The Brownian and Poisson transport maps

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Then μ satisfies log-Sobolev with constant $C_{\mathbb{P}}$:

$$\begin{aligned}\mathrm{Ent}_{\mu}(g^2) = \mathrm{Ent}_{\mathbb{P}}(g^2 \circ T) &\leq C_{\mathbb{P}} \int |\nabla(g \circ T)|^2 \,\mathrm{d}\mathbb{P} \\ &\leq C_{\mathbb{P}} \int |(\nabla g) \circ T)|^2 \,\mathrm{d}\mathbb{P} = C_{\mathbb{P}} \int |\nabla g|^2 \,\mathrm{d}\mu.\end{aligned}$$

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- Kim-E. Milman established the same result (and more) for a transport map based on the Langevin dynamics of the Gaussian.
- Many more 1-Lipschitz results for the Kim-E. Milman map, including on manifolds, but very few for the Brenier map.

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- Trying to use the transport approach in discrete settings faces two problems: (a) Splitting of mass (b) Lack of chain rule.

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"In fact, the effective dimension of $(\mathbb{R}^n, \gamma^{(n)})$ *is infinite, in a certain sense, whatever n. I admit that this perspective may look strange, and might be the result of lack of imagination; but in any case, it will fit very well into the picture (in terms of sharp constants for geometric inequalities, etc.)."*

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The theme of the talk today is that this perspective is not so strange: by taking the source measure to be **infinite-dimensional** we can overcome the challenges of the transport method.

The Brownian transport map

"The Brownian transport map" (Dan Mikulincer, Yair Shenfeld).
Probab. Th. Rel. Fields (2024)

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Then, $Y_1 \sim \mu$.

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Let $\mathbb{P}$ be the Wiener measure on the space

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Lipschitz transport maps and functional inequalities

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We say that Y_1, which transports $\mathbb{P}$ to μ, is 1-Lipschitz if $|\mathrm{D}Y_1| \leq 1$, where $\mathrm{D}Y_1$ is the **Malliavin derivative**, and $|\cdot|$ is the appropriate norm.

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The point is that the Wiener measure $\mathbb{P}$ satisfies functional inequalities with the same constant as the Gaussian γ on $\mathbb{R}^d$.

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Example. $\kappa > 0$: If $\kappa S^2 \geq 1$ we get the analogue of Caffarelli/Kim-E. Milman, and if $\kappa S^2 < 1$ we get improvement.

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- No average-Lipschitz bounds (even with good dimension-dependent constants) for the Brenier/Kim-E. Milman maps are known.
- In contrast, due to stochastic localization, we were able to establish such bounds, with dimension-dependent constants, for the Brownian transport map. In fact, we proved general moments bounds which have applications to Stein kernels and Central Limit Theorems.

The Poisson transport map

“The Poisson transport map” (Pablo López-Rivera, Yair Shenfeld).
arXiv preprint (2024)

Lipschitz transport maps and functional inequalities

Recall the argument

$$\begin{aligned}\mathsf{Ent}_\mu(g^2) = \mathsf{Ent}_{\mathbb{P}}(g^2 \circ T) &\le C_{\mathbb{P}} \int |\nabla(g \circ T)|^2 \, \mathrm{d}\mathbb{P} \\ &\le C_{\mathbb{P}} \int |(\nabla g) \circ T)|^2 \, \mathrm{d}\mathbb{P} = C_{\mathbb{P}} \int |\nabla g|^2 \, \mathrm{d}\mu.\end{aligned}$$

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First defined by Budhiraja, Dupuis, and Maroulas, but further elaborated in dimension 1, and used for functional inequalities, by Klartag and Lehec.

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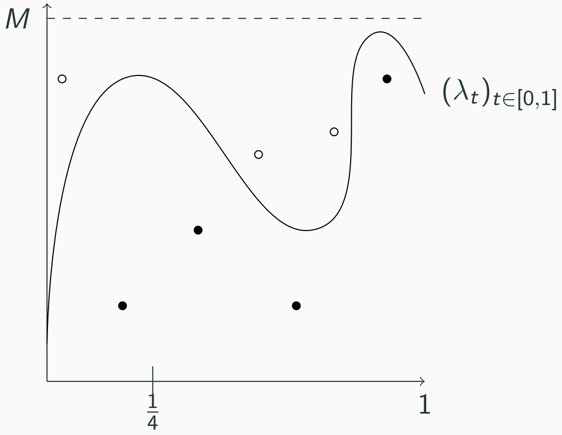

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Then,

$$X_1 \sim \mu = f\pi.$$

The Poisson transport map

Let $\mathbb{P}$ be the Poisson measure on the space

$$\Omega := \{\omega : \omega = \sum_i \delta_{(t_i, z_i)} \text{ for } t_i \in [0,1], z_i \in [0,M]\}.$$

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Note. This construction avoids the problem of splitting the mass!

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The correct notion of X_1 being **1-Lipschitz** is that for almost every ω,

$$|\mathrm{D}_{(t,z)} X_1(\omega)| \leq 1 \quad \forall (t,z) \in [0,1] \times [0,M],$$

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where $\mathrm{D}_{(t,z)}X_1(\omega)$ is the **Malliavin derivative** defined by

$$\mathrm{D}_{(t,z)}X_1(\omega) := X_1(\omega + \delta_{(t,z)}) - X_1(\omega).$$

Transfer of functional inequalities

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Modified LSI on Poisson space (Wu):

$$\mathsf{Ent}_{\mathbb{P}}(G) \leq \mathbb{E}_{\mathbb{P}}\left[\int \Psi(G, \mathrm{D}_{(t,z)}G)\,\mathrm{d}t\,\mathrm{d}z\right],$$

where $\Psi(u, v) := (u+v)\log(u+v) - u\log u - (\log u + 1)v$.

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Proposition. Suppose that $\mathrm{D}_{(t,z)}X_1(\omega) \in \{0,1\}$ for all (t,z). Then, for any test function g,

$$\mathsf{Ent}_{\mu}(g) \leq |\log \mu(0)|\mathbb{E}_{\mu}[\Psi(g, DG)],$$

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Note. We overcome the chain rule obstacle but we need $\mathrm{D}_{(t,z)}X_1(\omega) \in \{0, 1\}$ rather than just $\mathrm{D}_{(t,z)}X_1(\omega) \in \{-1, 0, 1\}$.

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We expect the same phenomenon in discrete spaces: transport maps between a Poisson source π and a target μ which is more log-concave than π should be 1-Lipschitz.

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Due to mass splitting issue this doesn't make sense for transport maps from π to μ. What about the Poisson transport map from $\mathbb{P}$ to μ?

Ultra-log-concave measures

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Definition. μ is more log-concave than π if $\mu = f\pi$ where f is log-concave,

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Such measures μ are called **ultra-log-concave** $\Longleftrightarrow$

$$k\mu^2(k) \geq (k+1)\mu(k+1)\mu(k-1), \quad \text{for all } k \in \{1, 2, \ldots\}.$$

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Theorem [López-Rivera, S.]

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Let μ be an ultra-log-concave measure on $\mathbb{N}$. Then, the Poisson transport map between the Poisson measure $\mathbb{P}$ and μ is 1-Lipschitz:

Lipschitz properties of the Poisson transport map

Theorem [López-Rivera, S.]

Let μ be an ultra-log-concave measure on $\mathbb{N}$. Then, the Poisson transport map between the Poisson measure $\mathbb{P}$ and μ is 1-Lipschitz:$\mathbb{P}$-almost-surely,

$$\mathrm{D}_{(t,z)} X_1 \in \{0, 1\} \quad \text{for all} \quad t \in [0,1], z \in [0, M].$$

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$$\mathrm{Ent}_\mu(g) \leq |\log \mu(0)|\, \mathbb{E}_\mu[\Psi(g, DG)],$$

where $Dg(k) := g(k+1) - g(k)$, and
$\Psi(u, v) := (u+v)\log(u+v) - u\log u - (\log u + 1)v$.

Modified log-Sobolev inequalities for ultra-log-concave measures

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Remark 1. Previous modified LSI for ultra-log-concave measures (Johnson) hold with the worse constant $\frac{\mu(1)}{\mu(0)}$.

Remark 2. In light of concentration result of Aravinda, Marsiglietti, and Melbourne it is natural to conjecture that the inequality holds with the even better constant $\mathbb{E}[\mu]$.

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- We also prove transport-entropy inequalities for ultra-log-concave measures.

A few words on proofs

Establishing Lipschitz bounds for the Brownian transport map

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- Hence, if we can bound $\nabla^2 \log \mathrm{H}_{1-s} f$ we can derive differential inequalities for $t \mapsto |\mathrm{D} Y_t|$, which imply Lipschitz bounds on $\mathrm{D} Y_1$.
- Use properties of $\mu = f\gamma$ to bound $\nabla^2 \log \mathrm{H}_{1-s} f$.

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The key point is to exploit the log-concavity of f to show that for all $s \in [0,1]$,

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This is done by comparing $\lambda_s(\omega + \delta_{(t,z)})$ to $\lambda_s(\omega)$, where we recall

$$\lambda_s = \frac{\mathrm{P}_{1-s} f(X_s + 1)}{\mathrm{P}_{1-s} f(X_s)}.$$

Entropic representation formulas and functional inequalities

"Stability of Wu's logarithmic Sobolev inequality via the Poisson-Föllmer process" (Shrey Aryan, Pablo López-Rivera, Yair Shenfeld).

arXiv preprint (2024)

Entropy representation formula for the Gaussian

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Recall the Föllmer process: $Y_1 \sim \mu = f\gamma$ where

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Then,

$$\mathrm{H}(\mu|\gamma) = \frac{1}{2}\int_{\mathbb{R}^d} \mathbb{E}\left[|\nabla \log \mathrm{H}_{1-t} f(Y_t)|^2\right] \mathrm{d}t,$$

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The entropy representation formula is due to Boué & Dupuis/Borell/Lehec.

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$$\mathrm{H}(\mu|\gamma) \leq \frac{1}{2}\mathbb{E}\left[|\nabla \log f(Y_1)|^2\right] = \frac{1}{2}\mathrm{I}(\mu|\gamma),$$

where $\mathrm{I}(\mu|\gamma) = \int |\nabla \log f|^2 \,\mathrm{d}\mu$ is the relative Fisher information w.r.t. the Gaussian.

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$$\mathrm{H}(\mu|\pi) \leq \int_{\mathbb{N}} f(k+1)\left\{\log\left(\frac{f(k+1)}{f(k)}\right) - 1 + \frac{f(k)}{f(k+1)}\right\}\mathrm{d}\pi(k).$$

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Recall the Poisson-Föllmer process: $X_1 \sim \mu = f\pi$ where

$X_t :=$ number of points in $[0, t] \times [0, \infty)$ that fall below the curve λ,

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The entropy representation formula is due to Budhiraja & Dupuis & Maroulas/Klartag & Lehec.

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Since $t \mapsto \lambda_t$ is a martingale,

$$\begin{aligned}\mathrm{H}(\mu|\pi) &\leq \mathbb{E}[\lambda_1 \log \lambda_1 - \lambda_1 + 1] \\ &= \int_{\mathbb{N}} f(k+1)\left\{\log\left(\frac{f(k+1)}{f(k)}\right) - 1 + \frac{f(k)}{f(k+1)}\right\}\mathrm{d}\pi(k).\end{aligned}$$

Stability of the Gaussian log-Sobolev inequality and Wu's inequality

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- In *"Stability of the logarithmic Sobolev inequality via the Föllmer Process" (Ronen Eldan, Joseph Lehec, Yair Shenfeld)* we used the Gaussian entropy representation formula to derive **stability** results for the Gaussian log-Sobolev inequality.

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- In *'Stability of Wu's logarithmic Sobolev inequality via the Poisson-Föllmer process" (Shrey Aryan, Pablo López-Rivera, Yair Shenfeld)* we used the Poisson entropy representation formula to derive new **stability** results of the inequality when $\mu = f\pi$, assuming f is ultra-log-concave.

Comparison between the Gaussian and Poisson settings

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- In the Gaussian setting the stability results are obtained by deriving **differential inequalities** for $|\nabla \log f(Y_1)|^2$.
- The discrete nature of the Poisson setting does not allow for such differential inequalities for λ_t.
- Instead, we find a useful representation formula as a replacement for the lack of differential inequalities, from which we derive our stability results.

Thank You