ICERM: Harmonic Analysis and Convexity 12/9/24

Dimension-free discretizations of the uniform norm

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Based on the work:

Lars Becker, Ohad Klein, J. S., Alexander Volberg, and Haonan Zhang, *Dimension-free discretizations of the uniform norm by small product sets*, Invent. Math. (to appear).

Motivation. Harmonic analysis on $\mathbb{Z}_K^n$.

$$\begin{aligned} n\text{-torus} \quad & \mathbb{T}^n := \{z \in \mathbb{C} : |z| = 1\}^n \\ \text{Discrete } n\text{-torus} \quad & \Omega_K^n := \{e^{2\pi i k/K} : k = 0, 1, \ldots, K-1\}^n \\ \text{Hypercube} \quad & \Omega_2^n \; = \{-1, 1\}^n \end{aligned}$$

Bohnenblust–Hille-type inequality

Let $f : \Omega_K^n \to \mathbb{C}$ have $\deg(f) = d$. Then

$$\|\widehat{f}\|_{\frac{2d}{d+1}} \lesssim_{d,K} \|f\|_{\Omega_K^n} .$$

Bounded homogeneous projection

Let $f : \Omega_K^n \to \mathbb{C}$ have $\deg(f) = d$. Then

$$\|f_\ell\|_{\Omega_K^n} \lesssim_{d,K} \|f\|_{\Omega_K^n} ,$$

where f_ℓ is the ℓ-homogeneous part of f.

A silver bullet. Suppose we had for $f : \mathbb{T}^n \to \mathbb{C}$ of degree d and individual degree $K-1$ that

$$\|f\|_{\mathbb{T}^n} \lesssim_{K,d} \|f\|_{\Omega_K^n}. \qquad (*)$$

Then as corollaries:

Cyclic-group Bohnenblust–Hille:

$$\|\widehat{f}\|_{\frac{2d}{d+1}} \overset{\text{[BH31]}}{\lesssim_{d,K}} \|f\|_{\mathbb{T}^n} \overset{(*)}{\lesssim_{d,K}} \|f\|_{\Omega_K^n}.$$

Bounded ℓ-homogeneous projection:

$$\|f_\ell\|_{\Omega_K^n} \leq \|f_\ell\|_{\mathbb{T}^n} \overset{\text{(Cauchy est.)}}{\leq} \|f\|_{\mathbb{T}^n} \overset{(*)}{\lesssim_{d,K}} \|f\|_{\Omega_K^n}.$$

Let us prove $(*)$.

Theorem (Bernstein-type discretization inequality for Ω_K^n).

$f : \mathbb{T}^n \to \mathbb{C}$ a polynomial, $\deg(f) \leq d$.

$$\|f\|_{\mathbb{T}^n} \leq \mathcal{O}(\log K)^d \|f\|_{\Omega_K^n},$$

where $K - 1 \leq d$ is the *individual degree* of f.

Proposition. (1-D Interpolation)

Fix $K \geq 0$ and $z \in \mathbb{T}$. There are complex $a_\omega^{(z)}$'s with

$$f(z) = \sum_{\omega \in \Omega_K} a_\omega^{(z)} f(\omega) \qquad \text{and} \qquad \sum_{\omega} |a_\omega^{(z)}| \leq \mathcal{O}(\log K)\,.$$

for any polynomial $f : \mathbb{T} \to \mathbb{C}$, $\deg(f) < K$.

Proof. Lagrange interpolation:

$$f(z) = \sum_{\omega \in \Omega_K} \left(\prod_{\xi \in \Omega_K : \xi \neq \omega} \frac{z - \xi}{\omega - \xi} \right) f(\omega)\,.$$

□

So, with $z^* \in \mathbb{T}$ a maximizer of $|f|$,

$$\|f\|_{\mathbb{T}} = |f(z^*)| = \left| \textstyle\sum_{\omega \in \Omega_K} a_\omega^{(z)} f(\omega) \right| \leq \|a^{(z)}\|_1 \|f\|_{\Omega_K} \leq \mathcal{O}(\log K) \|f\|_{\Omega_K}\,.$$

Naive induction.

Repeat coordinatewise:

$$f(z) = \sum_{\omega_1 \in \Omega_K} \cdots \sum_{\omega_n \in \Omega_K} \left(\prod_{j=1}^{n} a_j^{(z)}(\omega_j) \right) f(\omega_1, \ldots, \omega_n)$$

yields

$$\|f\|_{\mathbb{T}^n} \leq \mathcal{O}(\log K)^n \|f\|_{\Omega_K^n}.$$

Exponential dependence on n...

Idea: d-wise independent distributions "fool" degree-d polynomials.

Goal: Find probabilistic expression for interpolation.

Originally: $f(z) = \sum_{\omega \in \Omega_K} a_\omega^{(z)} f(\omega).$

If $a_\omega^{(z)} \subseteq \mathbb{R}^{\geq 0}$ $\quad f(z) = C \mathbb{E}_{\boldsymbol{w}} f(\boldsymbol{w}) \quad \boldsymbol{w} \in \Omega_K$

If $a_\omega^{(z)} \subseteq \mathbb{R}$ $\quad f(z) = C \mathbb{E}_{\boldsymbol{w},\boldsymbol{r}} \boldsymbol{r} f(\boldsymbol{w}) \quad \boldsymbol{w} \in \Omega_K, \boldsymbol{r} \in \{1, -1\}$

If $a_\omega^{(z)} \subseteq \mathbb{C}$ $\quad f(z) = C \mathbb{E}_{\boldsymbol{w},\boldsymbol{r}} \boldsymbol{r} f(\boldsymbol{w}) \quad \boldsymbol{w} \in \Omega_K, \boldsymbol{r} \in \{1, i, -1, -i\}.$

Lemma. (1-D Interpolation, probabilistic)

There is $r : [0,1] \to \Omega_4$ s.t. for any $d \geq 0$ and $z \in \mathbb{C}$, there is $w : [0,1] \to \Omega_K$ s.t. for any polynomial $f : \mathbb{C} \to \mathbb{C}$, $\deg(f) < K$,

$$f(z) = C \mathop{\mathbb{E}}_{T \sim [0,1]} \Big[r(T) f\big(w(T)\big) \Big] \quad \text{and} \quad 0 < C \leq \mathcal{O}(\log K).$$

Naive induction (probabilistic).

With $\boldsymbol{z} \in \mathbb{T}^n$ a maximizer of $|f|$,

$$f(\boldsymbol{z}) = C^n \mathop{\mathbb{E}}_{T \sim \mathcal{U}[0,1]^n} \Big[r_1(T_1) \cdots r_n(T_n) \cdot f\big(w_1(T_1), w_2(T_2), \ldots, w_n(T_n)\big) \Big]$$

yields

$$|f(\boldsymbol{z})| \leq \mathcal{O}(\log K)^n \|f\|_{\Omega_d^n}.$$

New problem: integrand is no longer low-degree!

The idea. Instead of using

$$T_1, \ldots, T_n \overset{\text{iid}}{\sim} \mathcal{U}[0,1]$$

to evaluate

$$\mathop{\mathbb{E}}_{T_1,\ldots,T_n}\Big[r_1(T_1)\cdots r_n(T_n)\cdot f\big(w_1(T_1), w_2(T_2), \ldots, w_n(T_n)\big)\Big],$$

consider only m-many $T_1, \ldots, T_m \overset{\text{iid}}{\sim} \mathcal{U}[0,1]$ for $m \ll n$.

Sample unif. random map $P : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ to obtain

$$R_m := \prod_{j=1}^m r(T_j) \qquad \text{and} \qquad W_m := \big(w_1(T_{P(1)}), w_2(T_{P(2)}), \ldots, w_n(T_{P(n)})\big)$$

and consider $\mathbb{E}[R_m f(W_m)]$.

Result:

$$f(\mathbf{z}) = C^m\, \mathbb{E}[R_m f(W_m)] + \text{error} = C^m\, \mathbb{E}[R_m f(W_m)] + p(\tfrac{1}{m})$$

with $\deg(p) < d$ and $p(0) = 0$.

Proof. Argue for single monomial: $\boldsymbol{z}^\alpha = C^m\,\mathbb{E}[RW^\alpha] + p\left(\frac{1}{m}\right)$; that is,

$$\mathbb{E}[RW^\alpha] = C^{-m}\left(\boldsymbol{z}^\alpha - p\left(\tfrac{1}{m}\right)\right).$$

Notation

$$\boldsymbol{z}^\alpha := \prod_{j=1}^{n} z^{\alpha_j}, \quad \alpha \in \{0, 1, \ldots, K-1\}^n$$

Condition on choice of function $P : [n] \to [m]$. P *induces* partition S of $\mathrm{supp}(\alpha)$ in the natural way:

Example on board.

Notation

$P \rightarrowtail S \;:\; P$ induces S.

Goal: $\mathbb{E}[RW^\alpha] = C^{-m}\left(\boldsymbol{z}^\alpha - p\left(\frac{1}{m}\right)\right).$

Case 1. P induces the "singleton partition" $S^* := \big\{\{j\} : j \in \operatorname{supp}(\alpha)\big\}$.

$$\mathbb{E}[RW^\alpha | P \rightarrowtail S^*] = C^{-m}\boldsymbol{z}^\alpha$$

Case 2. P induces some $S \neq S^*$...only easy to say for all $m \geq |S|$,

$$\mathbb{E}[RW^\alpha | P \rightarrowtail S \neq S^*] =: C^{-m} \cdot \operatorname{stuff}(S),$$

where stuff(S) is <u>independent of m.</u>

Goal: $\mathbb{E}[RW^\alpha] = C^{-m}\left(\mathbf{z}^\alpha - p\left(\frac{1}{m}\right)\right).$

However, $\Pr[P \rightarrowtail S]$ is nice. For any partition S,

$$\Pr[P \rightarrowtail S] = \frac{m(m-1)\cdots(m-|S|+1)}{m^{|\mathrm{supp}(\alpha)|}} =: \begin{cases} 1 + q_{|S^*|}\left(\frac{1}{m}\right) & S = S^*, \\ q_{|S|}\left(\frac{1}{m}\right) & S \neq S^* \end{cases}$$

and $\mathbf{deg}(q_{|S|}) < d$, $q_{|S|}(0) = 0$.

$$\mathbb{E}[RW^\alpha] = C^{-m}\mathbf{z}^\alpha\left(1 + q_{|S^*|}\left(\tfrac{1}{m}\right)\right) + C^{-m}\sum_{S \neq S^*} \mathrm{stuff}(S)\, q_{|S|}\left(\tfrac{1}{m}\right)$$
$$=: C^{-m}\left(\mathbf{z}^\alpha - p\left(\tfrac{1}{m}\right)\right). \qquad \square$$

Proposition. (Interpolation with error)

Given f, $\mathbf{z}$, there is

- $p = p_{f,z} : \mathbb{R} \to \mathbb{R}$ a polynomial

such that for any $m \in \mathbb{Z}^+$, there are dependent random variables

- $R = R_m$ on $\mathbf{\Omega}_4$
- $W = W_{\mathbf{z},m}$ on $\mathbf{\Omega}_K^n$

$$f(\mathbf{z}) = C^m\, \mathbb{E}[R\, f(W)] + p\left(\tfrac{1}{m}\right)$$

for $C = \mathcal{O}(\log K)$. Moreover, $\deg(p) < d$ and $p(0) = 0$.

How to remove the error: Consider a linear combination of $m = 1, 2, \ldots, d$.

Have:

$$f(\mathbf{z}) = C^m\,\mathbb{E}[R\,f(W)] + p\left(\tfrac{1}{m}\right)\ .$$

There are $b_1, \dots, b_d \in \mathbb{C}$ such that for any polynomial p with $\deg(p) < d$ and $p(0) = 0$,

$$\sum_{m=1}^{d} b_m p\left(\tfrac{1}{m}\right) = 0, \qquad \sum_{m=1}^{d} b_m = 1, \qquad \text{and} \qquad \|b\|_\infty \lesssim \exp(d)\,.$$

Thus

$$\begin{aligned} f(\mathbf{z}) = \sum_{m=1}^{d} b_m f(\mathbf{z}) &= \sum_{m=1}^{d} b_m \Big(C^m\,\mathbb{E}[R\,f(W)] + p\left(\tfrac{1}{m}\right)\Big) \\ &= \sum_{m=1}^{d} b_m C^m\,\mathbb{E}[R\,f(W)] \\ &\leq \textstyle\sum_{m=1}^{d} |b_m C^m|\,\|f\|_{\Omega_K^n} \ \leq\ \mathcal{O}(\log K)^d \|f\|_{\Omega_K^n}\,. \end{aligned}$$

□

Theorem (Becker, Klein, S., Volberg, Zhang).

Consider $Y = X_1 \times X_2 \times \ldots X_n \subset \mathbb{D}^n$ with

- $|X_j| = K$ for all $j = 1, \ldots, n$
- All $y \neq y' \in Y$ have $\|y - y'\|_\infty \geq \eta > 0$

Then for any f with $\mathrm{deg}(f) \leq d$ and individual degree $K - 1$,

$$\|f\|_{\mathbb{D}^n} \;\leq\; C(K, \eta)^d \|f\|_Y \,.$$

- Holds in the real category too
- About the constant:
  - Exponential dependence on degree d is necessary
  - Correct dependence on K not understood
- About the cardinality $|Y|$
  - Exponential dependence on dimension n necessary
  - Y can be subsampled down to $|\widetilde{Y}| = (1 + \varepsilon)^n \cdot C(\varepsilon, d)$

For more, see:

For a concise proof of the Ω_K^n case:

Section 2 of Klein, S., Volberg, and Zhang, *Quantum and classical low-degree learning via a dimension-free Remez inequality*. ITCS 2024, TQC 2024. `arXiv:2301.01438`

Full version (with extensions and discussion):

Becker, Klein, S., Volberg, and Zhang, *Dimension-free discretizations of the uniform norm by small product sets*, Invent. Math. (to appear). `arXiv:2310.07926`

⚠ Out of date, to be updated very shortly!