Bohnenblust–Hille inequalities: from classical to quantum and back

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Joint work with Alexander Volberg (Michigan State University) and Joseph Slote (Caltech)

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History: Bohnenblust–Hille inequalities for $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

Bohr's problem on the convergence of Dirichlet series 1913

$$\vdots$$

(polynomial) Bohnenblust–Hille inequalities 1931

$$\|\widehat{f}\|_{\ell^{\frac{2d}{d+1}}} \leq C_d \|f\|_{L^\infty(\mathbb{T}^n)}$$

for all degree-d analytic polynomial $f : \mathbb{T}^n \to \mathbb{C}$

(d-linear BH inequalities; going back to Littlewood 1930)

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. . .

functional analysis/harmonic analysis...(Sidon constant, Bohr's radius...)

Book: Dirichlet Series and Holomorphic Functions in High Dimensions, Defant et al. 2019

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$$\Downarrow$$

replace $\mathbb{T}$ with $\{-1, 1\}$?

Plan

$\{\pm 1\}$

Plan

$$\begin{tikzcd}
\{\pm 1\} \arrow[d, dashed] \\
\Omega_K
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Motivation

How many samples do we need to recover $f : \{-1,1\}^n \to \{-1,1\}$ nicely?

- Might be too ambitious to consider arbitrary functions
- Some structures are needed: structures $\approx$ low complexity
- In this talk: low complexity $\approx$ low degree

Same question if we replace f with $A \in M_2(\mathbb{C})^{\otimes n}$?

Notation: classical VS quantum

	Classical	Quantum
functions	$f:\{-1,1\}^n \to \mathbb{C}$	$A \in M_2(\mathbb{C})^{\otimes n} = M_{2^n}(\mathbb{C})$
Fourier expansion	$f = \sum_{S\subset\{1,\ldots,n\}} \widehat{f}(S)\chi_S$ $\chi_S(x) := \prod_{j\in S} x_j$	$A = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \widehat{A}_{\mathbf{s}}\sigma_{\mathbf{s}}$ $\sigma_{\mathbf{s}} := \sigma_{s_1}\otimes\cdots\otimes\sigma_{s_n}$
degree-d	$\widehat{f}(S) = 0$ if $\lvert S\rvert > d$ $\lvert S\rvert :=$cardinality of S	$\widehat{A}_{\mathbf{s}} = 0$ if $\lvert\mathbf{s}\rvert > d$ $\lvert\mathbf{s}\rvert := \lvert\{j : s_j \neq 0\}\rvert$
probability measure	uniform	normalized (trace)
Parseval's identity	$\lVert f\rVert_2 = \lVert\widehat{f}\rVert_2$	$\lVert A\rVert_2 = \lVert\widehat{A}\rVert_2$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Learning: random query model

Our (bounded) functions with low complexity

$$\mathcal{F}_n^{\leq d} := \{f : \{-1,1\}^n \to [-1,1] \text{ degree-}d\}\,.$$

Fix $\epsilon, \delta > 0$. Let $N(\mathcal{F}_n^{\leq d}, \epsilon, \delta)$ be the least $N > 0$ such that for any

- function $f \in \mathcal{F}_n^{\leq d}$
- i.i.d. random variables $X_1, \ldots, X_N$ uniformly distributed on $\{-1,1\}^n$
- and queries
$$(X_1, f(X_1)), \ldots, (X_N, f(X_N)),$$

we can construct a random function h such that

$$\Pr\left(\|f-h\|_2^2 \leq \epsilon\right) \geq 1-\delta.$$

What is $N(\mathcal{F}_n^{\leq d}, \epsilon, \delta)$? How does it depend on the dimension n?

The LMN algorithm: Parseval + Chernoff

- Linial–Mansour–Nisan (1993) low-degree algorithm

$$N(\mathcal{F}_n^{\leq d}, \epsilon, \delta) \leq \frac{2n^d}{\epsilon} \log\left(\frac{2n^d}{\delta}\right) = O_{d,\epsilon,\delta}(n^d \log n)$$

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We know: random $(X_1, f(X_1)), \ldots, (X_N, f(X_N))$.

$$\overset{\text{unknown}}{f = \sum_{|S|\leq d} \widehat{f}(S)\chi_S} \qquad \approx \qquad \overset{\text{random}}{h = \sum_{|S|\leq d} \widehat{h}(S)\chi_S} \qquad \text{in } L^2$$

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$\widehat{f}(S) = \langle f, \chi_S \rangle$	$\overset{\text{w.h.p.}}{\approx}$	$\widehat{h}(S) = \frac{1}{N} \sum_{j=1}^{N} f(X_j)\chi_S(X_j)$	Chernoff

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$$\overset{\text{w.h.p.}}{\Longrightarrow} \|f-h\|_2^2 = \sum_{\lvert S\rvert \leq d} \lvert \widehat{f}(S) - \widehat{h}(S)\rvert^2 \leq \sum_{j=0}^{d} \binom{n}{j} \cdot \text{small} = O(n^d) \cdot \text{small}.$$

Learning low-degree polynomials

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- Eskenazis–Ivanisvili–Streck (2023)

$$N(\mathcal{F}_n^{\leq d}, \epsilon, \delta) = \Omega_{d,\epsilon,\delta}(\log n)$$

EI algorithm: LMN + BH

However, we lose too much in the previous estimate. Recall:

$$\sum_{S\subset[n]:|S|\le d} |\widehat{f}(S)|^2 = \|f\|_2^2 \le \|f\|_\infty^2 \le 1.$$

So many of $|\widehat{f}(S)|^2 \lesssim n^{-d}$ are too small to contribute:

- keep $\widehat{f}(S)$'s that are influential (e.g. $|\widehat{f}(S)| > 1/10$)

$$|\{S : |\widehat{f}(S)| > 1/10\}| \le 100\|f\|_2^2 \le 100$$

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$$|\{S : |\widehat{f}(S)| > 1/10\}| \leq 100\|f\|_2^2 \leq 100$$

- forget about $\widehat{f}(S)$'s that are non-influential (e.g. $|\widehat{f}(S)| \leq 1/10$)

$$\sum_{S:|\widehat{f}(S)|\leq 1/10} |\widehat{f}(S)|^2 \leq (1/10)^{2-p} \sum_{S:|\widehat{f}(S)|\leq 1/10} |\widehat{f}(S)|^p \overset{?}{\leq} (1/10)^{2-p} C$$

Bohnenblust–Hille inequality for $\{-1,1\}^n$

BH inequalities for Boolean cubes $\{-1,1\}^n$

Bohnenblust–Hille inequality for $\{-1,1\}^n$ (Blei 2001)

For $d \geq 1$, there exists $C_d > 0$ s.t. for all $n \geq 1$ and all $f : \{-1,1\}^n \to \mathbb{R}$ of degree-d we have

$$\|\widehat{f}\|_{\frac{2d}{d+1}} = \left(\sum_{|S| \leq d} |\widehat{f}(S)|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{2d}} \leq C_d \|f\|_\infty .$$

Defant–Mastyło–Pérez (2019)

The best constant satisfies $\mathrm{BH}^{\leq d}_{\{\pm 1\}} \leq C^{\sqrt{d \log d}}$ for some universal $C > 0$.

$$\begin{tikzcd}
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Bits to qubits?

Conjecture/Theorem: qubit BH (Rouzé–Wirth–Z. 2024)

For $d \geq 1$, $\exists C_d > 0$ s.t. for all $n \geq 1$ and all degree-d $A \in M_2(\mathbb{C})^{\otimes n}$

$$\|\widehat{A}\|_{\frac{2d}{d+1}} \leq C_d \|A\|. \tag{qBH}$$

Denote the best constant by $\mathrm{BH}^{\leq d}_{M_2}$.

Huang–Chen–Preskill (2023)

(qBH) holds and $\mathrm{BH}^{\leq d}_{M_2} \leq d^{\mathcal{O}(d)}$ (learning to predict arbitrary quantum processes).

A reduction method: Volberg–Z. (2023)

(qBH) holds and $\mathrm{BH}^{\leq d}_{M_2} \leq C^d$. In fact, $\mathrm{BH}^{\leq d}_{M_2} \leq 3^d \mathrm{BH}^{\leq d}_{\{\pm 1\}}$.

Idea of proving qubit BH via reduction

Goal: for any degree-d $A \in M_2(\mathbb{C})^{\otimes n}$,

$$\|\widehat{A}\|_{\frac{2d}{d+1}} \leq C_d \|A\|\,.$$

Idea: find degree-d $f_A : \{-1,1\}^m \to \mathbb{C}$ with some $m > 0$ such that

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\|\widehat{A}\|_{\frac{2d}{d+1}} & \lesssim_d \arrow[d] & \|\widehat{f_A}\|_{\frac{2d}{d+1}} & \lesssim_d \arrow[d] & \|f_A\|_\infty & \lesssim_d \arrow[d] & \|A\| \\
 & ? & & \text{BH for } \{-1,1\}^m & & ? &
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 & ! & & f_A(\vec{\epsilon}) = \mathrm{tr}[A\rho(\vec{\epsilon})] & & ! &
\end{tikzcd}$$

key: $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbf{1}$

In fact, we may choose $m = 3n$ and each $\rho(\vec{\epsilon})$ is an explicit density matrix.

Key density matrices

The density matrices $\rho(\vec{\epsilon})$ are defined as follows

$$\vec{\epsilon} = \left(\epsilon_1^{(1)}, \ldots, \epsilon_j^{(1)}, \ldots, \epsilon_n^{(1)}, \epsilon_1^{(2)}, \ldots, \epsilon_j^{(2)}, \ldots, \epsilon_n^{(2)}, \epsilon_1^{(3)}, \ldots, \epsilon_j^{(3)}, \ldots, \epsilon_n^{(3)}\right) \in \{\pm 1\}^{3n}$$

$$\Big\downarrow \rho$$

$$\rho(\vec{\epsilon}) := \rho_1(\vec{\epsilon}) \otimes \cdots \otimes \rho_j(\vec{\epsilon}) \otimes \cdots \otimes \rho_n(\vec{\epsilon}) \in M_2(\mathbb{C})^{\otimes n},$$

where for each $1 \leq j \leq n$

$$\rho_j(\vec{\epsilon}) = \frac{1}{3} P^{(1)}_{\epsilon_j^{(1)}} + \frac{1}{3} P^{(2)}_{\epsilon_j^{(2)}} + \frac{1}{3} P^{(3)}_{\epsilon_j^{(3)}}$$

and $P^{(j)}_{\pm 1}$ is the eigen-projection of σ_j corresponding to eigenvalue ± 1.

Key observation: $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbf{1}$

Since

$$\sigma_i\sigma_j = -\sigma_j\sigma_i, \qquad 1 \leq i \neq j \leq 3,$$

we have

$$\left\langle \sigma_i, P^{(j)}_{\pm 1} \right\rangle = 0, \qquad 1 \leq i \neq j \leq 3.$$

So for $1 \leq i \leq 3$

$$\left\langle \sigma_i, \frac{1}{3}P^{(1)}_{\epsilon_1} + \frac{1}{3}P^{(2)}_{\epsilon_2} + \frac{1}{3}P^{(3)}_{\epsilon_3} \right\rangle = \frac{1}{3}\left\langle \sigma_i, P^{(i)}_{\epsilon_i} \right\rangle = \frac{1}{3}\epsilon_i.$$

Thus

$$\left\langle \cdots \otimes \mathbf{1} \otimes \underbrace{\sigma_{\kappa_1}}_{i_1\text{-th}} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{\sigma_{\kappa_l}}_{i_l\text{-th}} \otimes \mathbf{1} \otimes \cdots, \rho(\vec{\epsilon}) \right\rangle = 3^{-l}\epsilon^{(\kappa_1)}_{i_1} \cdots \epsilon^{(\kappa_l)}_{i_l}$$

$\rightsquigarrow$ one-one correspondence: $\widehat{A}_{\mathbf{s}} \approx_d \widehat{f_A}(S) \rightsquigarrow \|\widehat{A}\|_p \approx_d \|\widehat{f_A}\|_p$.

$$\begin{tikzcd}
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\end{tikzcd}$$

Learning qudit quantum observables?

2×2 matrices $\rightsquigarrow$ $K \times K$ matrices?

Qudit system: Heisenberg–Weyl basis

Let $K \geq 3$ and $\omega_K = e^{\frac{2\pi i}{K}}$. The Heisenberg–Weyl basis of $M_K(\mathbb{C})$ is given by

$$X^\ell Z^m, \qquad 1 \leq \ell, m \leq K,$$

with (shift & clock matrices)

$$Xe_j = e_{j+1}, \qquad Ze_j = \omega_K^j e_j, \qquad j \in \mathbb{Z}_K.$$

In particular, $X^K = Z^K = \mathbf{1}$ and $ZX = \omega_K XZ$.

Question: do we have qudit BH and the reduction?

Reduction from qudit systems to cyclic groups

$\Omega_K = \{1, \omega_K, \ldots, \omega_K^{K-1}\}$: the multiplicative cyclic group of order K.

Reduction: Slote–Volberg–Z. (2023, 2024)

Suppose that $K \geq 3$ is prime. For any $A \in M_K(\mathbb{C})^{\otimes n}$ of degree d, we can find $f_A : \Omega_K^{(K+1)n} \to \mathbb{C}$ of degree d such that

$$\begin{tikzcd} \|\widehat{A}\|_{\frac{2d}{d+1}} & \lesssim_{d,K} \arrow[d] & \|\widehat{f_A}\|_{\frac{2d}{d+1}} & \lesssim_{d,K} \arrow[d] & \|f_A\|_\infty & \leq \arrow[d] & \|A\| \\ & ! & & ? & & ! & \end{tikzcd}$$

Works for non-prime case as well but looks more ugly.

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Warning: no cyclic BH was known when $2 < K < \infty$

While the proofs for $\Omega_2 = \{-1, 1\}$ and $\Omega_\infty = \mathbb{T}$ are similar, they do not work for $2 < K < \infty$ due to an obstacle.

The obstacle: dimension-free approximation/discretization

If you recap $K = 2, \infty$ proof: for analytic $f : \Omega_K^n \to \mathbb{C}$ of degree d

$$\|\widehat{f}\|_{\frac{2d}{d+1}} \lesssim_{d,K} \|f\|_{L^\infty(\mathrm{conv}(\Omega_K)^n)} \overset{?}{\lesssim}_{d,K} \|f\|_{L^\infty(\Omega_K^n)}$$

with conv(Ω_K) being the convex hull of Ω_K. Is it true that

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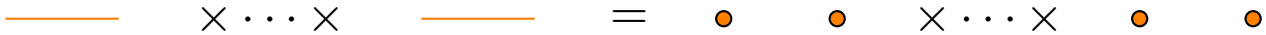
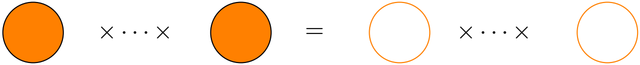
This is trivial when $K = 2$ since f is multi-affine



and when $K = \infty$ because of maximum module principle:



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but is it true?

Good news

Slote–Volberg–Z. (2023, 2024)

We do have the dimension-free estimate for low-degree [1] polynomials

As consequences:

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[1] both the total degree and individual degree are small

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⇝ low-degree learning in qudit systems

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More recent work

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See the next talk by Joseph Slote.

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Arunachalam–Dutt–Gutiérrez–Palazuelos (2024)

When $f : \{-1,1\}^n \to \{-1,1\}$: for all f of degree-d

$$\|\widehat{f}\|_{\frac{2d}{d+1}} \le 2^{\frac{d-1}{d}} \|f\|_\infty = 2^{\frac{d-1}{d}},$$

and $2^{\frac{d-1}{d}}$ is optimal.

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and $2^{\frac{d-1}{d}}$ is optimal.

Question: $\mathrm{BH}^{\leq d}_{\{\pm 1\}} \leq$ Poly(d)? Or even $\mathrm{BH}^{\leq d}_{\{\pm 1\}} \leq 2$?

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Becker–Klein–Slote–Volberg–Z. (2024)

More is true

See the next talk by Joseph Slote.

Arunachalam–Dutt–Gutiérrez–Palazuelos (2024)

When $f:\{-1,1\}^n\to\{-1,1\}$: for all f of degree-d

$$\|\widehat{f}\|_{\frac{2d}{d+1}}\le 2^{\frac{d-1}{d}}\|f\|_\infty = 2^{\frac{d-1}{d}},$$

and $2^{\frac{d-1}{d}}$ is optimal.

Question: $\mathrm{BH}^{\le d}_{\{\pm1\}}\le \mathrm{Poly}(d)$? Or even $\mathrm{BH}^{\le d}_{\{\pm1\}}\le 2$?

Thank you for your attention!