, \mu\Big(([X_1 \cdots X_N] C)^\circ \Big) \le \mathbb{E}_{\underline{X}}\, \mu\Big(([X_1^* \cdots X_N^*] C)^\circ \Big)}$$

Apply the theorem with $f = \chi_K$ (so $f^* = \chi_{K^*}$)

$C = B_1^k$

$\mu = \gamma_n$.

Then
$$\mathbb{E}_{\underline{X}} \mathbb{E}_g \left(\bar{\rho}_{t,N}(g) \right)^n = \mathbb{E}_{\underline{X}} \frac{1}{N^n} \sum_{k=1}^{n} \sum_{I=(I_1 \cdots I_k)} \gamma_n \left(\left(t^{-1} \underline{X}_I B_1^k \right)^\circ \right)$$

$$\le \mathbb{E}_{\underline{X}} \frac{1}{N^n} \sum_{k=1}^{n} \sum_{I=(I_1, \dots, I_k)} \gamma_n \left(\left(t^{-1} \underline{X}_I^* B_1^k \right)^\circ \right)$$

$$= \mathbb{E}_{\underline{X}} \mathbb{E}_g \left(\bar{\rho}_{t,N}^*(g) \right)^n .$$