Sources of low degree points on curves

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Arithmetic problems

A Vague Goal

Given a nice curve $X/\mathbb{Q}$, understand the structure of $X(\overline{\mathbb{Q}})$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set.

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An application

A polynomial $p(x)$ gives a map $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$.
Two intuitions:

1. For most α, $\mathbb{Q}(p(\alpha)) = \mathbb{Q}(\alpha)$;
2. For most α, $[\mathbb{Q}(p^{-1}(\alpha)) : \mathbb{Q}(\alpha)] = \deg p$.

A theorem on polynomials

Theorem (K.– Neftin)

Suppose $p \in \mathbb{Q}[x]$ *is an indecomposable polynomial with* b *branch points. Then for all but finitely many* $\alpha \in \overline{\mathbb{Q}}$, $\deg \alpha \leqslant b/6$, *for any* $\beta \in p^{-1}(\alpha)$ *the degree* $[\mathbb{Q}(\beta) : \mathbb{Q}(\alpha)]$ *is either* $\deg p, \deg p - 1$, *or* 1.

Moral: Study of algebraic points on curves has useful consequences for concrete arithmetic.

Density degree set and sources of points

Methods apply to the study of

$$\delta(X) \coloneqq \{d \colon X \text{ has infinitely many degree } d \text{ points}\}.$$

Basic examples of $d \in \delta(X)$

1. A covering $\phi : X \to \mathbb{P}^1$, $\deg \phi = d$
2. A covering $\phi : X \to E$, $\deg \phi = d$
3. An incidence $I \subset X \times E$ of bidegree (e, d)
4. A covering $\phi : X \to Y$, Y has a mystery source of degree $d / \deg \phi$ points
5. Often $g \in \delta(X)$

The geometric approach

Claim

$\min \delta(X) = d$ is "small", implies that X is "special"

$$\begin{tikzcd} \operatorname{Sym}^d X \arrow[r, "\simeq"] \arrow[d, "\pi_{AJ}"] & X^d//S_d \\ W_d X \arrow[r, hook] & \operatorname{Pic}^d X \end{tikzcd}$$

$\pi_{AJ}(D) = \pi_{AJ}(D')$ whenever $\exists f\colon f^{-1}(0) = D, f^{-1}(\infty) = D'$

Remark: We will denote degree d points by D, D' etc., and view them as divisors.

Applying Mordell-Lang

Assume $\operatorname{gon}(X) > d$

$\operatorname{Sym}^d X \subset \operatorname{Pic}^d X$ has infinitely many rational points

$$\lightning$$

$\exists A \subset \operatorname{Sym}^d X$ a coset of $\operatorname{Pic}^d X$

A useful triviality

B and abelian group, $A \subset B$ a coset if and only if

$$\forall a_1, a_2, a_3 \in A,\ a_1 + a_2 - a_3 \in A.$$

Coset axiom and doubling

Choose $D_1, D_2, D_3 \in A(\mathbb{Q})$. Then $D_1 + D_2 - D_3 = D_4 \in A(\mathbb{Q})$.

Corollary

If $d = \min \delta(X)$, then $\operatorname{gon}(X) \leqslant 2d$.

Remark. A is divisible, so $D_1 + D_2 = 2D$. Thus have a large supply of linear systems $|2D|$ on X.

A closer look at the doubling construction

Theorem (K.–Vogt)

X a nice curve, $d = \min \delta(X)$. Then either there exists a nontrivial covering $\phi : X \to Y$ such that $d = \deg \phi \min \delta(Y)$, or $|2D|$ is birational for a general $D \in A$.

$$\{\text{low degree points}\} \rightsquigarrow \{\text{embeddings } X \hookrightarrow \mathbb{P}^n\}$$

Drawing $|2D|$

Main relation

For $D_1 \in A(\mathbb{Q})$, there exists a $D_2 \in A(\mathbb{Q})$ such that $2D = D_1 + D_2$

Debarre-Fahlaoui curves

Theorem (K.–Vogt)

If $\dim |2D| = 2$, *then* X *is a Debarre-Fahlaoui curve.*

Key point Embeddings $|3D|, |4D|$, etc. are even more special than $|2D|$. The *spans* of low degree points form remarkable combinatorial configurations.

More on Debarre-Fahlaoui

Case of interest

What if generically $\dim|2D| = 2$, $\dim|3D| = 5$? In that case $\mathrm{Span}_{|3D|}(D')$ is a 2-plane for a general D'.

A useful corollary

Corollary

If $d = \min \delta(X)$, then there is a covering $\phi : X \to Y$ of degree $e|d$ such that $\min \delta(Y) = d/e$ and

$$g(Y) \leqslant \frac{3}{4}\,(d/e)^2 + d/e.$$

Back to polynomials

Main problem

Given a degree ℓ polynomial $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ want to understand fibers $\phi^{-1}(P)$ above low degree d points P. Expect for low degrees all fibers are either irreducible or $(1, \ell - 1)$

Geometrically:

$$X_n = \tilde{X}/S_n \times S_{\ell-n} = \left(\left(\mathbb{P}^1 \times_\phi \mathbb{P}^1 \times_\phi \cdots \times_\phi \mathbb{P}^1\right) \setminus \Delta\right)/S_n$$

Simplest case

Simple branching, monodromy S_ℓ.

A rough estimate

Want to understand $\min \delta(X_n) \approx \operatorname{gon}(X_n)$. Let γ be the gonal map.

$$\begin{tikzcd} X_n \arrow[r, "\binom{\ell}{n}"] \arrow[d, "\gamma"] & \mathbb{P}^1 \\ \mathbb{P}^1 & \end{tikzcd}$$

So $g_n \lessapprox \operatorname{gon}(X_n)\binom{\ell}{n}$. But $g_n \gtrapprox (\ell-1)\binom{\ell-2}{n-1} + \binom{\ell}{n}$

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$$\mathrm{gon}(X_n) \gtrapprox n\frac{\ell - n}{\ell} \qquad :($$

Gonality and coverings

Useful fact

If $\psi\colon X \to Y$ is a covering then $\operatorname{gon}(Y) \leqslant \operatorname{gon}(X) \leqslant \operatorname{gon}(Y) \deg \psi$; same for $\min \delta(\bullet)$.

So the *diagram of covers* $\tilde{X}/H \to \mathbb{P}^1$ has to satisfy the above and the Castelnuovo-Severi constraints. Restricts the function $\gamma(H) = \operatorname{gon}(\tilde{X}/H)$.

The generic case

Let $Y_n = \tilde{X}/S_{\ell-n}$, a degree $n!$ covering of X_n. Castelnuovo-Severi for $Y_n \to Y_{n-1}$ and the gonal map gives

$$\operatorname{gon}(Y_n) \gtrsim \frac{(\ell-1)^{\underline{n}}}{\ell - n}.$$

This means

$$\operatorname{gon} X_n \gtrsim \frac{1}{\ell - n}\binom{\ell-1}{n}$$

Strategy of proof

Theorem (K.– Neftin)

Suppose $p \in \mathbb{Q}[x]$ is an indecomposable polynomial with b branch points. Then for all but finitely many $\alpha \in \overline{\mathbb{Q}}$, $\deg \alpha \leqslant b/6$, for any $\beta \in p^{-1}(\alpha)$ the degree $[\mathbb{Q}(\beta) : \mathbb{Q}(\alpha)]$ is either $\deg p, \deg p - 1$, or 1.

Strategy

1. Use combinatorics to understand the genera of $\tilde{X}/H$
2. Use the properties of the gonality function of the diagram to prove that gonality grows fast
3. Use refined methods to study $\min \delta$ for small quotients, most importantly for X_2

Remarks: Vojta inequality does not help. Applies to covering $X \to Y$, not just polynomial maps.