Twist parametrized points on modular curves

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joint work with Maarten Derickx (Zagreb)

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Images of Galois representations

Let $E/\mathbb{Q}$ be an elliptic curve and $\mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

$\mathrm{Gal}_{\mathbb{Q}}$ acts on $E_{tor} = \varprojlim_N E[N]$:
$\rho_E : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}\, E_{tor} \simeq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

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The level of an open $G \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is the largest integer N such that G is the inverse image of the reduction of G modulo N.

For an open $G \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ let X_G be the modular curve parametrizing elliptic curves E with $G_E \leq G$ (together with level structure).

Serre's uniformity conjecture

For a $G' \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ let $X_{G'} := X_G$, where $G \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is the largest subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ reducing to G' modulo N.

Let $\rho_{E,N} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}\, E[N] \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be the mod N Galois representation attached to E.

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Serre's open image theorem implies that for all $\ell > C_E$, where C_E depends on E we have $\rho_{E,\ell}(\mathrm{Gal}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ for non-CM E.

Conjecture (Serre's uniformity problem/conjecture)

There exists a C, not depending on E, such that for all $\ell > C$ and all non-CM $E/\mathbb{Q}$, we have $\rho_{E,\ell}(\mathrm{Gal}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$.

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Conjecture (Sutherland/Zywina)

One can take $C = 37$ above.

Some simplifying conditions

Suppose from now in the talk on $j(E) \neq 0, 1728$.

Suppose from now on that we add $-I$ to all matrix groups. Note: $X_G \simeq X_{\pm G}$. This has the benefit that all the quadratic twists correspond to the same point on the modular curve.

Isolated points on $X_1(N)$

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We also prove the conjecture for all $j(E)$, where E is of conductor ≤ 500000.

Isolated points on modular curves

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In this talk I will restrict to X_G with G having surjective discriminant.

Since we supposed that $-I \in H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ for $x \in X_H(\mathbb{Q})$ we can define G_x to be $\pm G_E$ for an elliptic curve $E/\mathbb{Q}$ with $j(E) = j(x)$.

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Generally in number theory, we expect that objects are as "complicated" as they are allowed to be.

Serre curves

Let's take the most obvious example: take $H = \mathrm{GL}_2(\widehat{\mathbb{Z}})$, so $X_H = X(1)$.

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But ρ_E can never be surjective!

We have that $\rho_E(\mathrm{Gal}_{\mathbb{Q}}) \leq H_\psi$, where H_ψ is the *Serre subgroup of* $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ *with character* ψ, defined by

$$H_\psi = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : \mathrm{sgn}(g) = \psi(\det g) \right\},$$

where $\mathrm{sgn} : \mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \{\pm 1\}$ is the map obtained by composing reduction modulo 2 and the sign map on $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \simeq S_3$.

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Theorem (N. Jones)

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Theorem (Zywina)

Let $K \neq \mathbb{Q}$ be a number field with $K \cap \mathbb{Q}^{cyc} = \mathbb{Q}$. Then 100% of E/K have $\rho_E(\mathrm{Gal}_K) = \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

So for $x \in X(1)(\mathbb{Q})$, we have typically $[H : G_x] = 2$. Furthermore, for 100% of such points we have $g(X_{G_x}) = 0$.

Elliptic curves with 25-isogenies

Let's take $H = B_0(25)$ now, so $X_H = X_0(25)$, so parametrizes elliptic curves with a 25-isogeny. We have $g(X_0(25)) = 0$.

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For $x \in X_0(25)(\mathbb{Q})$, what should we expect $g(X_{G_x})$ to be?

Label	Class	Class size	CM	Adelic index	Adelic genus	Weierstrass equation
11.a1	11.a	3		1200	37	$y^2 + y = x^3 - x^2 - 7820x - 263580$
99.d1	99.d	3		1200	37	$y^2 + y = x^3 - 70383x + 7187035$
121.d1	121.d	3		1200	37	$y^2 + y = x^3 - x^2 - 946260x + 354609639$
176.b1	176.b	3		1200	37	$y^2 = x^3 + x^2 - 125125x + 16994227$
275.b1	275.b	3		1200	37	$y^2 + y = x^3 + x^2 - 195508x - 33338481$
539.a1	539.a	3		1200	37	$y^2 + y = x^3 + x^2 - 383196x + 91174234$
550.e1	550.e	3		1200	37	$y^2 + xy + y = x^3 - 758201x + 254051548$
550.j1	550.j	3		1200	37	$y^2 + xy + y = x^3 + x^2 - 30328x + 2020281$
704.c1	704.c	3		1200	37	$y^2 = x^3 - x^2 - 31281x + 2139919$
704.h1	704.h	3		1200	37	$y^2 = x^3 + x^2 - 31281x - 2139919$
1089.b1	1089.b	3		1200	37	$y^2 + y = x^3 - 8516343x - 9565943918$
1342.b1	1342.b	3		1200	37	$y^2 + xy + y = x^3 + x^2 - 117257250x - 488766109679$
1584.g1	1584.g	3		1200	37	$y^2 = x^3 - 1126128x - 459970256$
1859.b1	1859.b	3		1200	37	$y^2 + y = x^3 - x^2 - 1321636x - 584371175$
1936.i1	1936.i	3		1200	37	$y^2 = x^3 + x^2 - 15140165x - 22679876749$
2475.a1	2475.a	3		1200	37	$y^2 + y = x^3 - 1759575x + 898379406$
3025.a1	3025.a	3		1200	37	$y^2 + y = x^3 + x^2 - 23656508x + 44278891894$
3179.a1	3179.a	3		1200	37	$y^2 + y = x^3 + x^2 - 2260076x - 1308527588$
3971.b1	3971.b	3		1200	37	$y^2 + y = x^3 + x^2 - 2823140x + 1824832093$
4400.i1	4400.i	3		1200	37	$y^2 = x^3 - x^2 - 3128133x + 2130534637$
4400.k1	4400.k	3		1200	37	$y^2 = x^3 - x^2 - 12131208x - 16259299088$

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Moreover we have a "parametrized family" of isolated points (!?).

Recall, objects are expected to be as complicated as they are allowed to be.

An explanation: geometry of modular curves

The map $X_1(25) \to X_0(25)$ is abelian with Galois group $(\mathbb{Z}/25\mathbb{Z})^\times/\langle -1 \rangle$.

By descent theory, this means that, for a $x \in X_0(25)(\mathbb{Q})$, there is a character ψ such that x lifts to a rational point on $X_1(25)^\psi$.

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Let N be the LCM of the conductor of ψ and 25.

$X_1(25)^\psi$ will parametrize, up to twist, elliptic curves E with ψ being their 25-isogeny character, or in other words

$$\rho_{E,N}(\mathrm{Gal}_\mathbb{Q}) \subseteq \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : 25 \mid c \text{ and } \pm a = \psi(\det A) \pmod{25} \right\}$$

$$\subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

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Note that $X_1(25)$ is $X_1(25)^\psi$ with ψ being the trivial character.

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Serre curves can be interpreted in this setting: we have an abelian map $X_{N_{ns}(2)} \to X(1)$ of degree 2, where $N_{ns}(2)$ is the unique subgroup of index 2 of $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

So any elliptic curve gives a rational point on some twist $X_{N_{ns}(2)}^\psi$.

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So an $x \in X_0(25)(\mathbb{Q})$ also gives a rational point on $X^{\psi_1}_{N_{ns}(2)} \times_{X(1)} X_1(25)^{\psi_2}$, for some characters ψ_1, ψ_2.

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Indeed $g\left(X_1(25) \times_{X(1)} X_{N_{ns}(2)}\right) = 37$.

Twist parametrized points

Definition (Twist parametrized points)

Let $H_1 \subset H_2 \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be open subgroups such that H_1 is normal in H_2 and H_2/H_1 is abelian. Then $x \in X_{H_2}(\mathbb{Q})$ lifts to a point $y \in X_{H_1}^{\chi}(\mathbb{Q})$ for some $\chi \in \mathrm{Hom}(\mathrm{Gal}_{\mathbb{Q}}, H_2/H_1)$. If $\#X_{H_2}(\mathbb{Q}) = \infty$ we say that y is *twist parameterized* over X_{H_2}.

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Definition

For a group $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ we say that a point $x \in X_G$ is twist parameterized if there exist groups $G \subseteq H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ such that x is twist parametrized over X_H.

A point that is not twist parametrized is called twist isolated.

We call a j-invariant $j_0 \in X(1)(\mathbb{Q}) \cong \mathbb{P}^1(\mathbb{Q})$ twist isolated if there exists $G \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and a twist isolated point $x \in X_G(\mathbb{Q})$ with $j(x) = j_0$.

Some remarks

a) By taking $H_1 = H_2$ it follows immediately that all points on $X_{H_1}(\mathbb{Q})$ are twist parametised when $\# X_{H_1}(\mathbb{Q}) = \infty$.

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c) On the other hand, if H is a maximal non-normal subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, then a rational isolated point on X_H will also be twist isolated.

d) E with $j(E) = -9317$ give isolated points on $X_0(37)$, and $g(X_0(37)) = 2$ and $B_0(37)$ is maximal, so this points is twist isolated.

Finitely many twist isolated points

We prove:

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Sketch of proof: We prove that any twist isolated point in $X_G(\mathbb{Q})$ will map to an isolated point on a $X_H(\mathbb{Q})$ where $G \leq H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is *agreeable*, using Zywina's terminology.

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There are 46 twist isolated rational j-invariants that we know of.

Lifting isolated points on modular curves

Given a CM j-invariant j_0, there are infinitely many modular curves $X_1(N)$ (with N varying) such that $x \in X_1(N)$ with $j(x) = j_0$ is isolated (proved by BELOV). Note that x here does not need to be rational.

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In the theorem above one can replace isolated with sporadic (a stronger property).

The result remains true if one restricts to just twists $X_1(N)^\psi$ of $X_1(N)$ instead of general X_G.

Algorithm for checking twist isolatedness

Whether a point is twist parameterized or isolated can be explicitly checked.

Theorem (Derickx, N.)

There exists an algorithm that given a $x \in X_G(\mathbb{Q})$ checks whether x is twist isolated.

This algorithm works well in practice.

Final remarks

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- Can one classify all agreeable groups G such that X_G has finitely/infinitely many degree d points?
- Is there a "geometric" explanation for the rational twist isolated points?

Thank you for your attention!