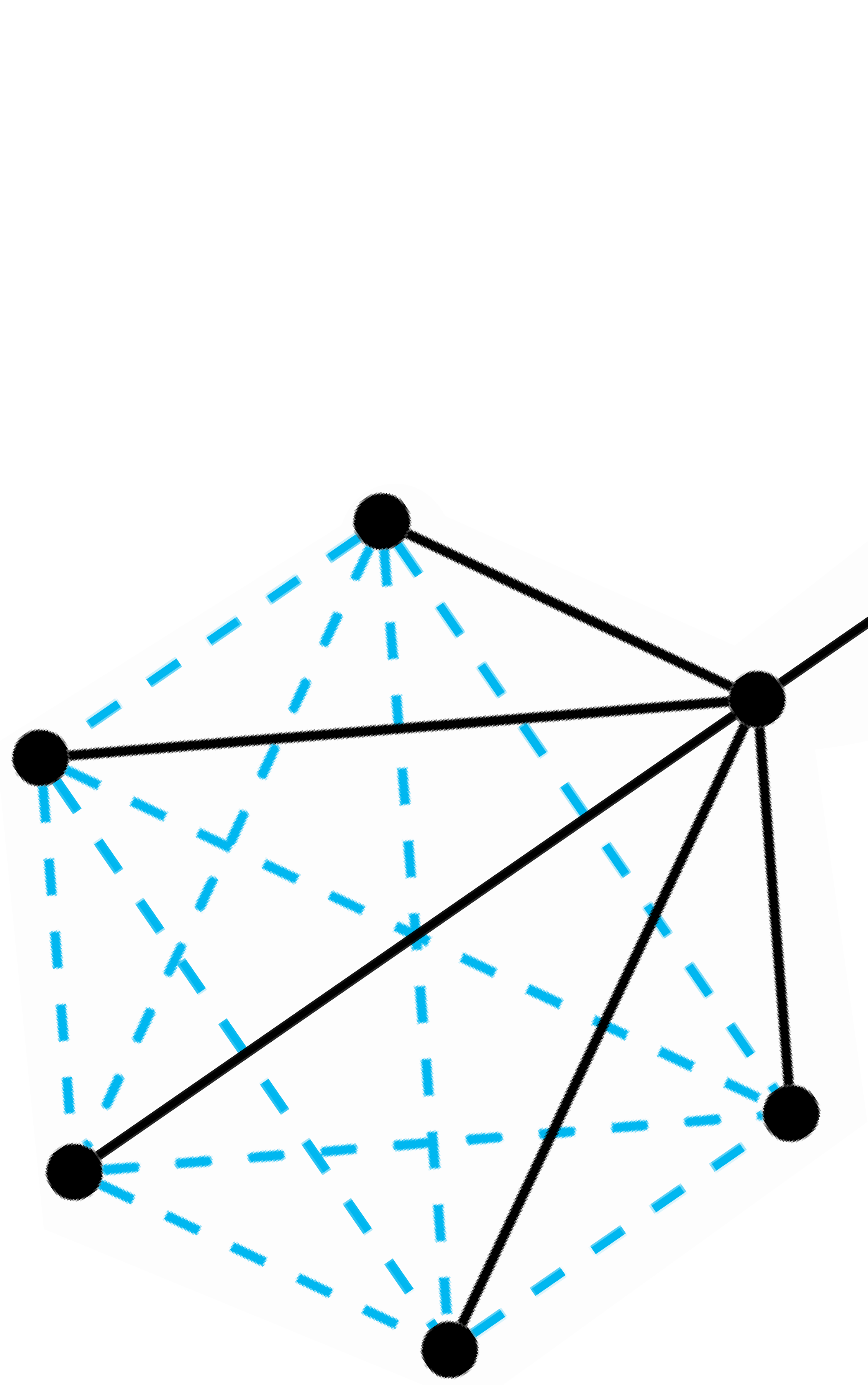
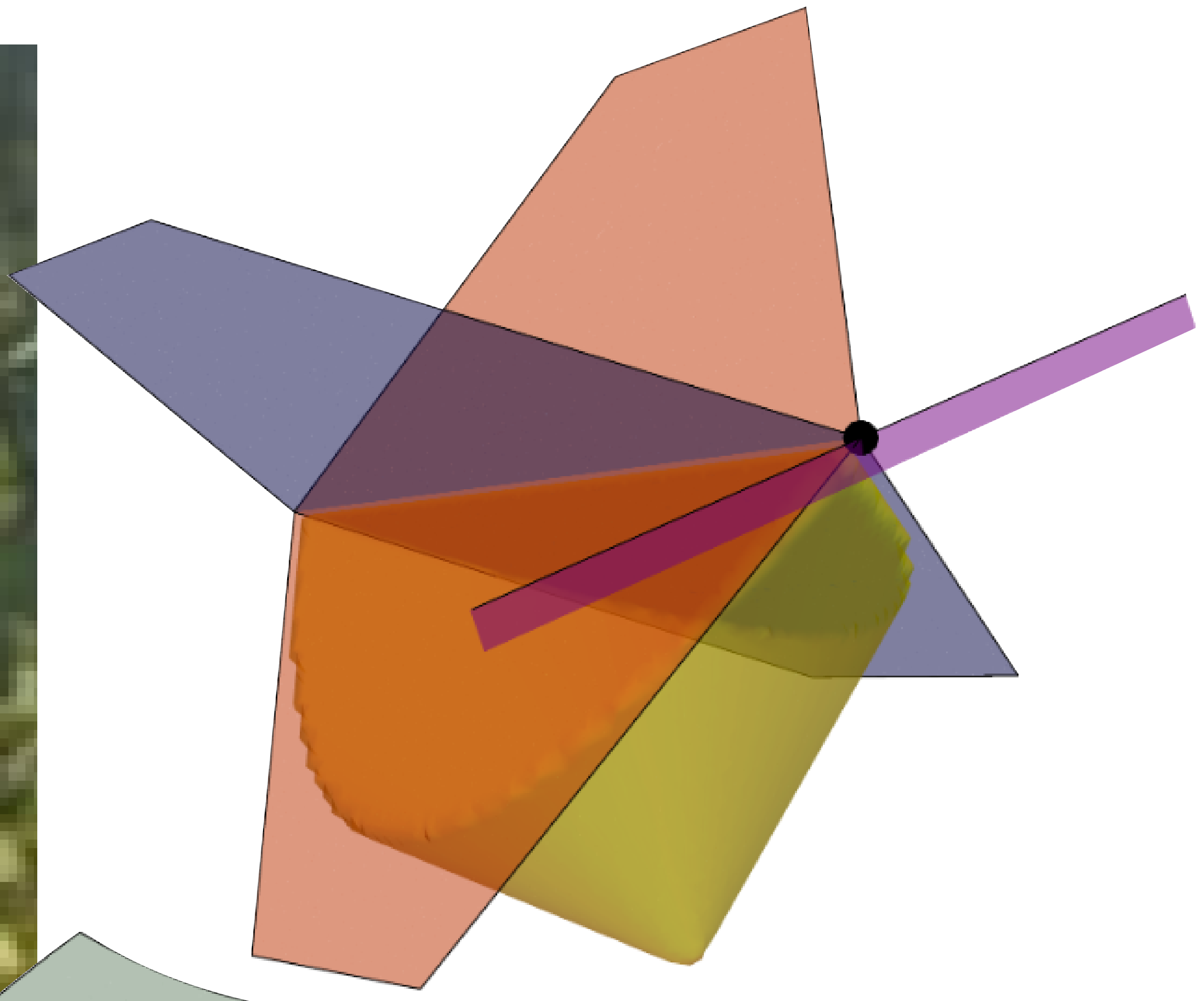
Spectrahedral Geometry of Graph Sparsifiers





Catherine Babecki

California Institute of Technology

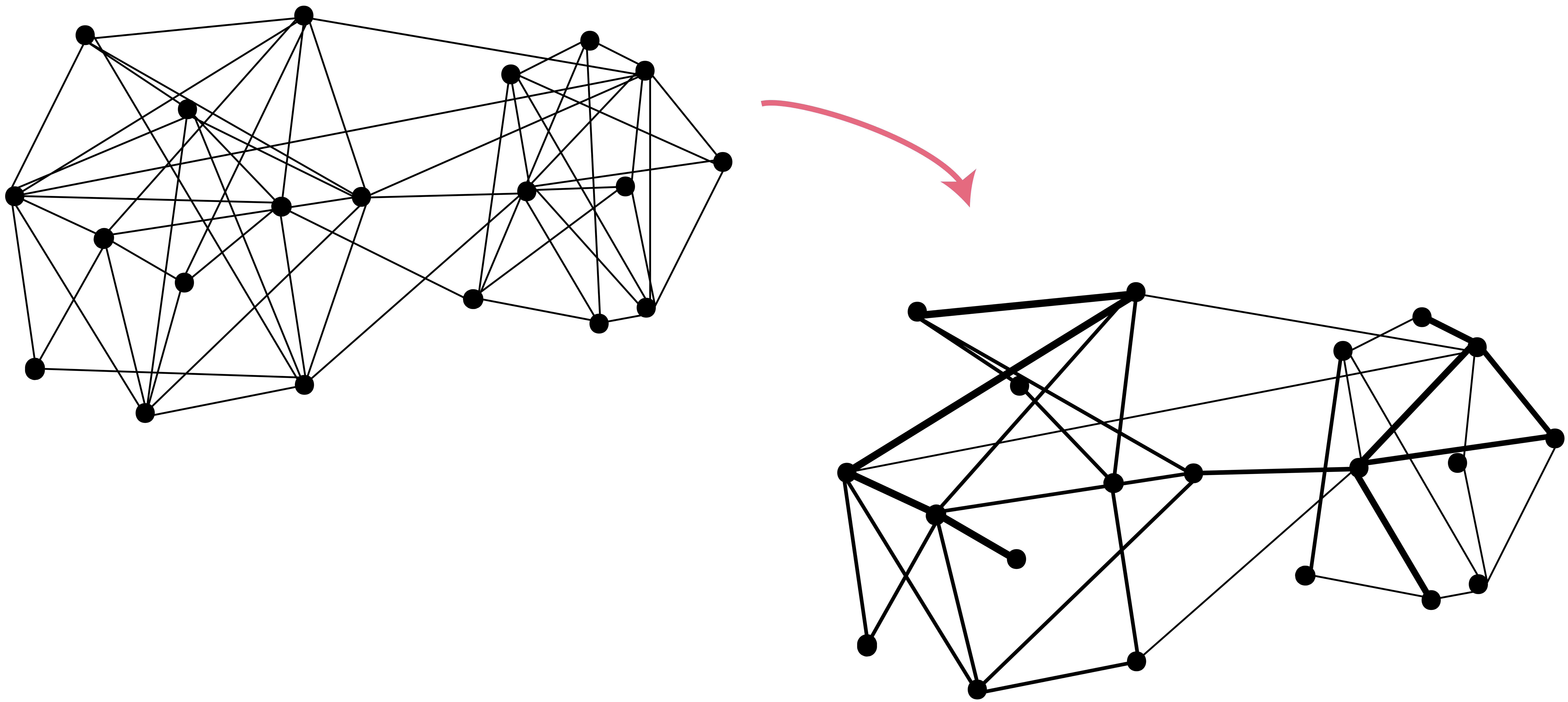
Joint with Stefan Steinerberger
and Rekha Thomas

What is graph sparsification?

Given a (weighted) graph $G = (V, E, w)$, find a graph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ which captures the `essence' of G but has *much* fewer edges.

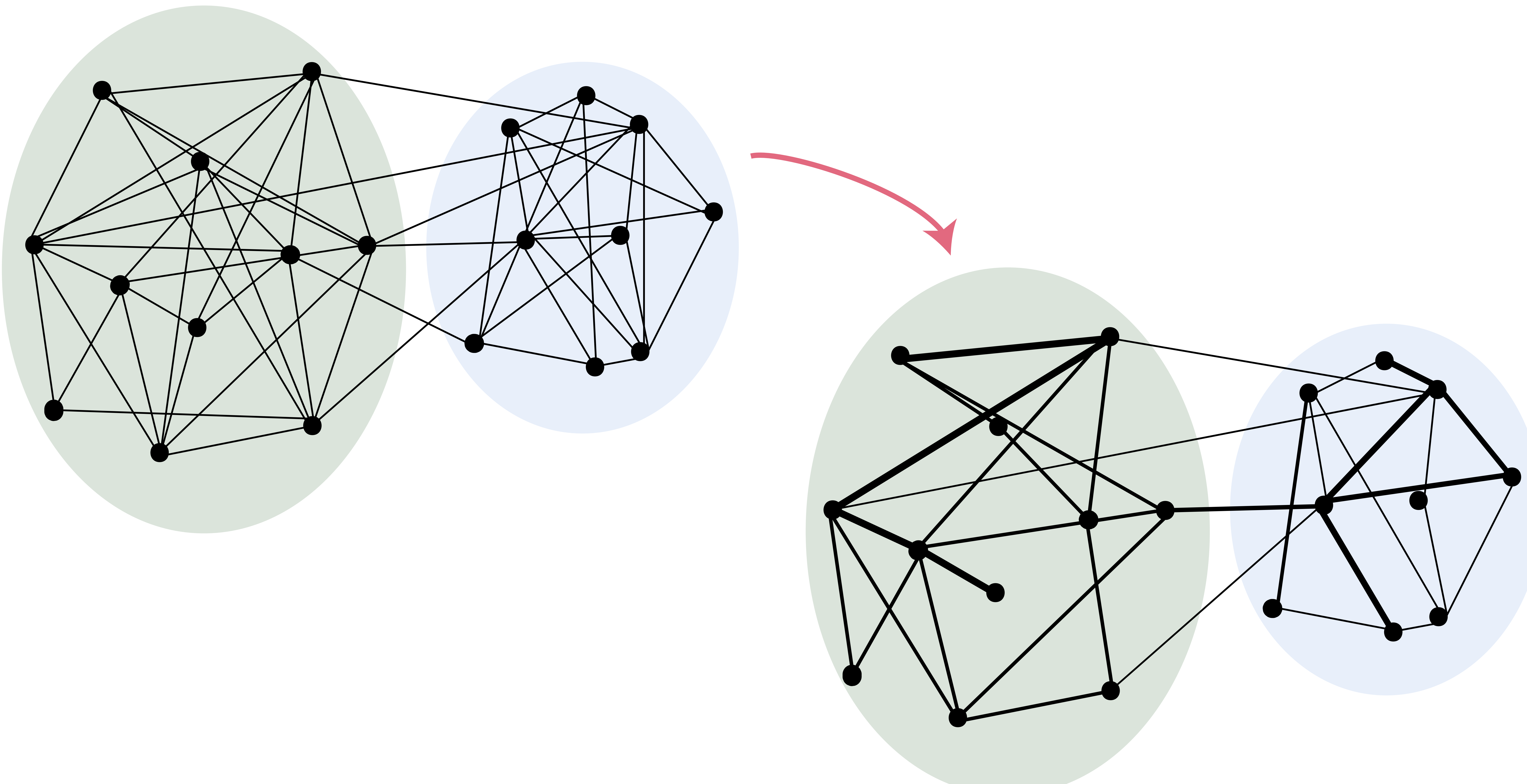
What is graph sparsification?

Given a (weighted) graph $G = (V, E, w)$, find a graph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ which captures the `essence' of G but has *much* fewer edges.



What is graph sparsification?

Given a (weighted) graph $G = (V, E, w)$, find a graph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ which captures the `essence' of G but has *much* fewer edges.



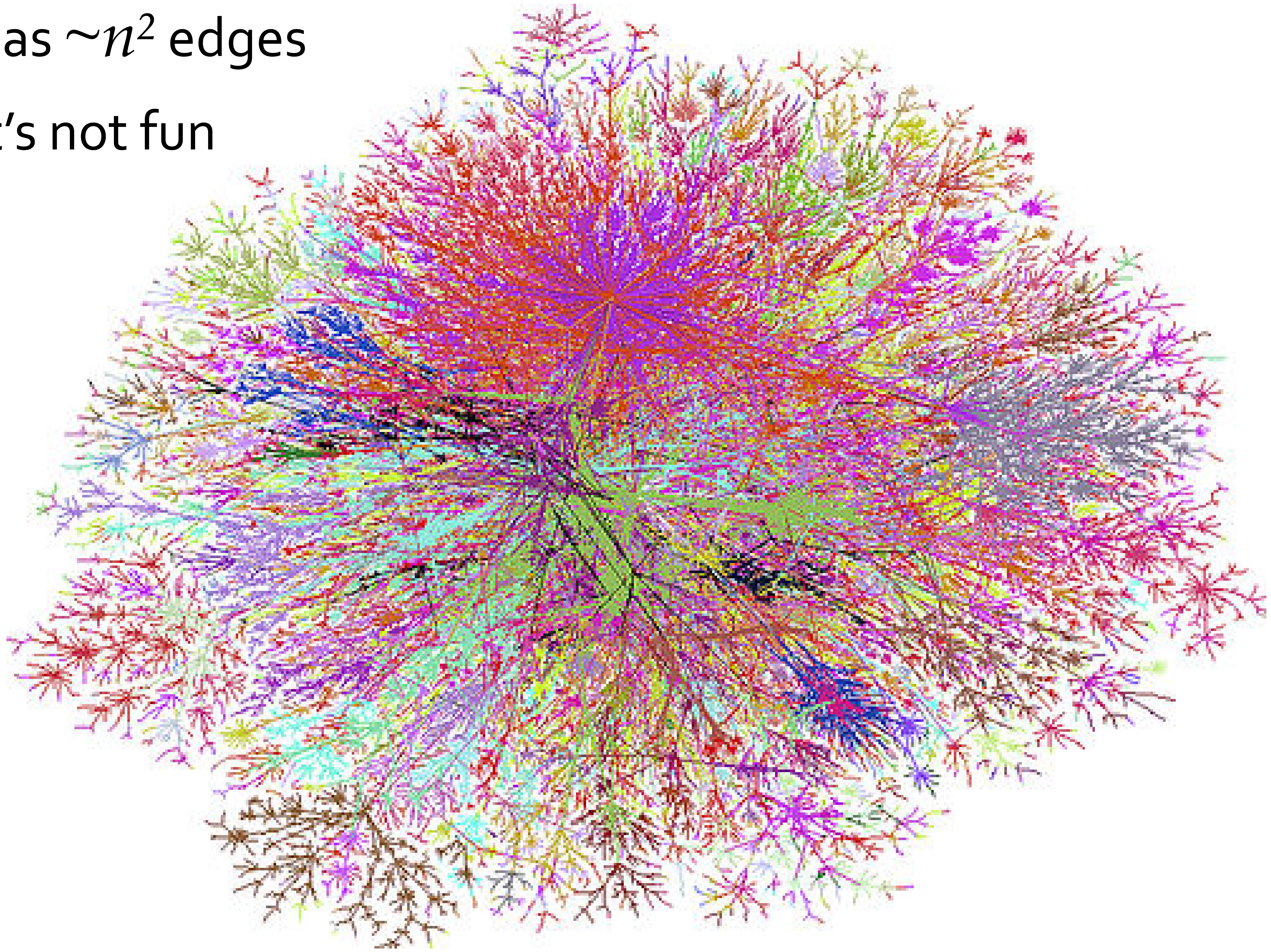
Why bother?

Why bother?

Data is too dang big!!

A dense graph has $\sim n^2$ edges

If n is huge, that's not fun



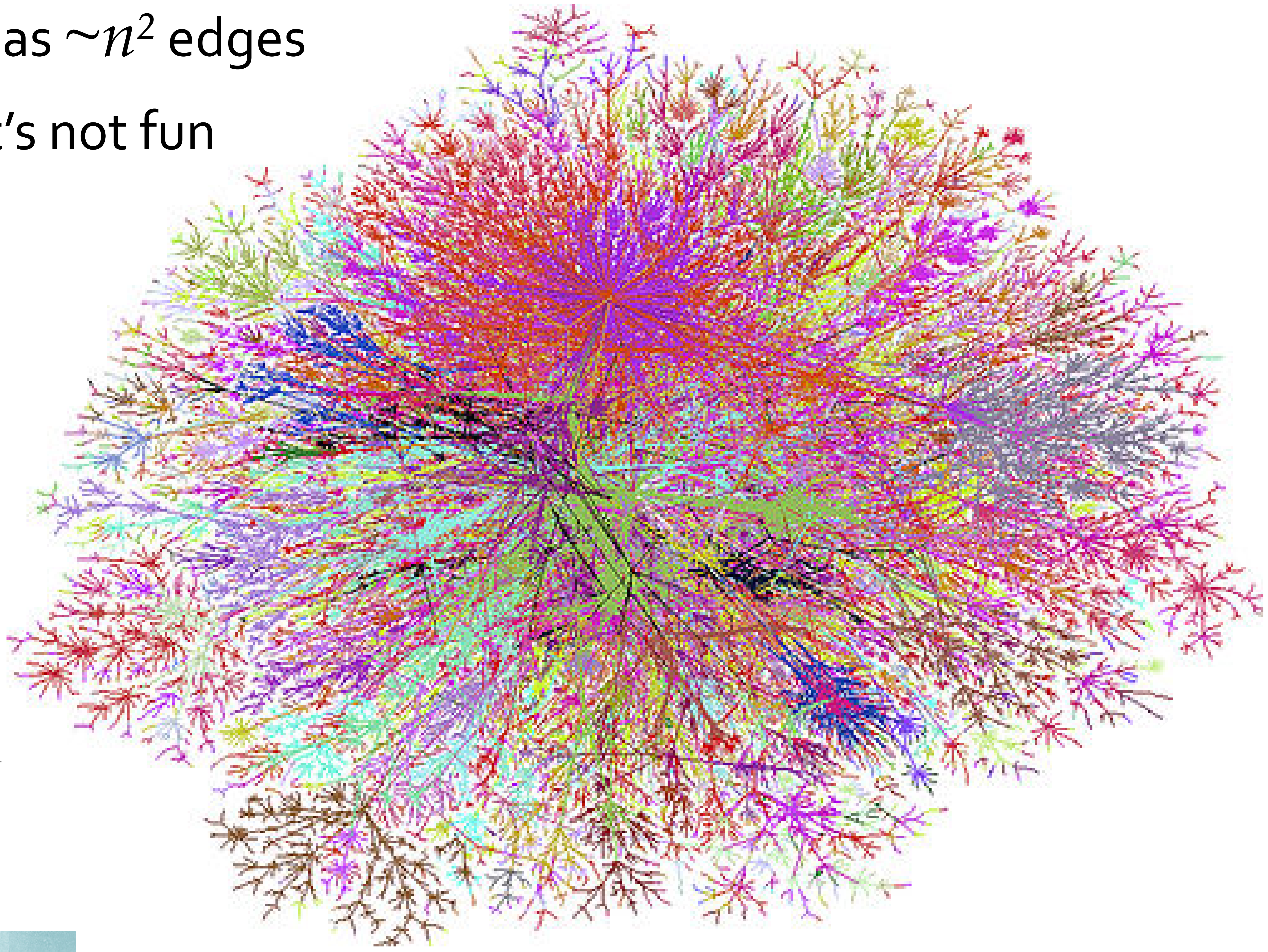
"Internet map 2004." From Math Insight. http://mathinsight.org/image/internet_map_jurvetson_2004

Why bother?

Data is too dang big!!

A dense graph has $\sim n^2$ edges

If n is huge, that's not fun



"Internet map 2004." From Math Insight. http://mathinsight.org/image/internet_map_jurvetson_2004

Erica Klarreich

Contributing Correspondent

November 24, 2015

VIEW PDF/PRINT MODE

computer science
mathematics networks
physics polynomials
quantum physics
traveling salesperson problem

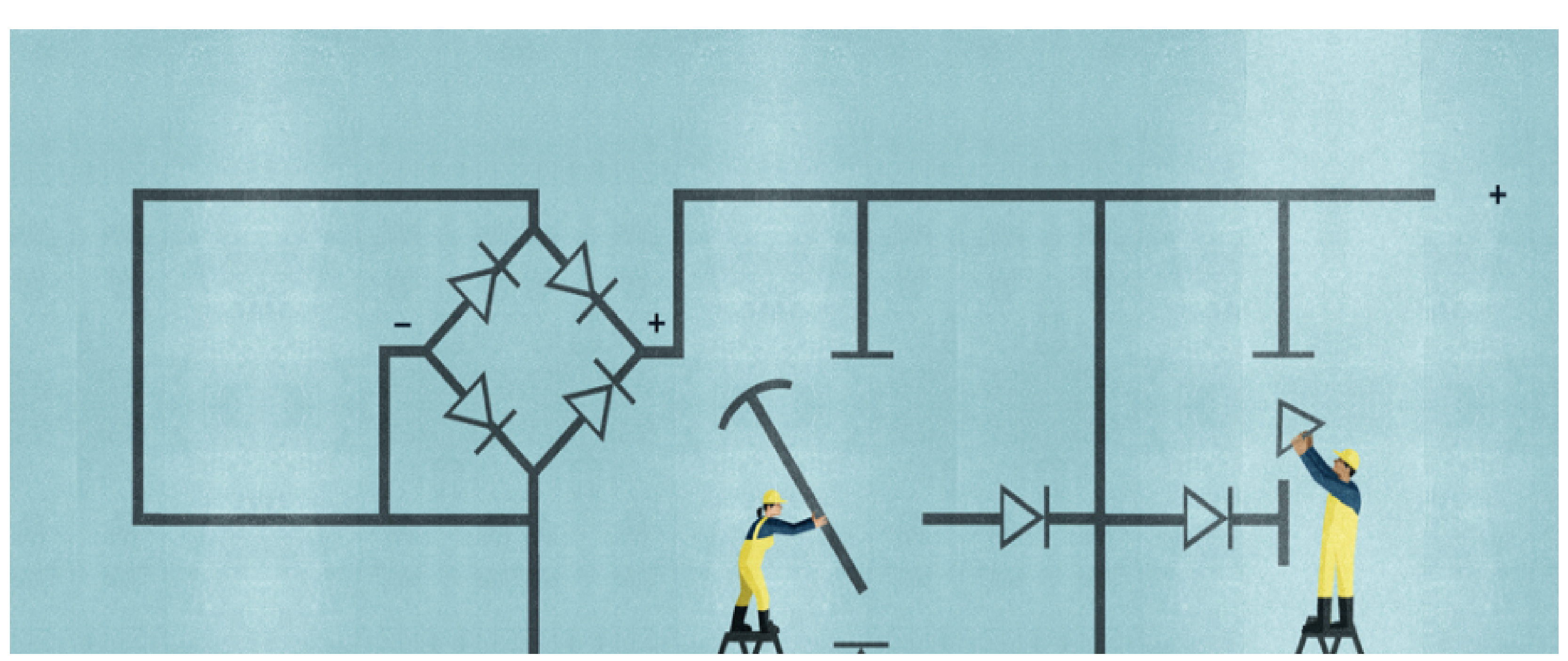
All topics →

MATHEMATICS

'Outsiders' Crack 50-Year-Old Math Problem

Three computer scientists have solved a problem central to a dozen far-flung mathematical fields.

20



Research is nonlinear

What is a graph Laplacian?

Definition: The Laplacian L of a graph $G = ([n], E, w)$ is the matrix $L = D - A$, where

$A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix $A_{ij} = w(ij)$ if $ij \in E$,

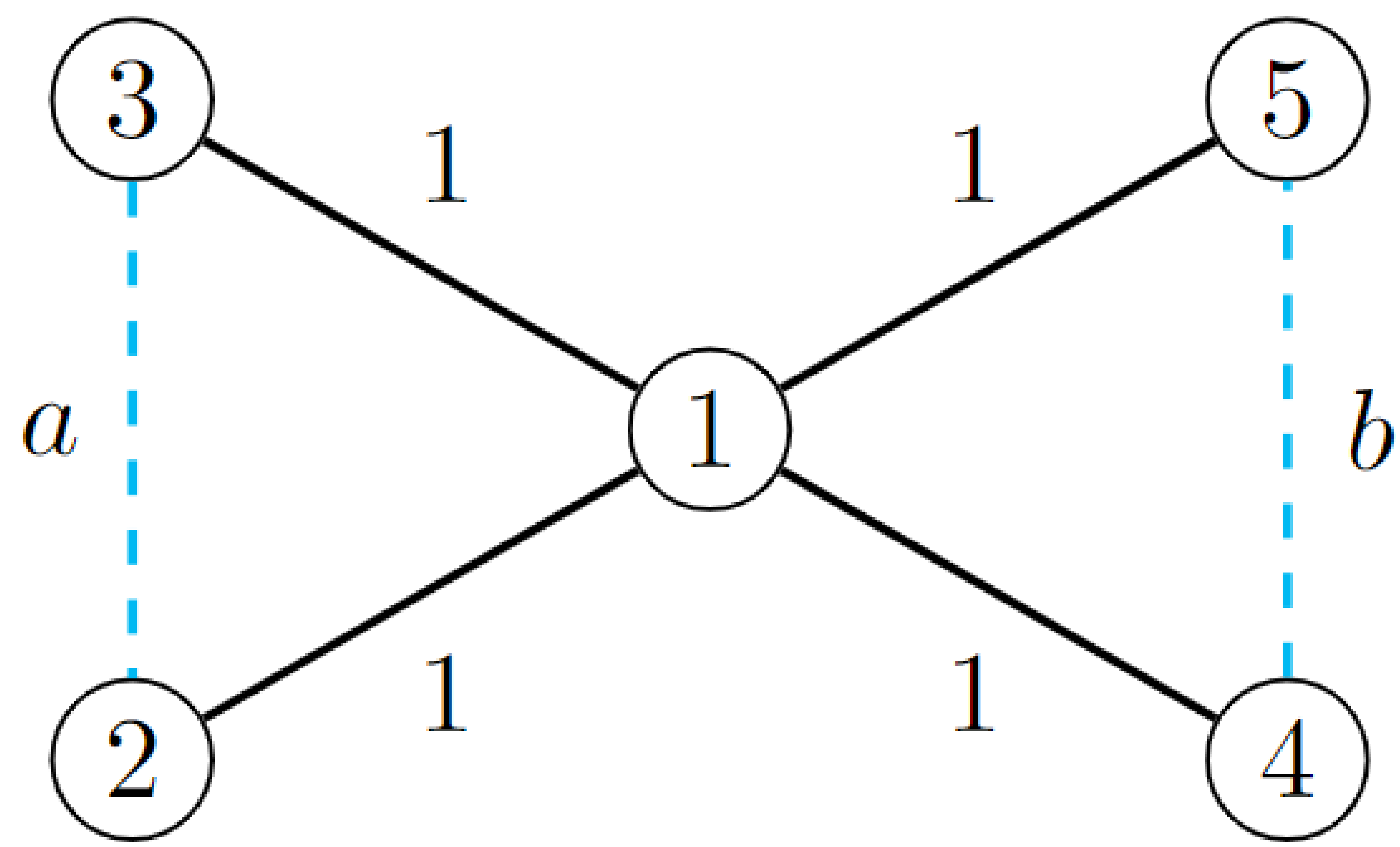
$D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $D_{ii} = \deg i = \sum_{ij\in E} w(ij)$.

What is a graph Laplacian?

Definition: The Laplacian L of a graph $G = ([n], E, w)$ is the matrix $L = D - A$, where

$A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix $A_{ij} = w(ij)$ if $ij \in E$,

$D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $D_{ii} = \deg i = \sum_{ij\in E} w(ij)$.

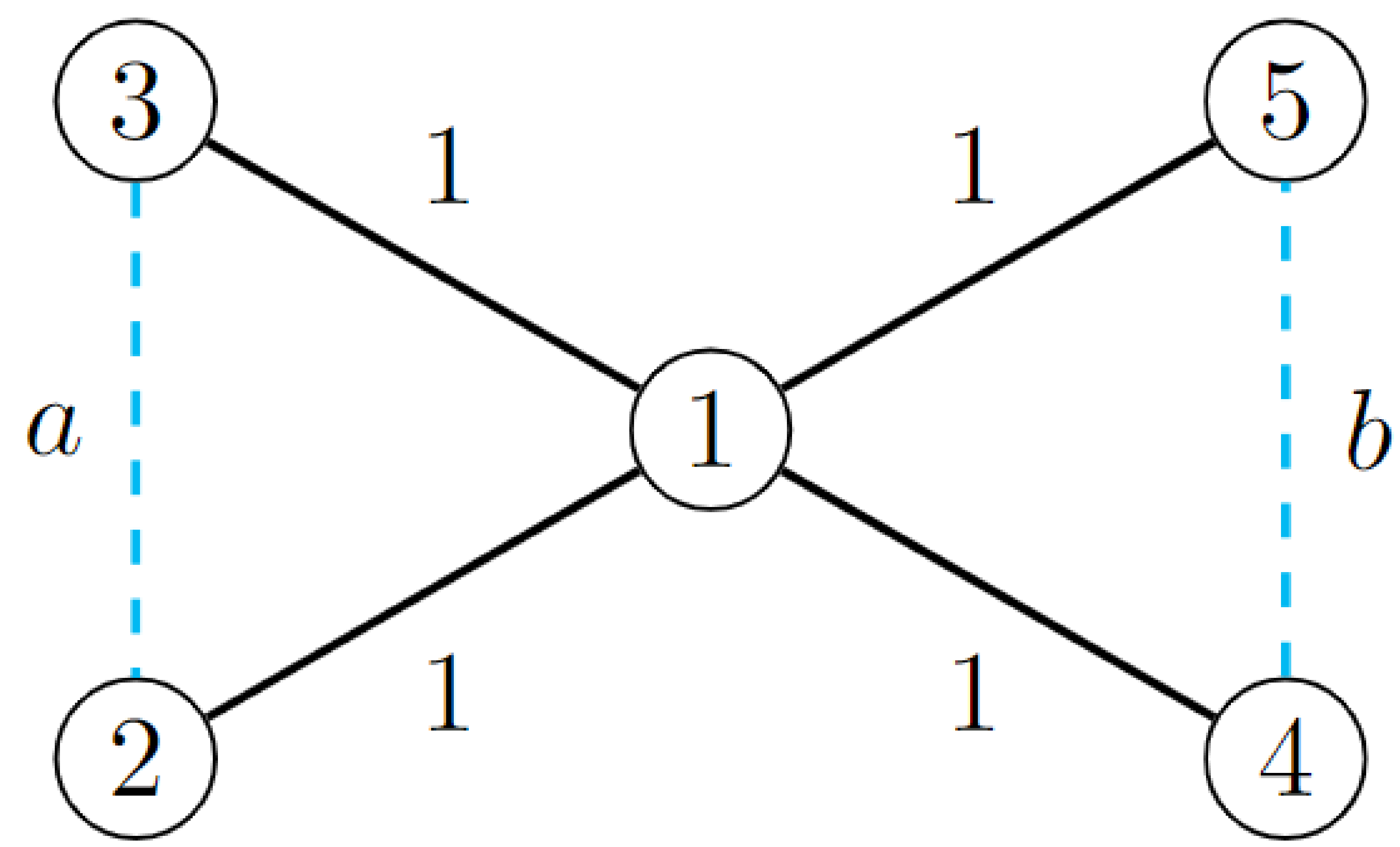


What is a graph Laplacian?

Definition: The Laplacian L of a graph $G = ([n], E, w)$ is the matrix $L = D - A$, where

$A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix $A_{ij} = w(ij)$ if $ij \in E$,

$D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $D_{ii} = \deg i = \sum_{ij\in E} w(ij)$.



$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ - & 1+a & -a & 0 & 0 \\ - & - & 1+a & 0 & 0 \\ - & - & - & 1+b & -b \\ - & - & - & - & 1+b \end{bmatrix}$$

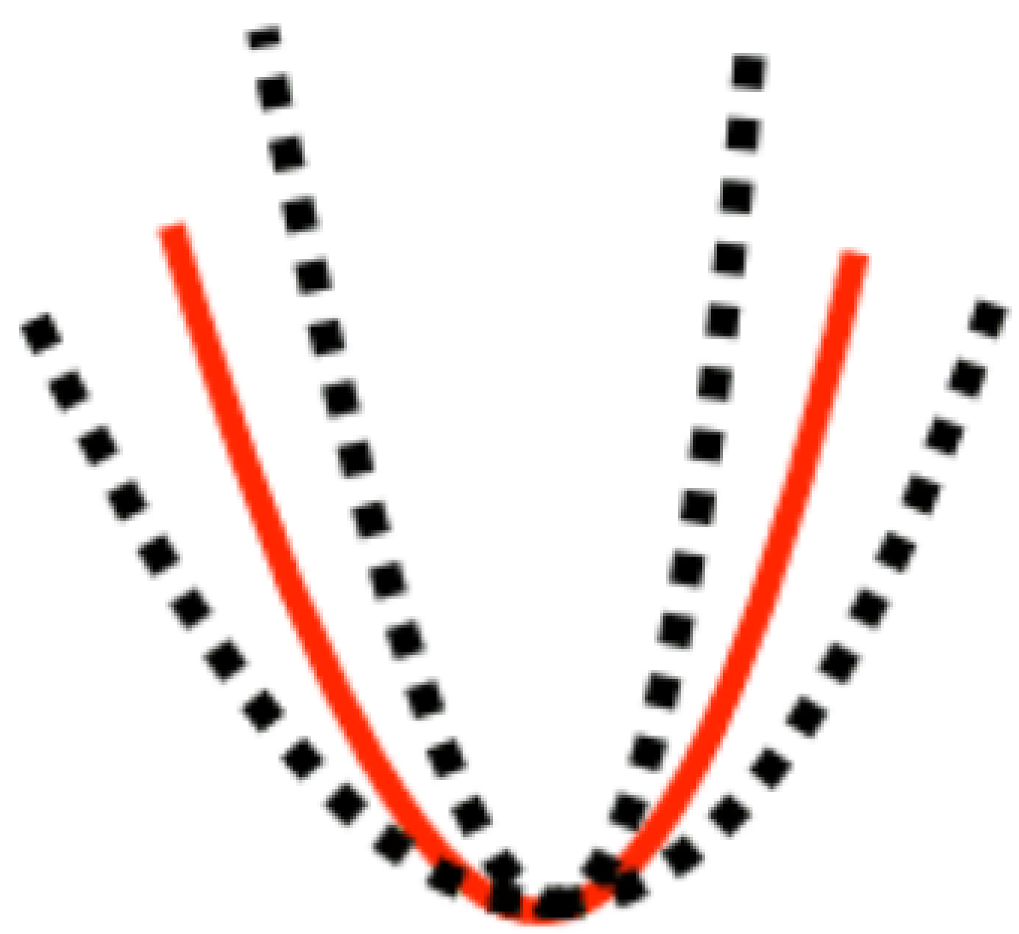
Definition: The quadratic form induced by L is

$$Q_G(x) = x^\top L x = \sum_{ij\in E} w(ij)(x_i - x_j)^2$$

Spectral Sparsification

Definition: A graph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ is an ε-sparsifier of $G = (V, E, w)$ if for all $x \in \mathbb{R}^n$

$$(1-\varepsilon)Q_G(x) \leq Q_{\tilde{G}}(x) \leq (1+\varepsilon)Q_G(x)$$



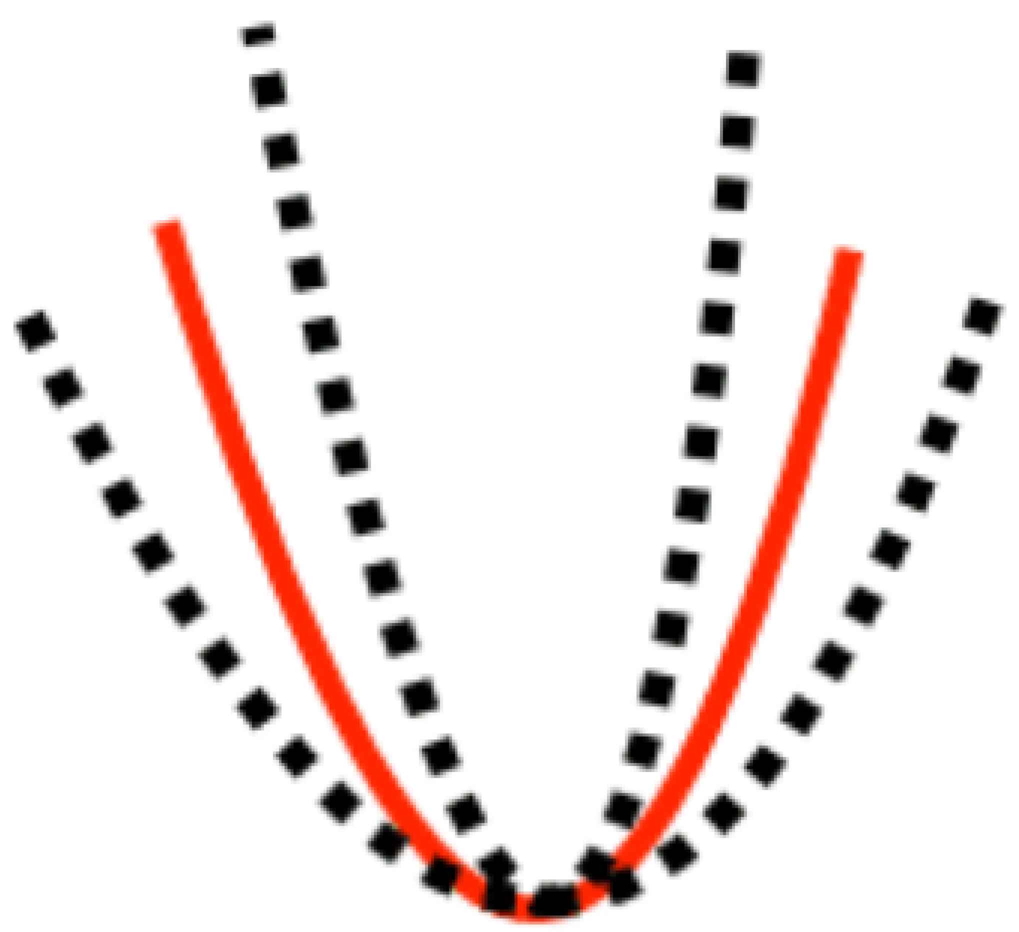
Spectral Sparsification

Definition: A graph $\widetilde{G} = (V,\widetilde{E},\widetilde{w})$ is an ε-sparsifier of $G = (V,E,w)$ if for all $x \in \mathbb{R}^n$

$$(1-\varepsilon)Q_G(x) \leq Q_{\tilde{G}}(x) \leq (1+\varepsilon)Q_G(x)$$

G and $\widetilde{G}$ have similar:

- eigenvalues
- cuts and clustering
- effective resistances



Spectral Sparsification

Definition: A graph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ is an ε-sparsifier of $G = (V,E,w)$ if for all $x \in \mathbb{R}^n$

$$(1-\varepsilon)Q_G(x) \leq Q_{\tilde{G}}(x) \leq (1+\varepsilon)Q_G(x)$$

G and $\widetilde{G}$ have similar:

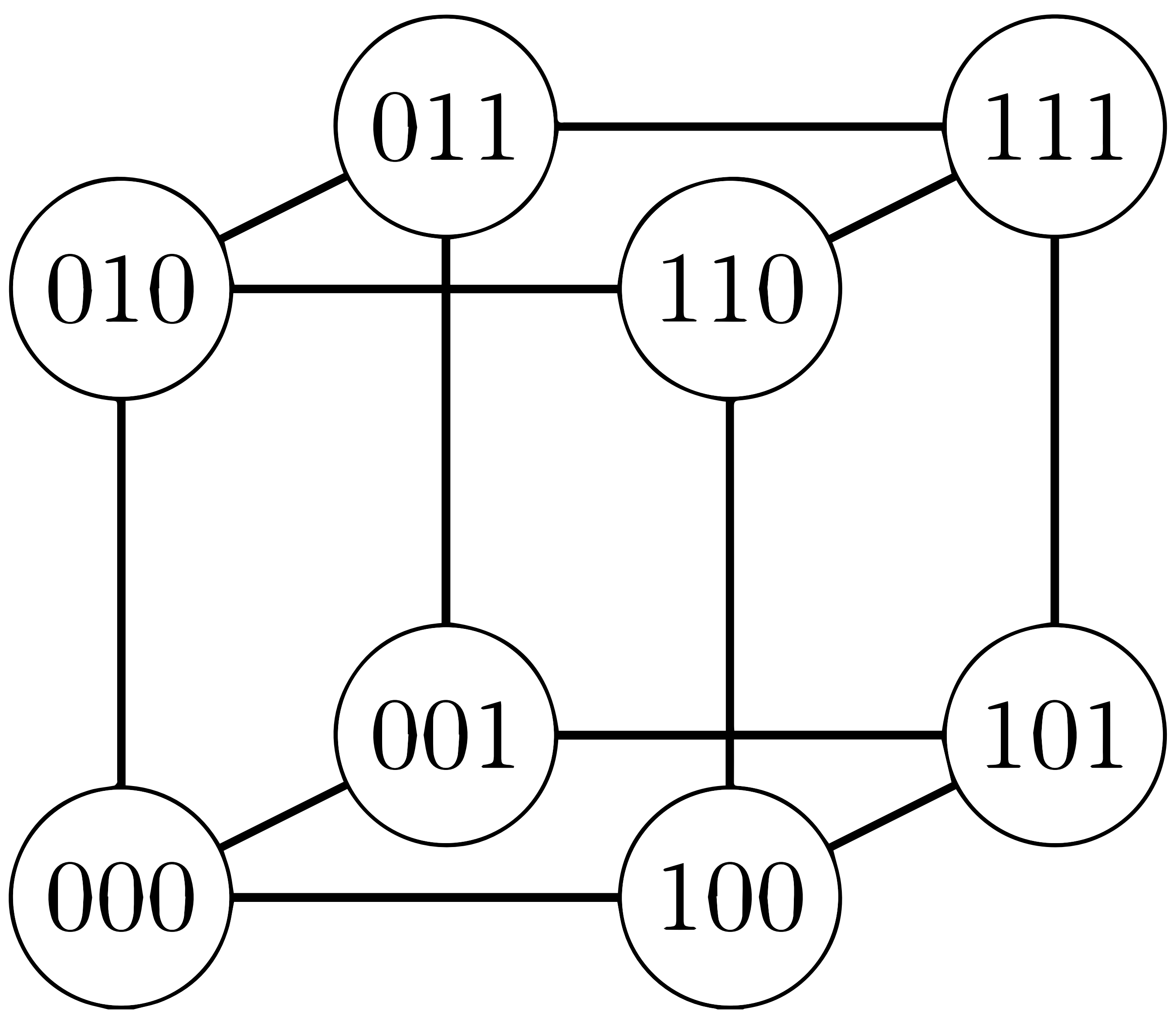
- eigenvalues
- cuts and clustering
- effective resistances

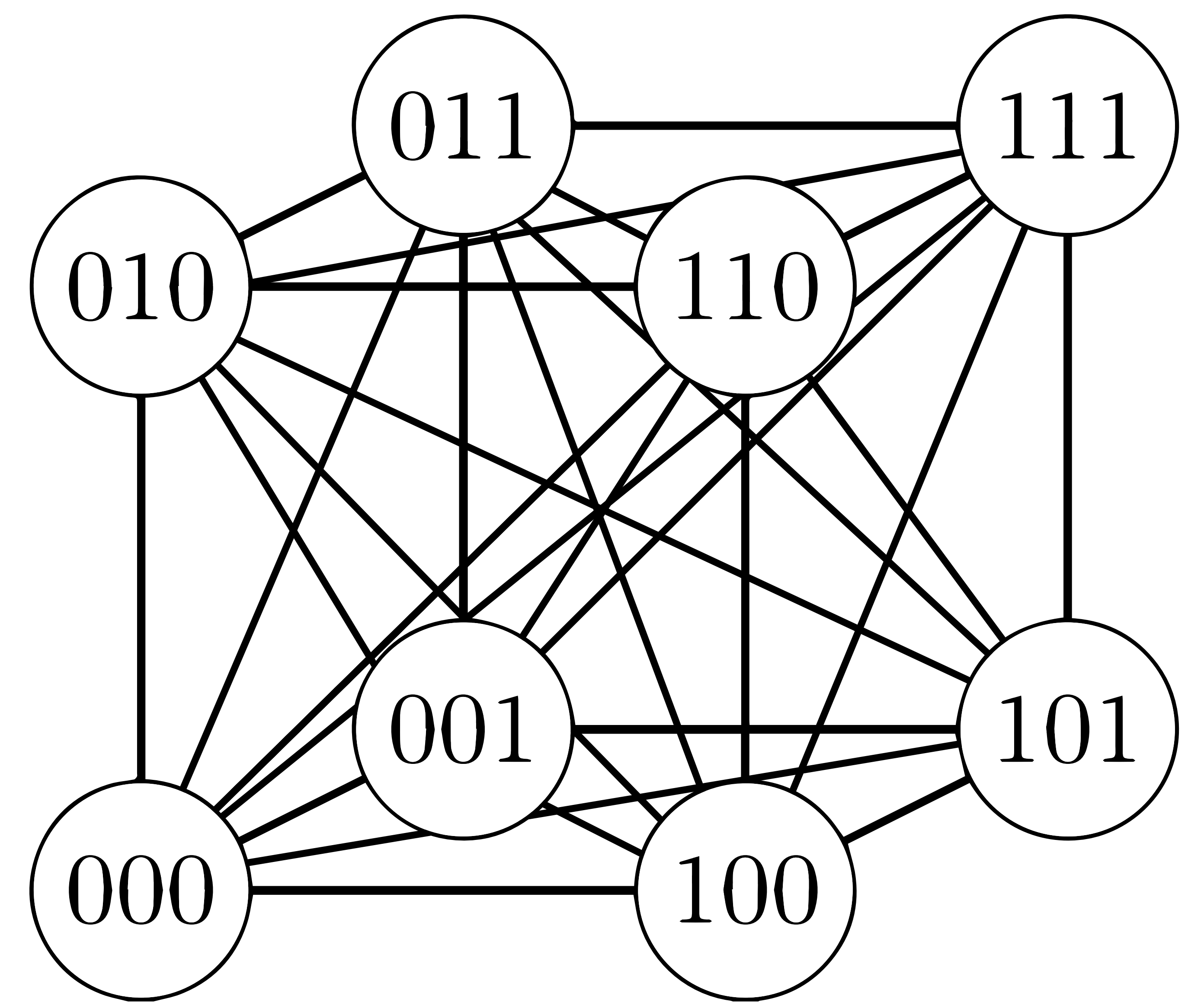
Theorem: Every graph has a near-optimal spectral sparsifier with $O(|V|)$ edges.

(2010's) Batson, Lee, Marcus, Peng, Spielman, Srivastava, Teng, Trevisan...

Spectral Sparsification - an example

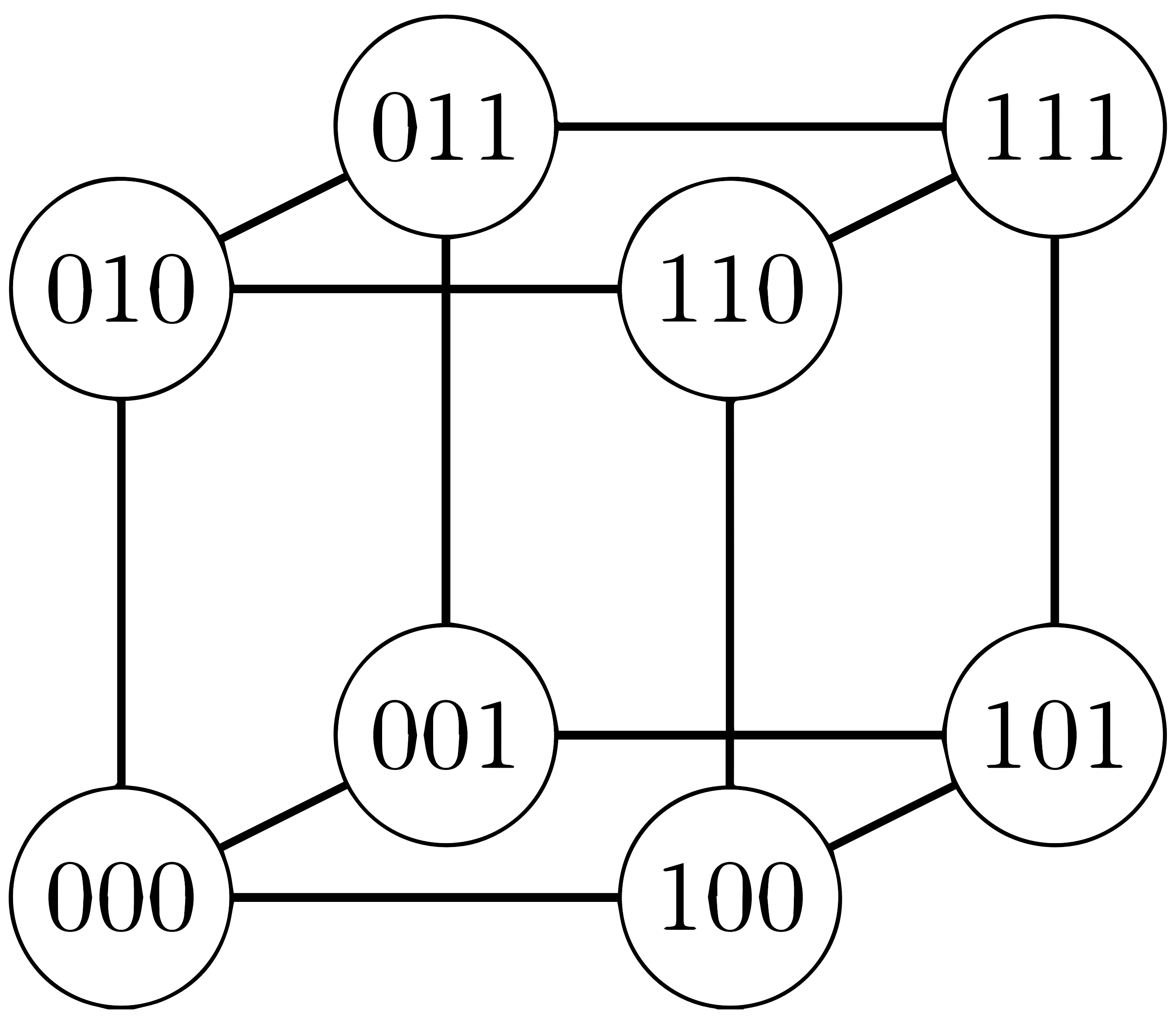
A rescaled d-dimensional cube is a $\sqrt{d}$-sparsifier of K_{2^d}

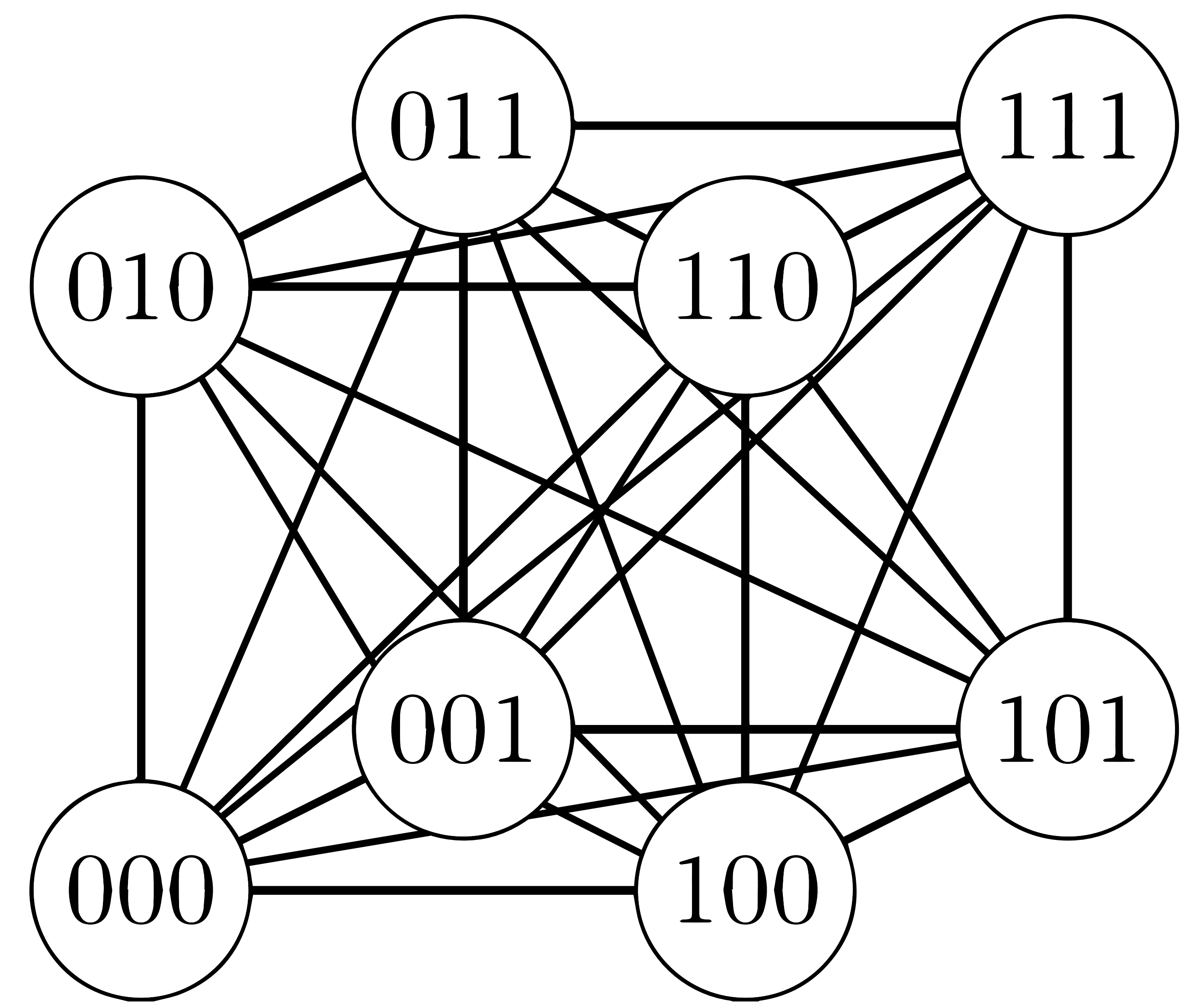




Spectral Sparsification - an example

A rescaled d-dimensional cube is a $\sqrt{d}$-sparsifier of K_{2^d}





d-cube spectrum

$$(2i)^{\binom{d}{i}} \text{ for } i = 0, \ldots, d$$

K_n spectrum

$$0^{(1)}, n^{(n-1)}$$

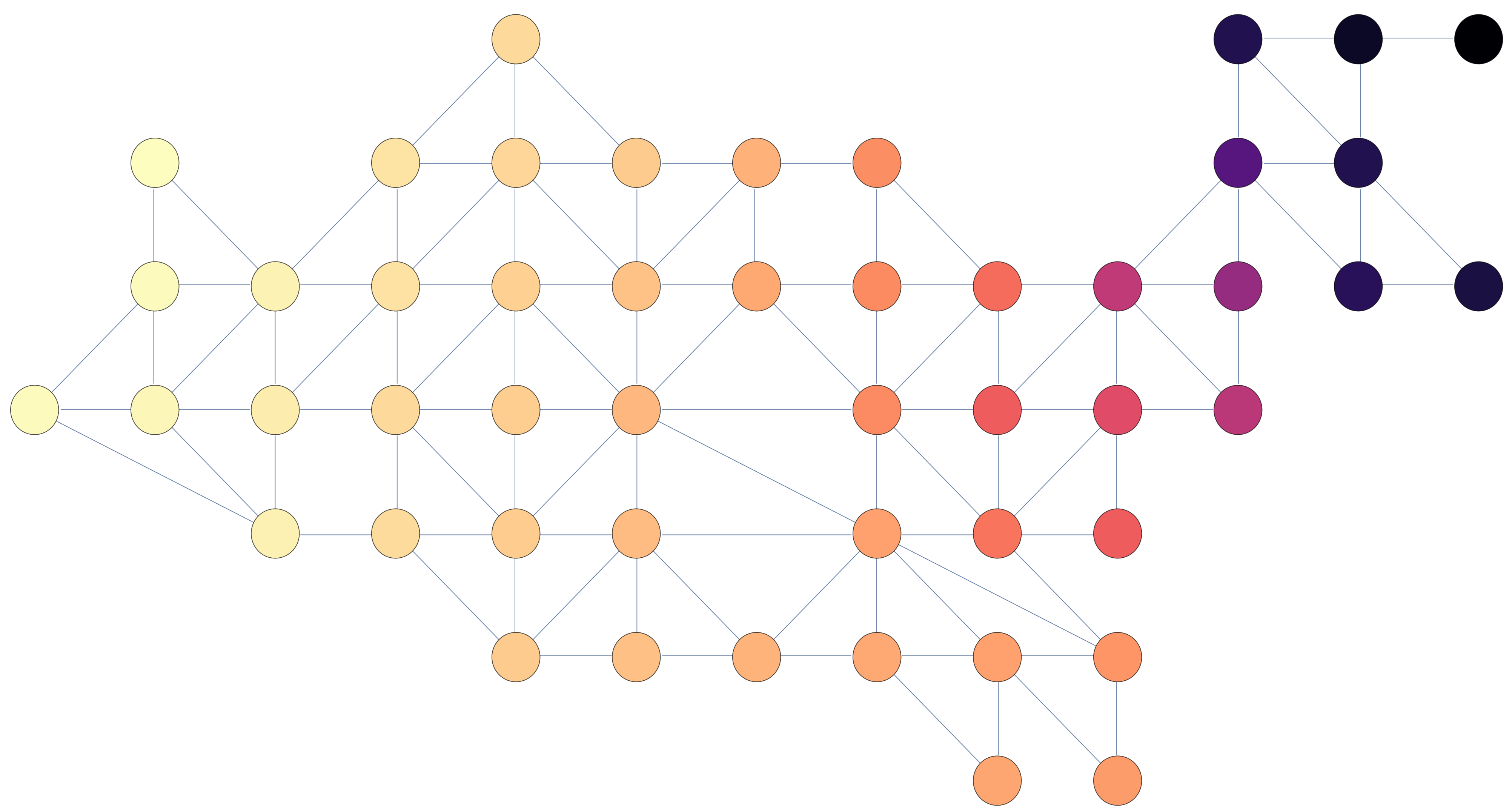
The Laplacian spectrum encodes graph structure

(Low Frequency) eigenvalues and eigenvectors of L:

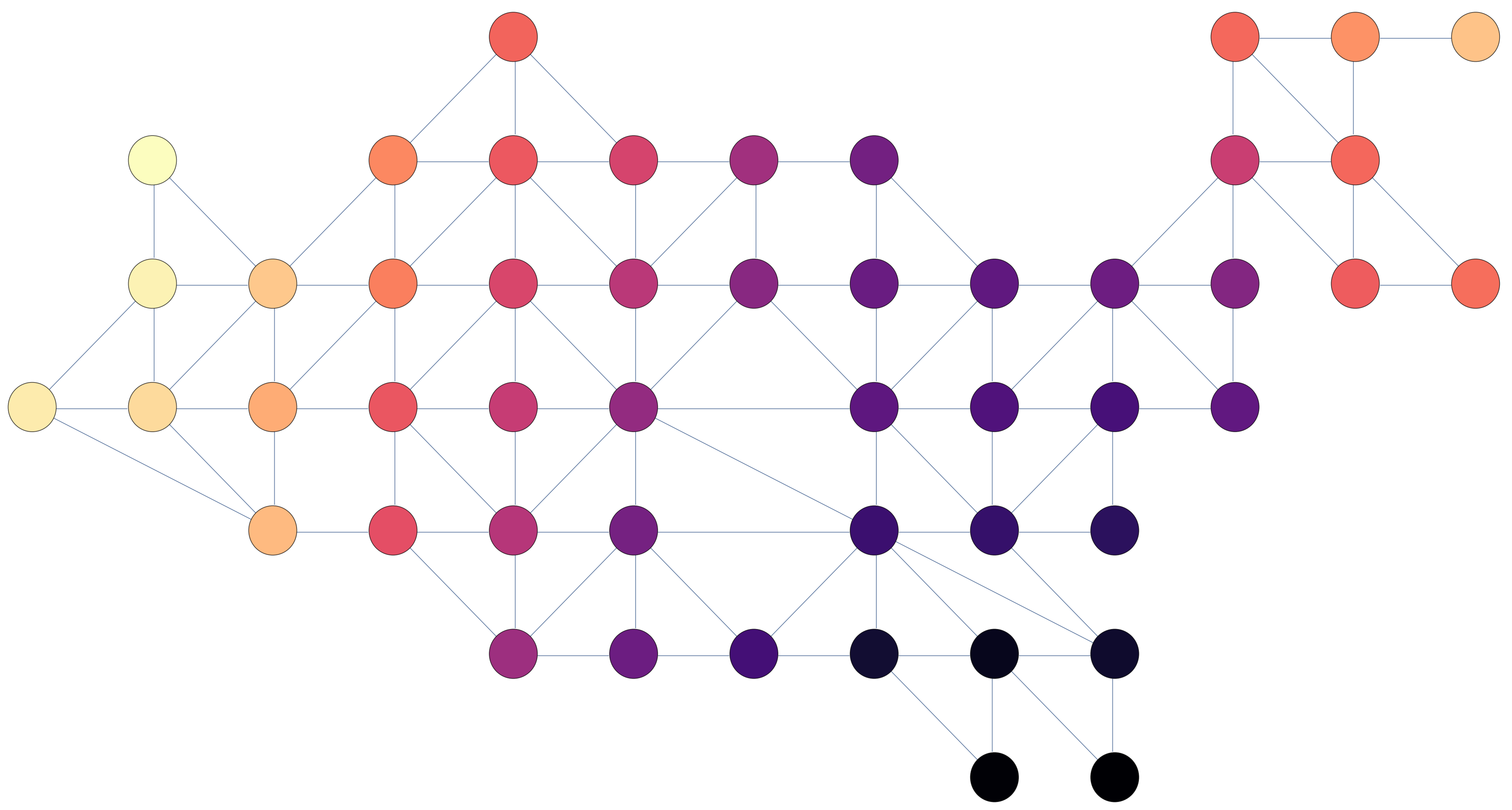
- connected components
- clustering
- mixing time of random walks
- sparsest cut
- spectral drawings

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$
$$L\varphi_i = \lambda_i \varphi_i$$

Low frequency eigenvectors

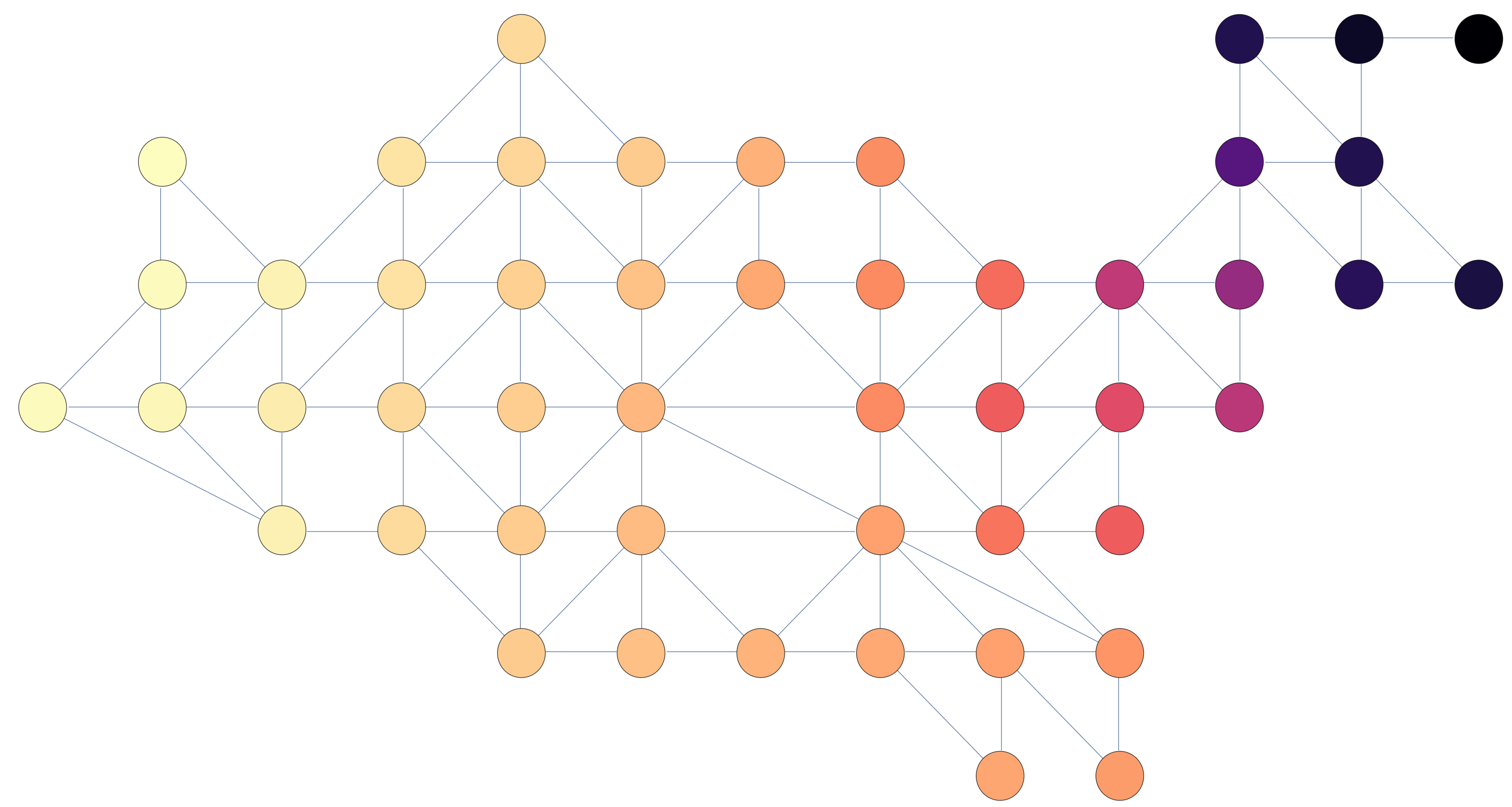


The first eigenvector

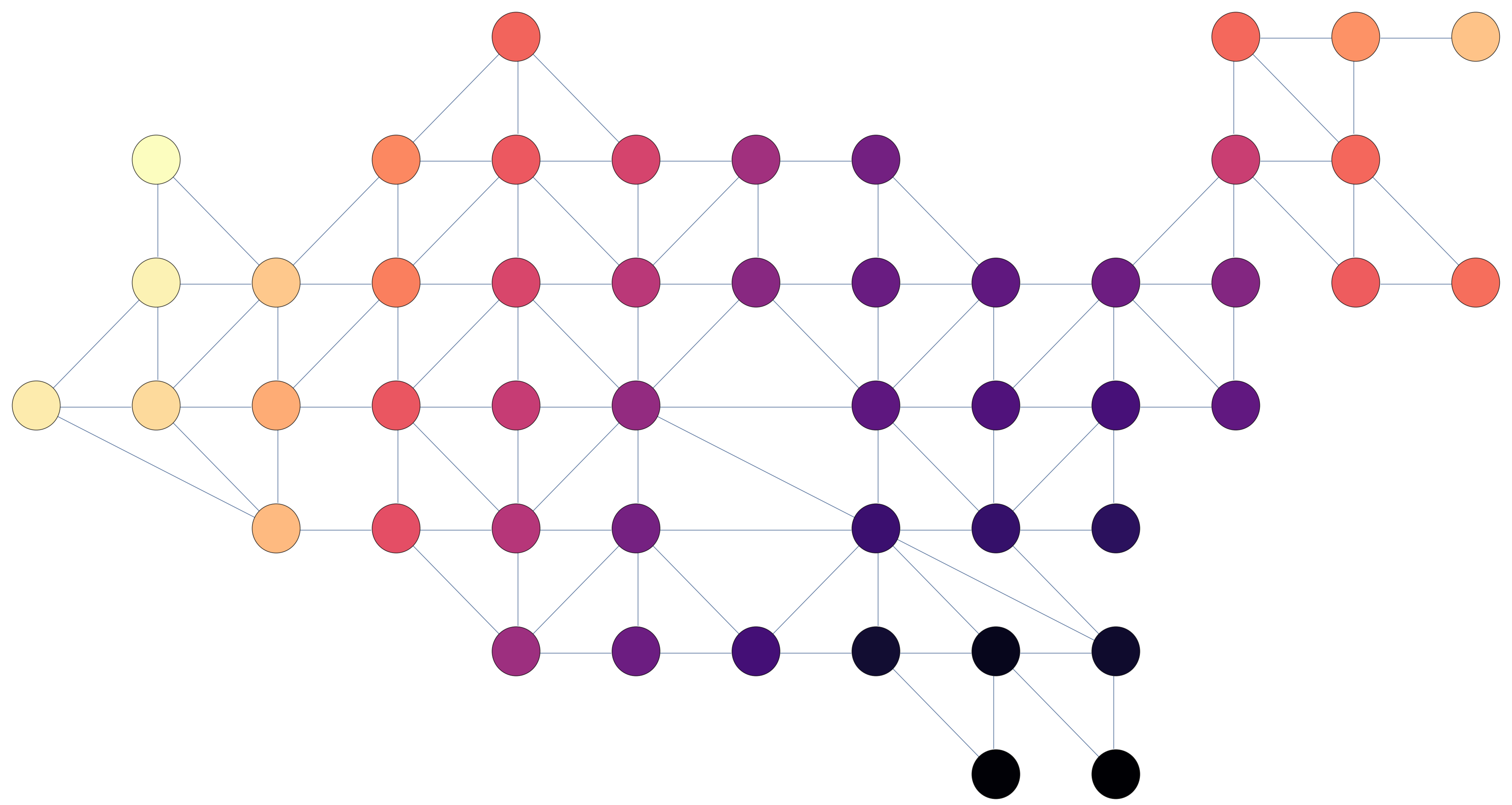


The second eigenvector

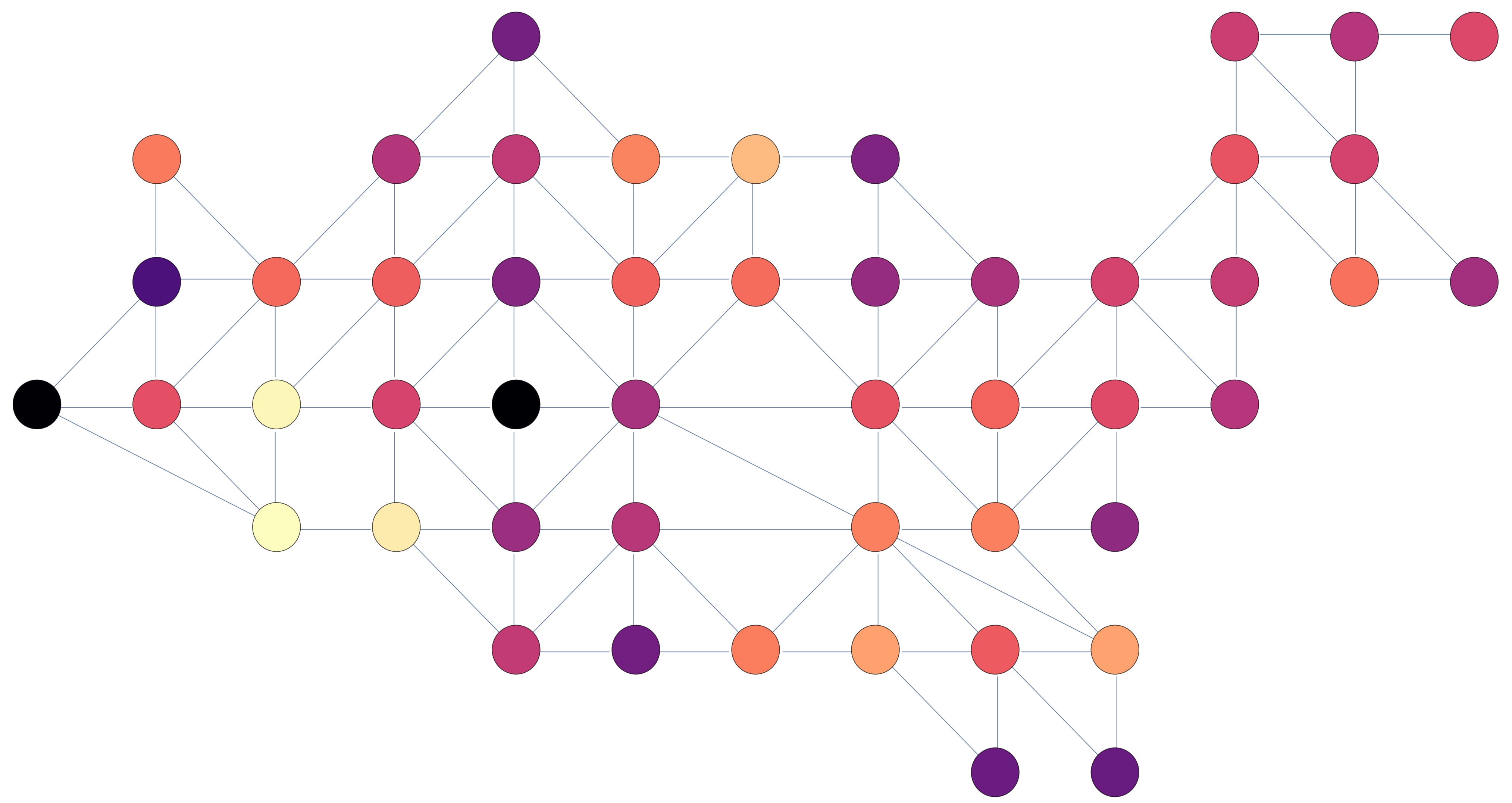
Low frequency eigenvectors



The first eigenvector

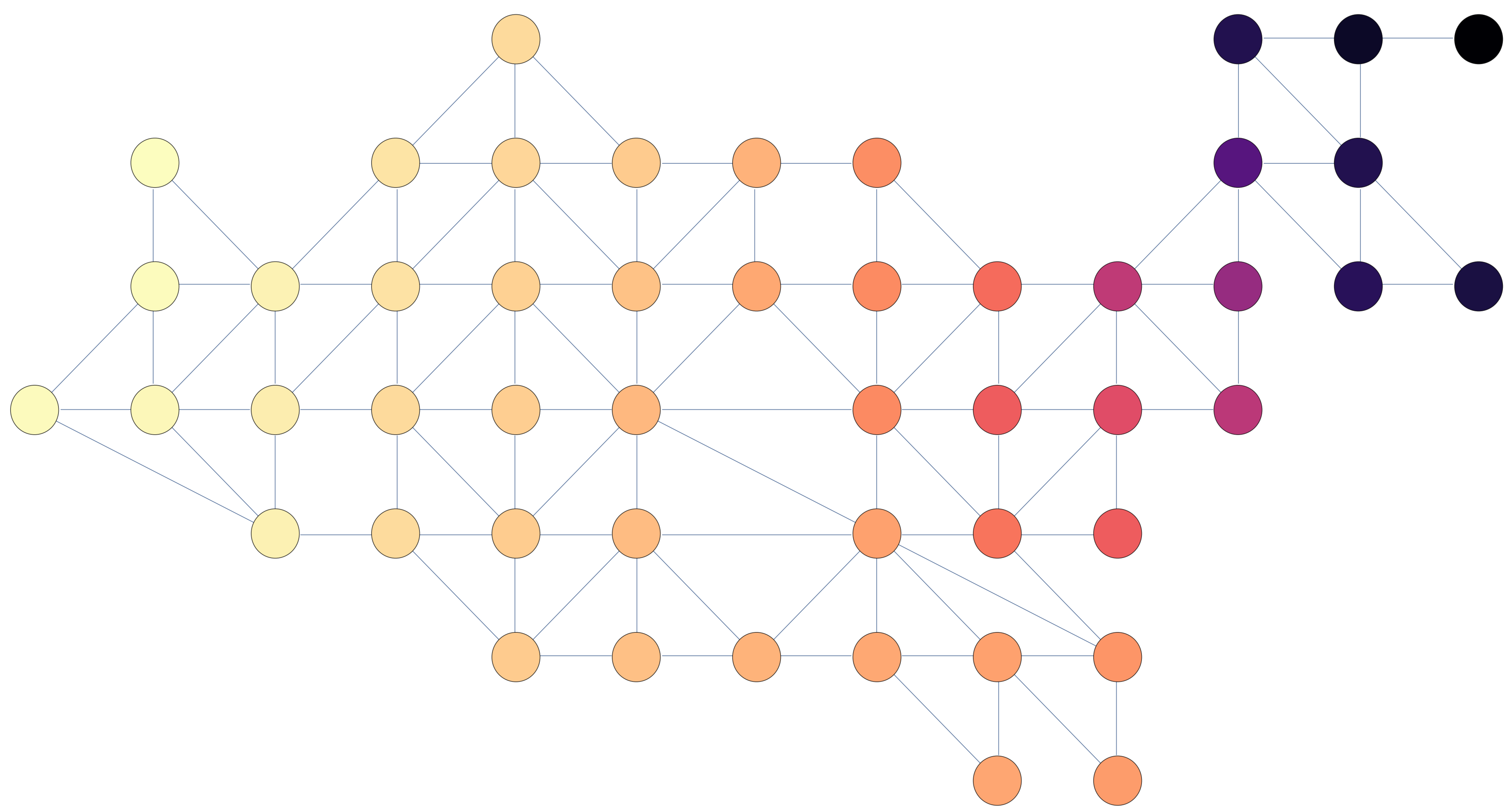


The second eigenvector

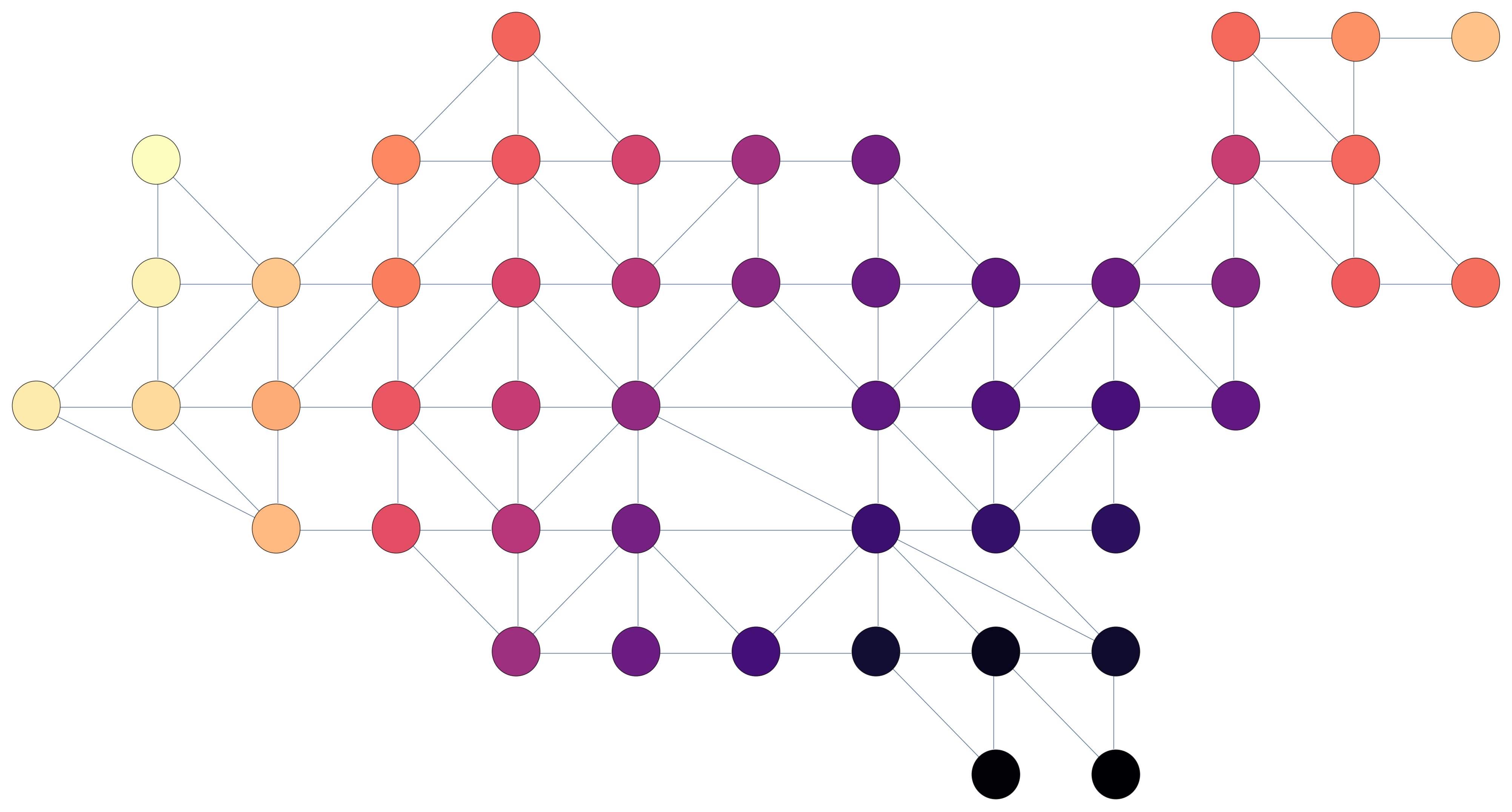


The 18th eigenvector

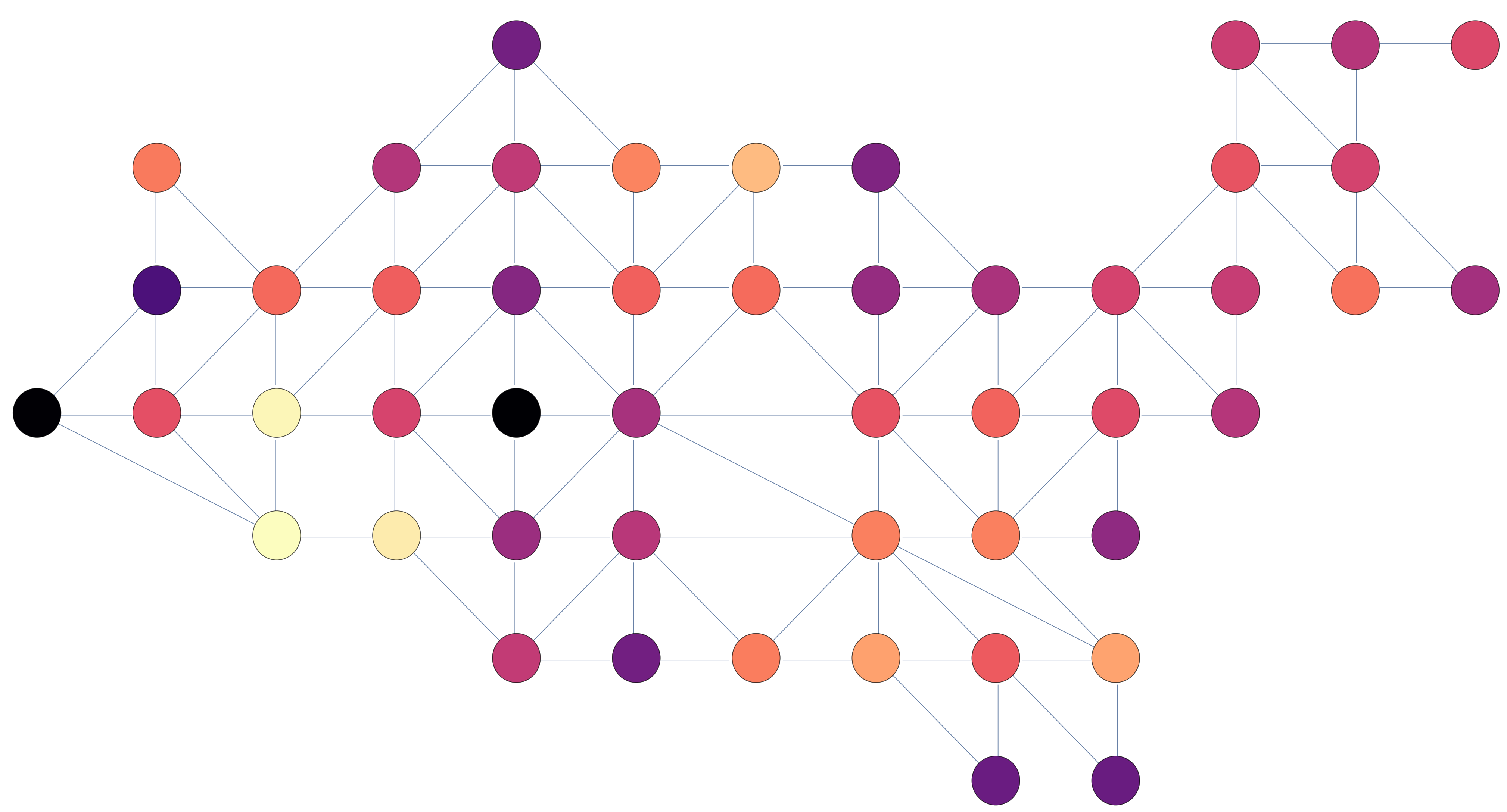
Low frequency eigenvectors



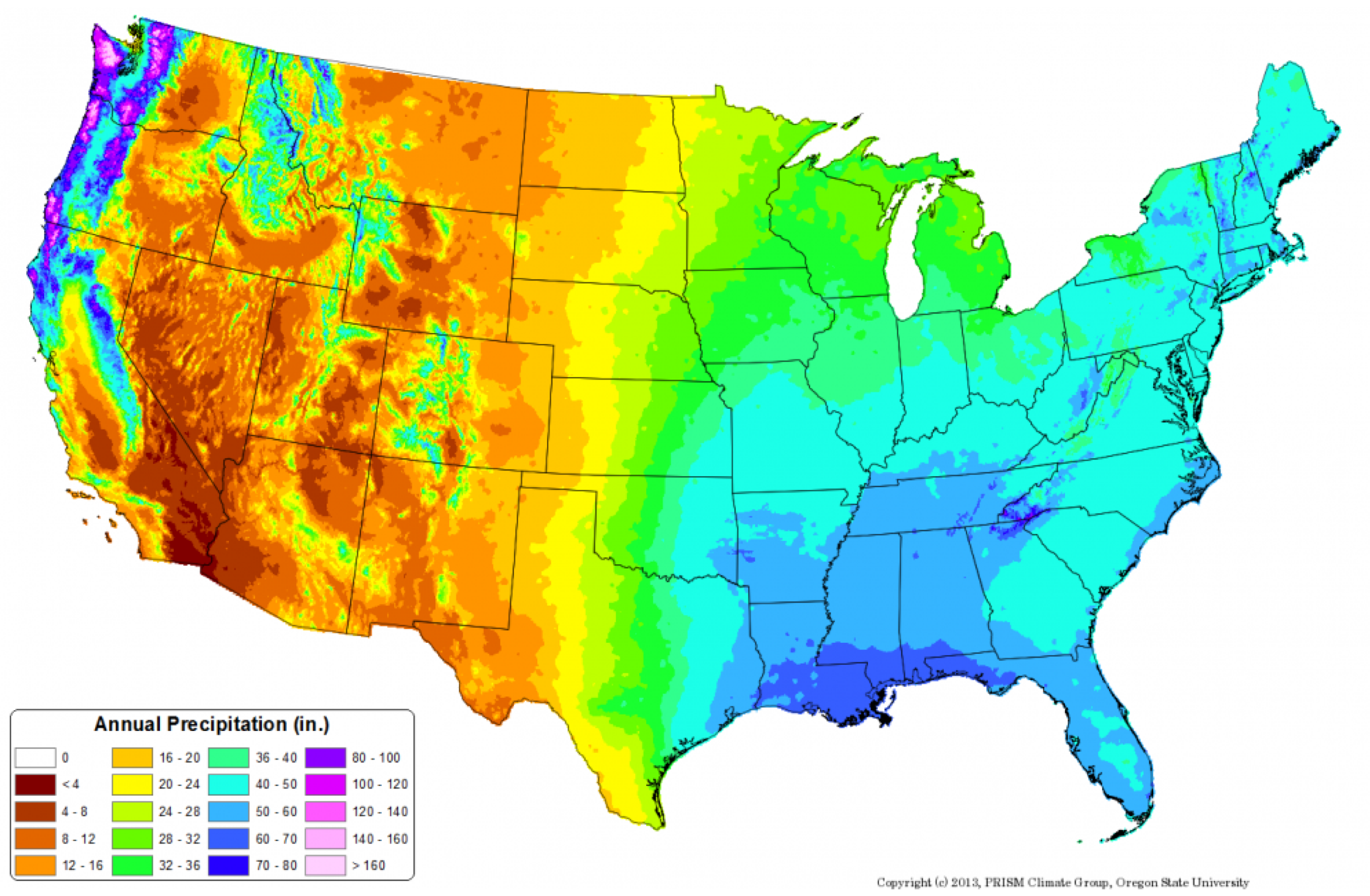
The first eigenvector



The second eigenvector



The 18th eigenvector



Average annual precipitation, 1981-2010

k-sparsifiers

Spectral Graph Theory Heuristic. The low-frequency eigenvalues (and eigenvectors) of L_G capture the global structure of G.

k-sparsifiers

Spectral Graph Theory Heuristic. The low-frequency eigenvalues (and eigenvectors) of L_G capture the global structure of G.

Spectral Sparsification Heuristic. A sparsification of G should preserve the spectrum of L.

k-sparsifiers

Spectral Graph Theory Heuristic. The low-frequency eigenvalues (and eigenvectors) of L_G capture the global structure of G.

\+ **Spectral Sparsification Heuristic.** A sparsification of G should preserve the spectrum of L.

Definition: A subgraph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ is ***k*-isospectral** to $G = (V, E, w)$ if they share the same first k eigenvalues and eigenvectors.

k-sparsifiers

Spectral Graph Theory Heuristic. The low-frequency eigenvalues (and eigenvectors) of L_G capture the global structure of G.

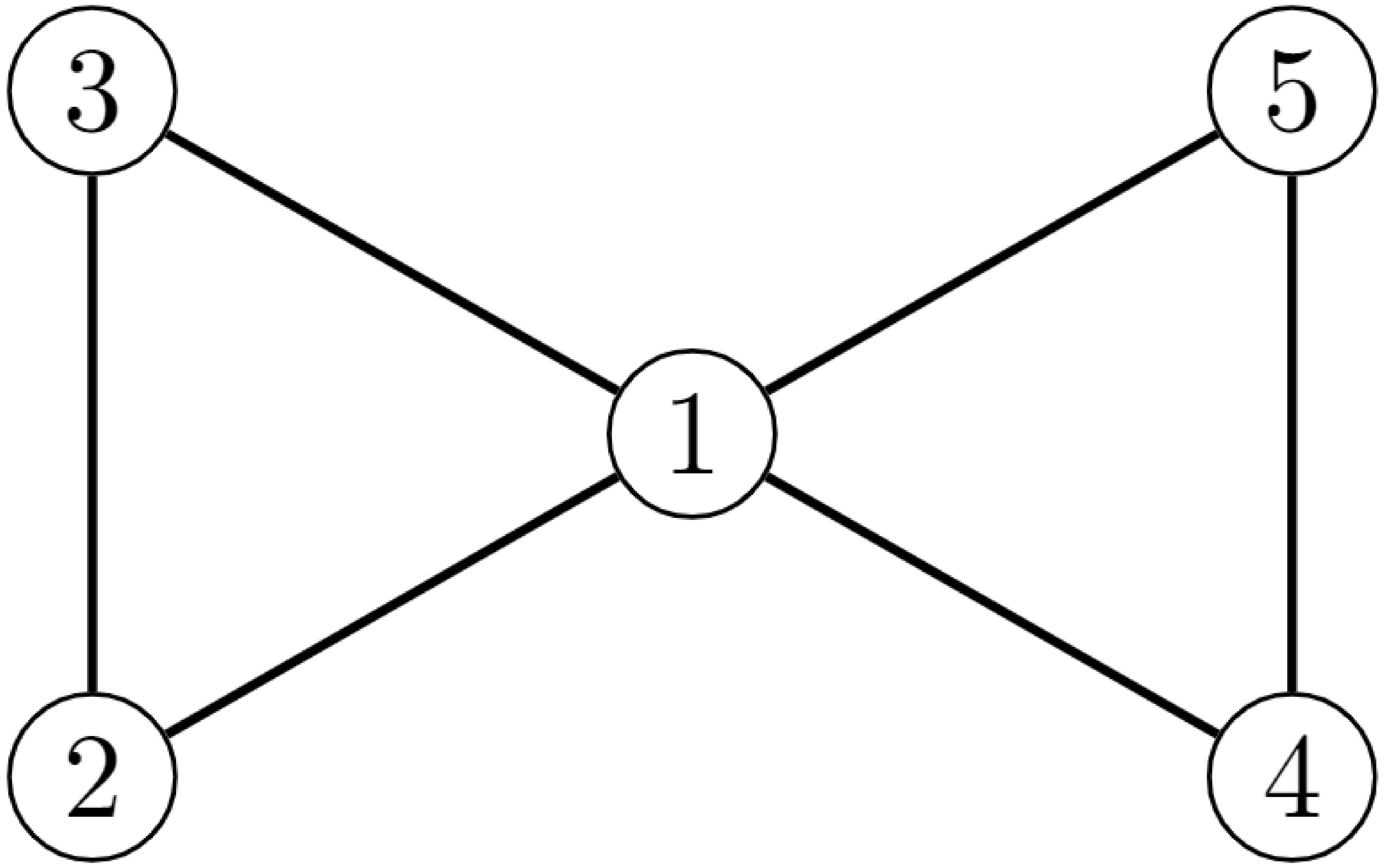
\+ **Spectral Sparsification Heuristic.** A sparsification of G should preserve the spectrum of L.

Definition: A subgraph $\widetilde{G} = (V,\widetilde{E},\widetilde{w})$ is ***k*-isospectral** to $G = (V,E,w)$ if they share the same first k eigenvalues and eigenvectors.

Definition: A k-isospectral subgraph $\widetilde{G} = (V,\widetilde{E},\widetilde{w})$ is a ***k*-sparsifier** of $G = (V,E,w)$ if

$$\tilde{E} \subsetneq E$$

A bare-hands example

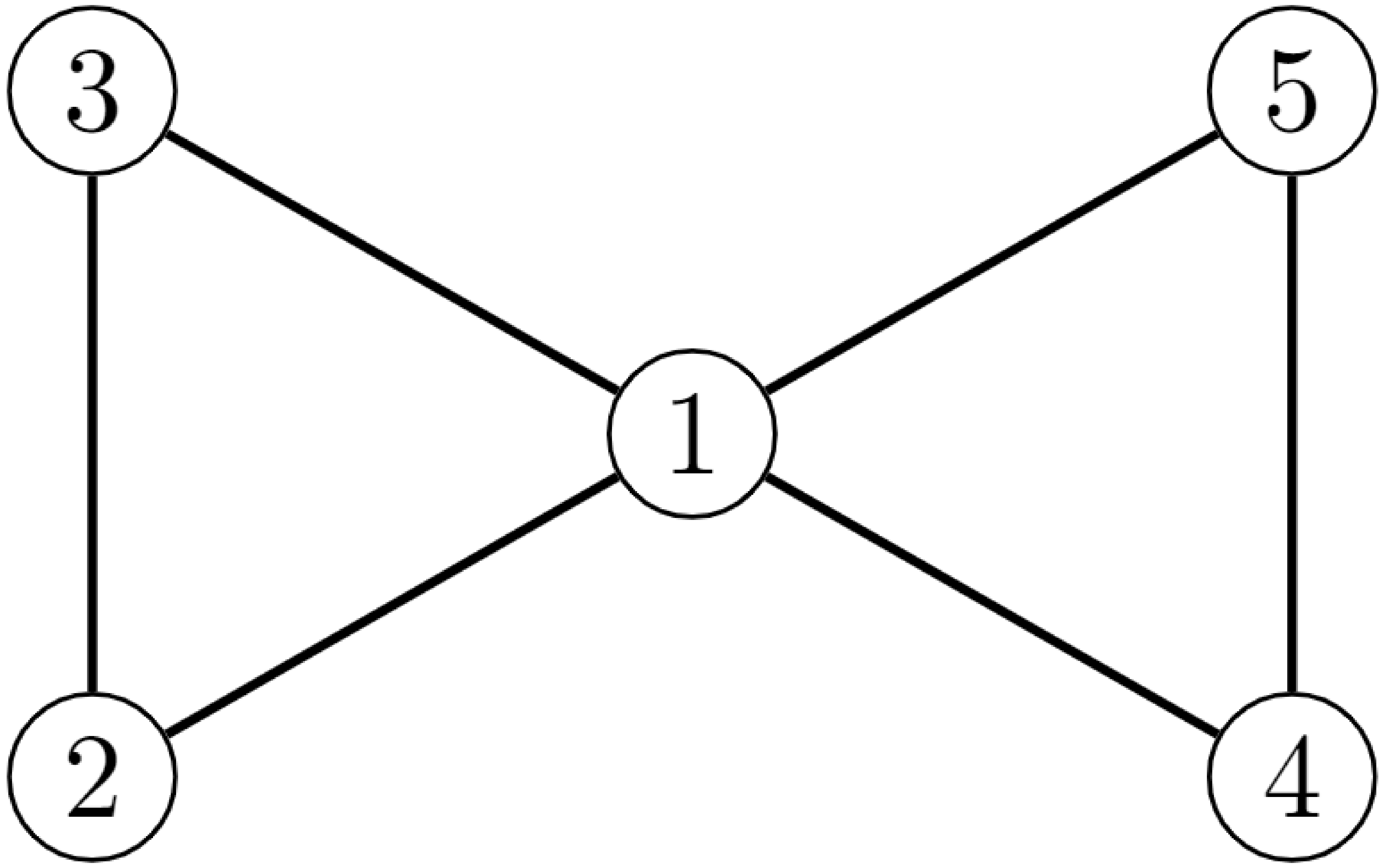


eigenvalues: $0, 1, 3, 3, 5,$

first 2 eigenvectors:

$$\varphi_1 = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)$$
$$\varphi_2 = \frac{1}{2}(0, -1, -1, 1, 1).$$

A bare-hands example

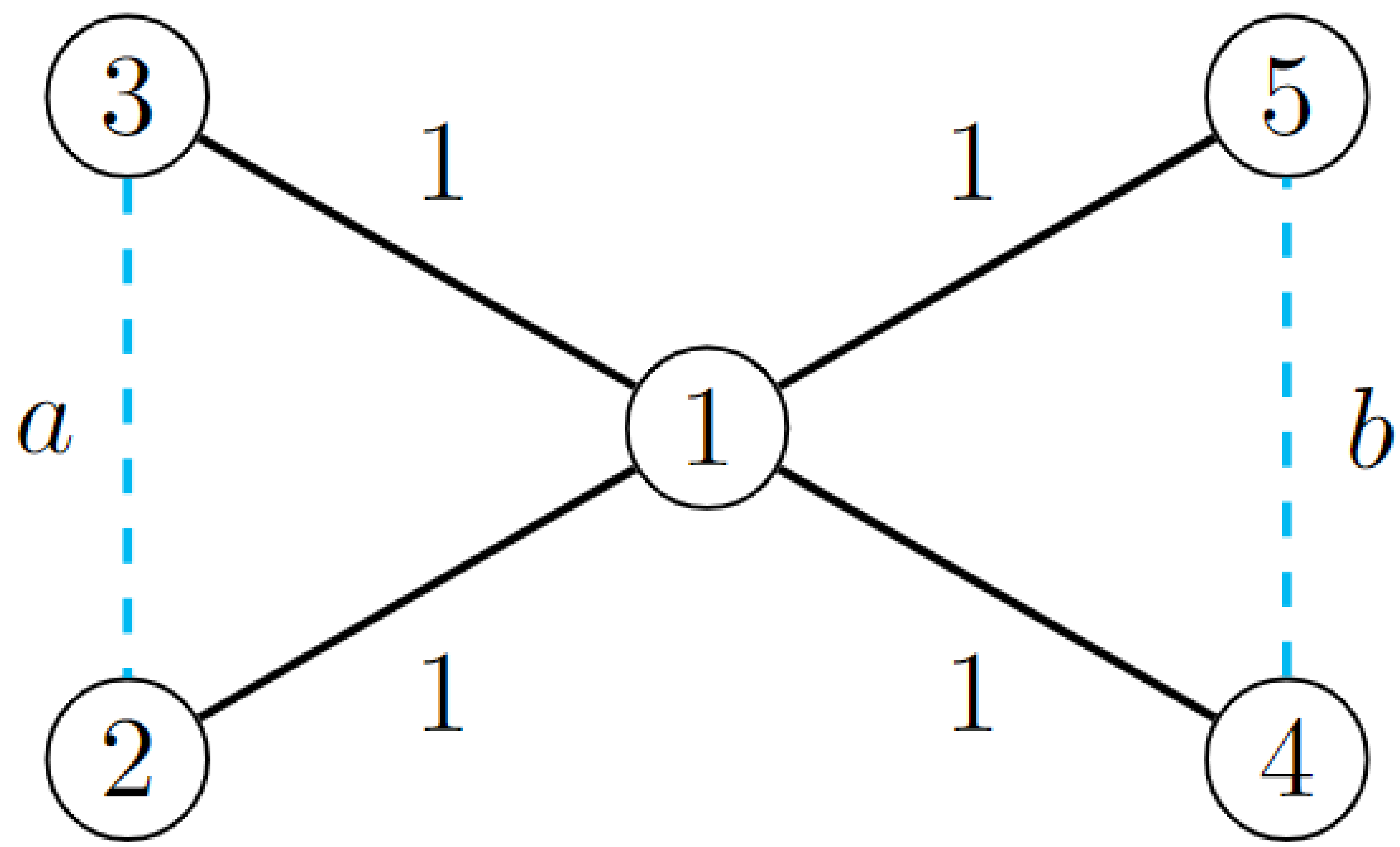


eigenvalues: $0, 1, 3, 3, 5,$

first 2 eigenvectors:

$$\varphi_1 = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)$$
$$\varphi_2 = \frac{1}{2}(0, -1, -1, 1, 1).$$

A bare-hands example

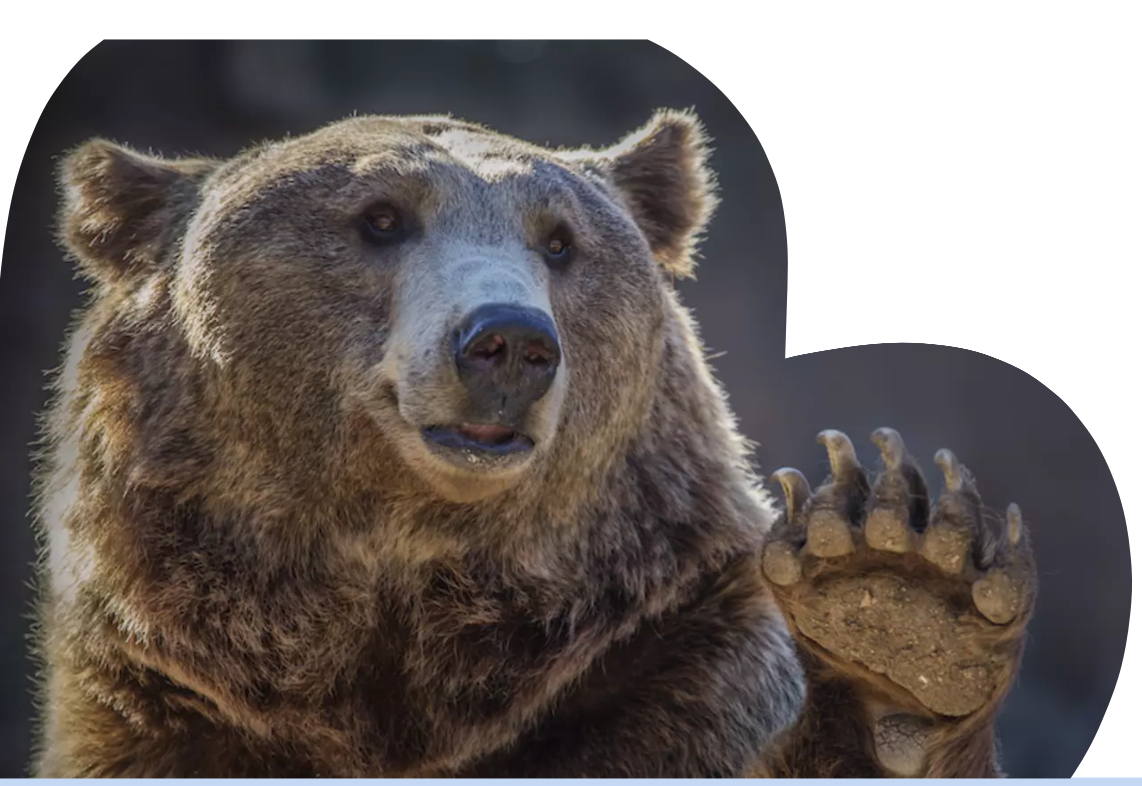


eigenvalues: $0, 1, 3, 3, 5,$

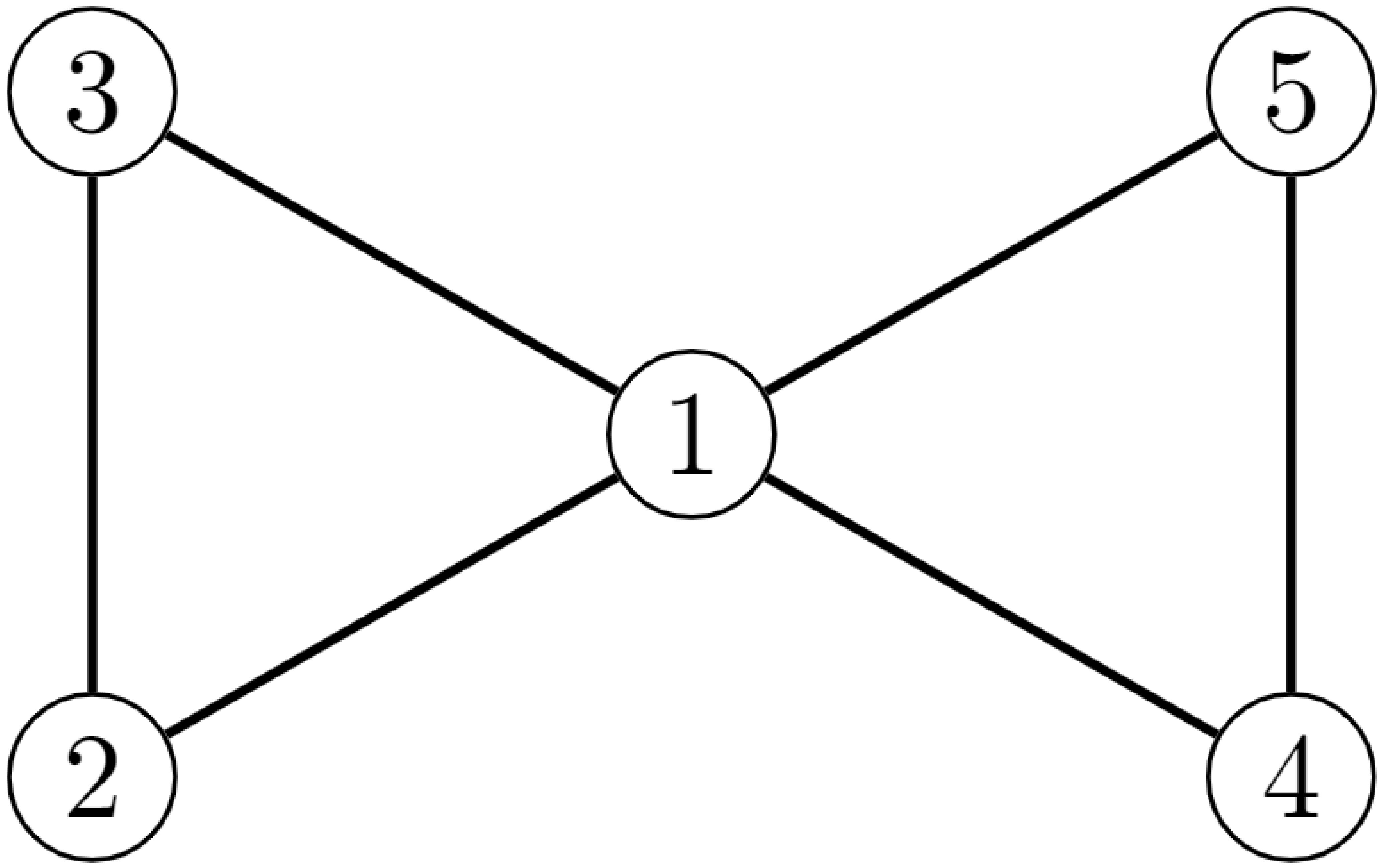
first 2 eigenvectors:

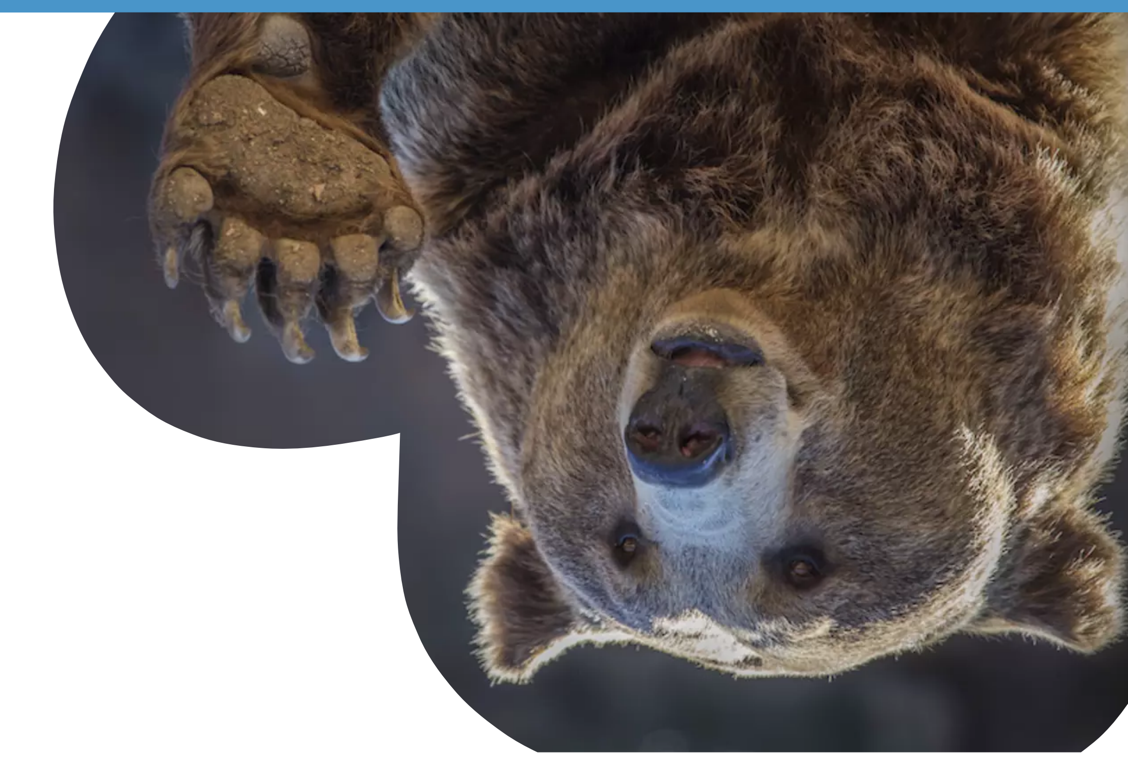
$$\varphi_1 = \frac{1}{\sqrt{5}}(1,1,1,1,1)$$
$$\varphi_2 = \frac{1}{2}(0,-1,-1,1,1).$$

$$\{\text{2-isospectral graphs}\} = \left\{ L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ - & 1+a & -a & 0 & 0 \\ - & - & 1+a & 0 & 0 \\ - & - & - & 1+b & -b \\ - & - & - & - & 1+b \end{bmatrix} : a, b \geq 0 \right\}$$



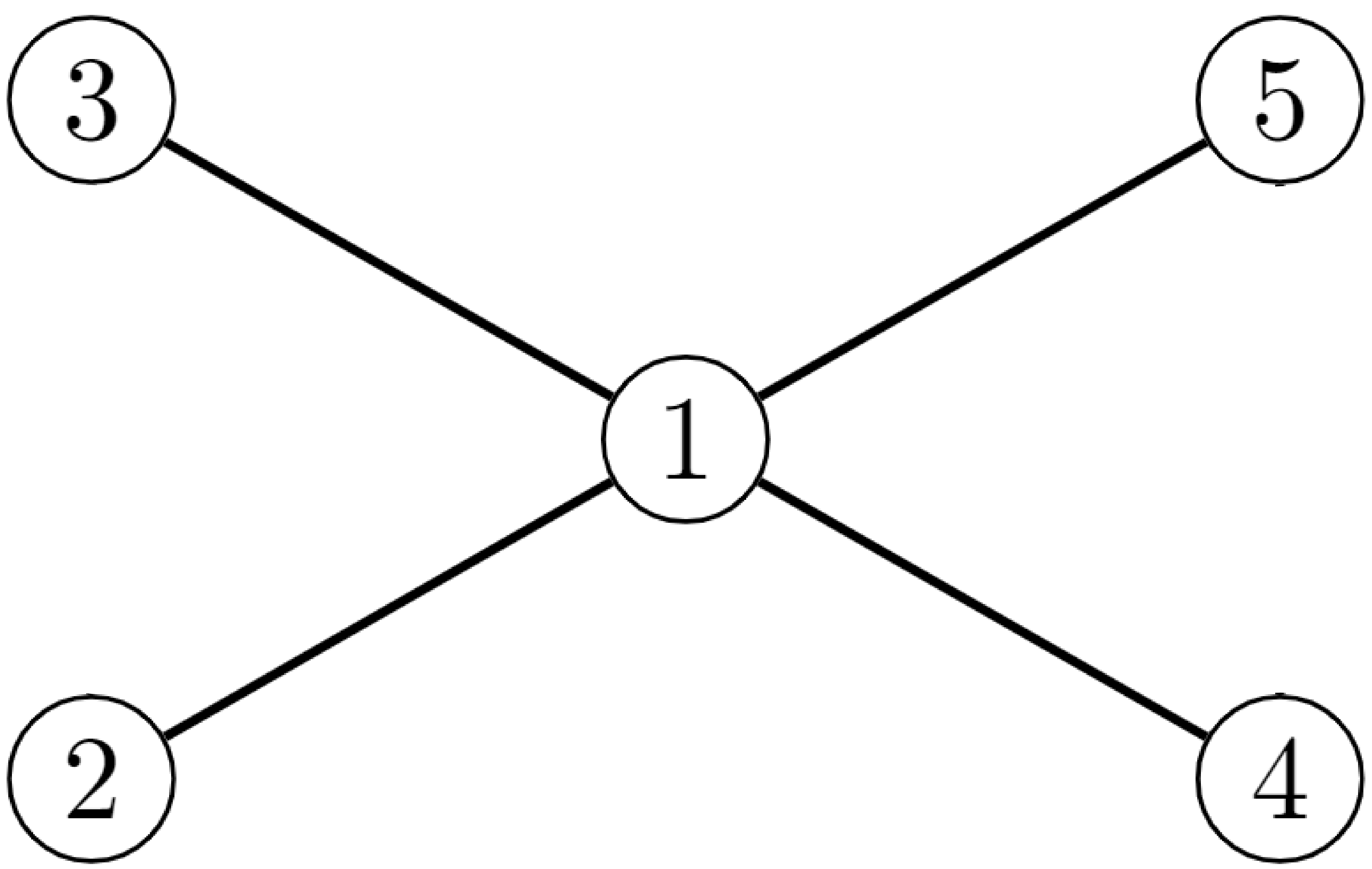
A bare-hands example





$$\left\{ L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ - & 1+a & -a & 0 & 0 \\ - & - & 1+a & 0 & 0 \\ - & - & - & 1+b & -b \\ - & - & - & - & 1+b \end{bmatrix} \; : \; a, b \geq 0 \right\}$$

$$a = b = 0$$



eigenvalues: $0, 1, 5, 6, 11$

first 2 eigenvectors:

$$\varphi_1 = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)$$

$$\varphi_2 = \frac{1}{2}(0, -1, -1, 1, 1).$$

Main Structure Theorem

Theorem 3.1. *Let $G = ([n], E, w)$ be a connected, weighted graph with eigenpairs $(0, \varphi_1), (\lambda_2, \varphi_2), \ldots, (\lambda_n, \varphi_n)$ where $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ and $\{\varphi_i\}_{i=1}^n$ are orthonormal. Fix $2 \leq k \leq n$ and define the matrices*

$$\Phi_k = \begin{bmatrix} \varphi_1 & \cdots & \varphi_k \end{bmatrix} \in \mathbb{R}^{n \times k}, \quad \Phi_{>k} = \begin{bmatrix} \varphi_{k+1} & \cdots & \varphi_n \end{bmatrix} \in \mathbb{R}^{n \times (n-k)},$$

$$\Lambda_k = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^{k \times k}.$$

Then the set of Laplacians of all $k-$isospectral subgraphs of G is

$$\mathrm{Sp}_G(k) = \left\{ L = \underbrace{\Phi_k \Lambda_k \Phi_k^\top + \lambda_k \Phi_{>k} \Phi_{>k}^\top}_{F} + \Phi_{>k} Y \Phi_{>k}^\top : \begin{array}{c} Y \in \mathcal{S}_+^{n-k} \\ L_{st} \leq 0 \ \forall \ (s,t) \in E \\ L_{st} = 0 \ \forall \ s \neq t, \ (s,t) \notin E \end{array} \right\}.$$

B., Steinerberger, Thomas '23

Main Structure Theorem

split the eigenbasis of L into the first k and last n-k eigenvectors

Theorem 3.1. *Let $G = ([n], E, w)$ be a connected, weighted graph with eigenpairs $(0, \varphi_1), (\lambda_2, \varphi_2), \ldots, (\lambda_n, \varphi_n)$ where $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ and $\{\varphi_i\}_{i=1}^n$ are orthonormal. Fix $2 \leq k \leq n$ and define the matrices*

$$\Phi_k = \begin{bmatrix} \varphi_1 & \cdots & \varphi_k \end{bmatrix} \in \mathbb{R}^{n \times k}, \quad \Phi_{>k} = \begin{bmatrix} \varphi_{k+1} & \cdots & \varphi_n \end{bmatrix} \in \mathbb{R}^{n \times (n-k)},$$

$$\Lambda_k = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^{k \times k}.$$

Then the set of Laplacians of all k-isospectral subgraphs of G is

$$\operatorname{Sp}_G(k) = \left\{ L = \underbrace{\Phi_k \Lambda_k \Phi_k^\top + \lambda_k \Phi_{>k} \Phi_{>k}^\top}_{F} + \Phi_{>k} Y \Phi_{>k}^\top \; : \; \begin{array}{c} Y \in \mathcal{S}_+^{n-k} \\ L_{st} \leq 0 \; \forall \; (s,t) \in E \\ L_{st} = 0 \; \forall \; s \neq t, \; (s,t) \notin E \end{array} \right\}.$$

fix the first k eigenpairs

force the last n-k eigenpairs to be at least as large

Main Structure Theorem

split the eigenbasis of L into the first k and last n-k eigenvectors

Theorem 3.1. *Let* $G = ([n], E, w)$ *be a connected, weighted graph with eigenpairs* $(0, \varphi_1), (\lambda_2, \varphi_2), \ldots, (\lambda_n, \varphi_n)$ *where* $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\{\varphi_i\}_{i=1}^n$ *are orthonormal. Fix* $2 \leq k \leq n$ *and define the matrices*

$$\Phi_k = [\varphi_1 \quad \cdots \quad \varphi_k] \in \mathbb{R}^{n \times k}, \quad \Phi_{>k} = [\varphi_{k+1} \quad \cdots \quad \varphi_n] \in \mathbb{R}^{n \times (n-k)},$$

$$\Lambda_k = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^{k \times k}.$$

Then the set of Laplacians of all $k-$*isospectral subgraphs of* G *is*

$$\mathrm{Sp}_G(k) = \left\{ L = \underbrace{\Phi_k \Lambda_k \Phi_k^\top + \lambda_k \Phi_{>k} \Phi_{>k}^\top}_{F} + \Phi_{>k} Y \Phi_{>k}^\top \ : \ \begin{array}{c} Y \in \mathcal{S}_+^{n-k} \\ L_{st} \leq 0 \ \forall \ (s,t) \in E \\ L_{st} = 0 \ \forall \ s \neq t, \ (s,t) \notin E \end{array} \right\}.$$

fix the first k eigenpairs

allow anything in the high frequencies

force the last n-k eigenpairs to be at least as large

Main Structure Theorem

split the eigenbasis of L into the first k and last n-k eigenvectors

Theorem 3.1. *Let* $G = ([n], E, w)$ *be a connected, weighted graph with eigenpairs* $(0, \varphi_1), (\lambda_2, \varphi_2), \ldots, (\lambda_n, \varphi_n)$ *where* $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\{\varphi_i\}_{i=1}^n$ *are orthonormal. Fix* $2 \leq k \leq n$ *and define the matrices*

$$\Phi_k = \begin{bmatrix} \varphi_1 & \cdots & \varphi_k \end{bmatrix} \in \mathbb{R}^{n \times k}, \quad \Phi_{>k} = \begin{bmatrix} \varphi_{k+1} & \cdots & \varphi_n \end{bmatrix} \in \mathbb{R}^{n \times (n-k)},$$

$$\Lambda_k = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^{k \times k}.$$

Then the set of Laplacians of all $k-$*isospectral subgraphs of* G *is*

$$\operatorname{Sp}_G(k) = \left\{ L = \underbrace{\Phi_k \Lambda_k \Phi_k^\top + \lambda_k \Phi_{>k} \Phi_{>k}^\top}_{F} + \Phi_{>k} Y \Phi_{>k}^\top \ : \ \begin{array}{c} Y \in \mathcal{S}_+^{n-k} \\ L_{st} \leq 0 \ \forall \ (s,t) \in E \\ L_{st} = 0 \ \forall \ s \neq t, \ (s,t) \notin E \end{array} \right\}.$$

fix the first k eigenpairs

no new edges

allow anything in the high frequencies

force the last n-k eigenpairs to be at least as large

positive edge weights

Main Structure Theorem (proof)

Theorem 3.1. *Let $G = ([n], E, w)$ be a connected, weighted graph with eigenpairs $(0, \varphi_1), (\lambda_2, \varphi_2), \ldots, (\lambda_n, \varphi_n)$ where $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ and $\{\varphi_i\}_{i=1}^n$ are orthonormal. Fix $2 \leq k \leq n$ and define the matrices*

$$\Phi_k = \begin{bmatrix} \varphi_1 & \cdots & \varphi_k \end{bmatrix} \in \mathbb{R}^{n \times k}, \quad \Phi_{>k} = \begin{bmatrix} \varphi_{k+1} & \cdots & \varphi_n \end{bmatrix} \in \mathbb{R}^{n \times (n-k)},$$

$$\Lambda_k = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^{k \times k}.$$

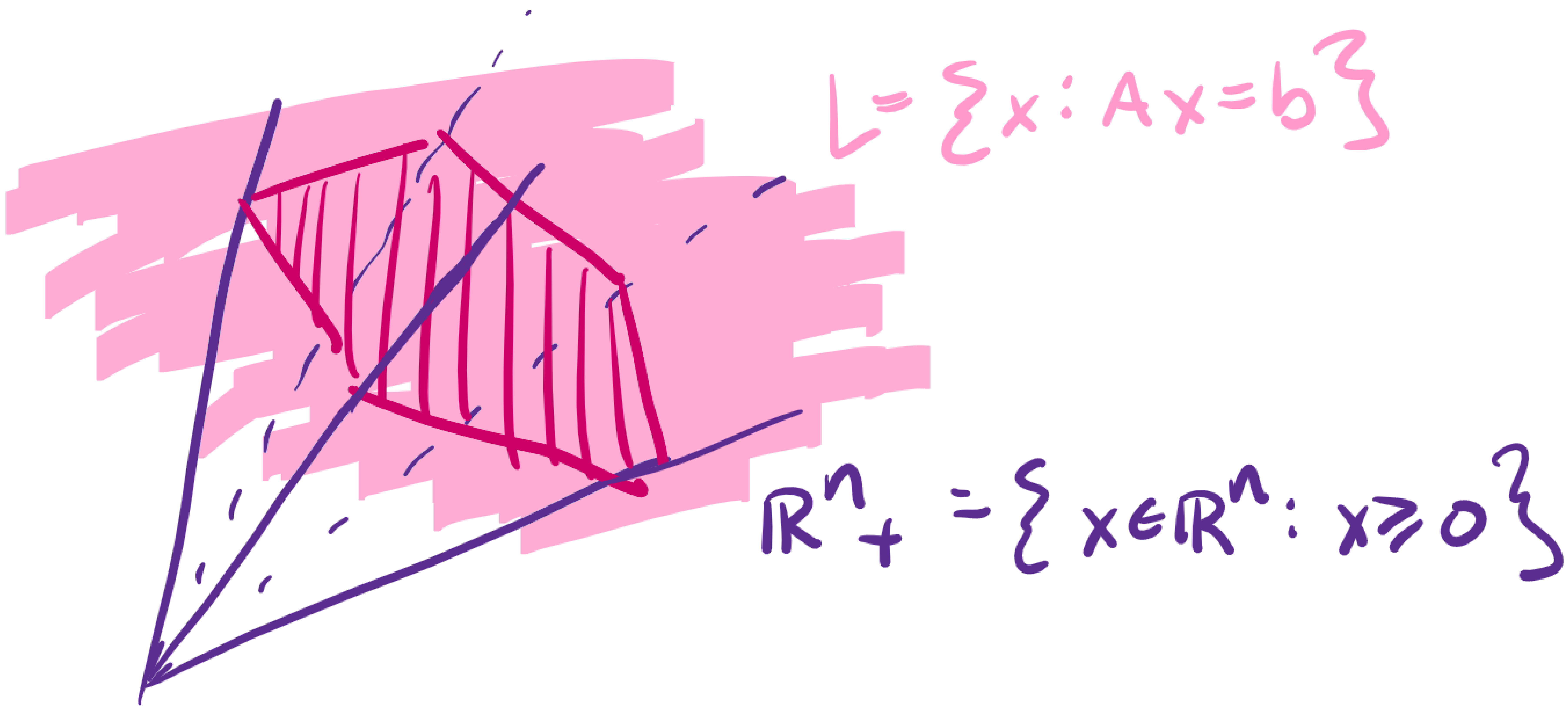
Then the set of Laplacians of all k-isospectral subgraphs of G is

$$\operatorname{Sp}_G(k) = \left\{ L = \underbrace{\Phi_k \Lambda_k \Phi_k^\top + \lambda_k \Phi_{>k} \Phi_{>k}^\top}_{F} + \Phi_{>k} Y \Phi_{>k}^\top \ : \begin{array}{c} Y \in \mathcal{S}_+^{n-k} \\ L_{st} \leq 0 \ \forall \ (s,t) \in E \\ L_{st} = 0 \ \forall \ s \neq t, \ (s,t) \notin E \end{array} \right\}.$$

Spectrahedra

Polyhedron = cone of nonnegative vectors intersected with an affine space

= feasible set of a linear program



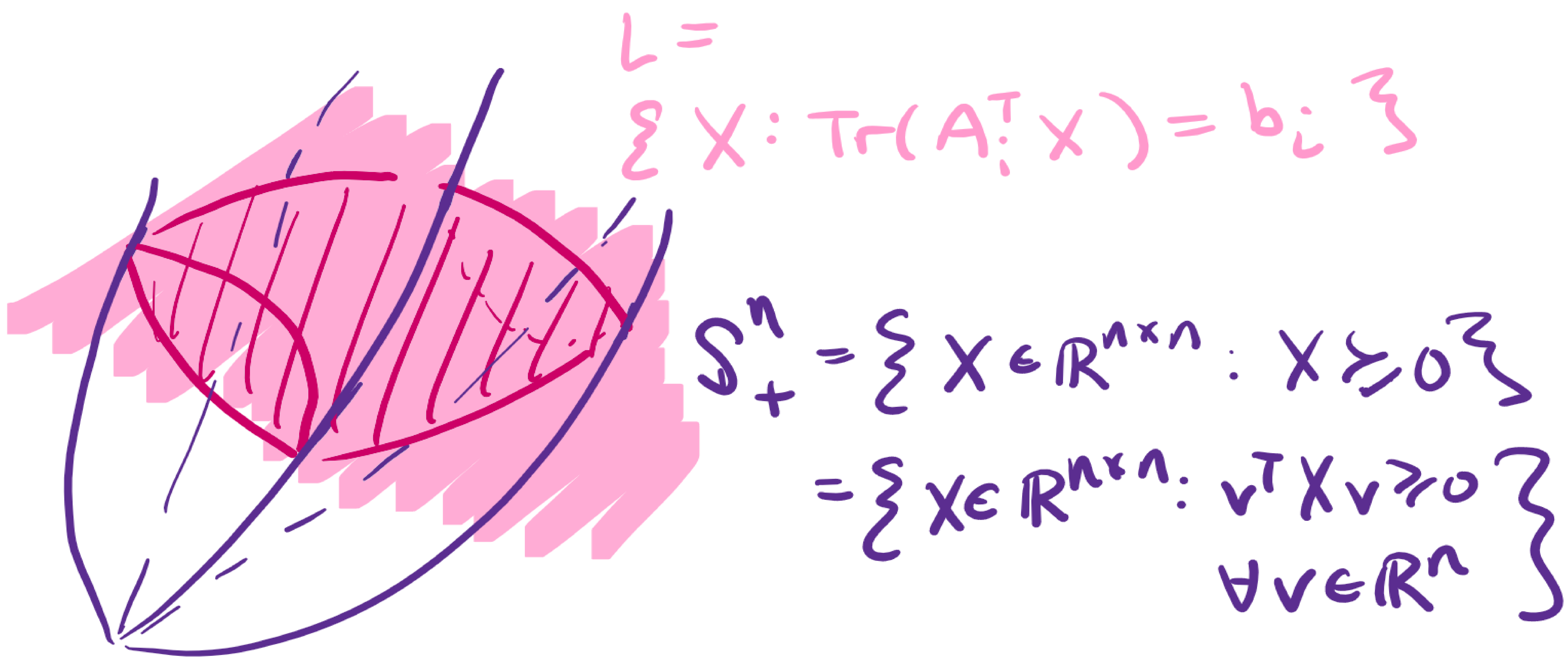
Spectrahedra

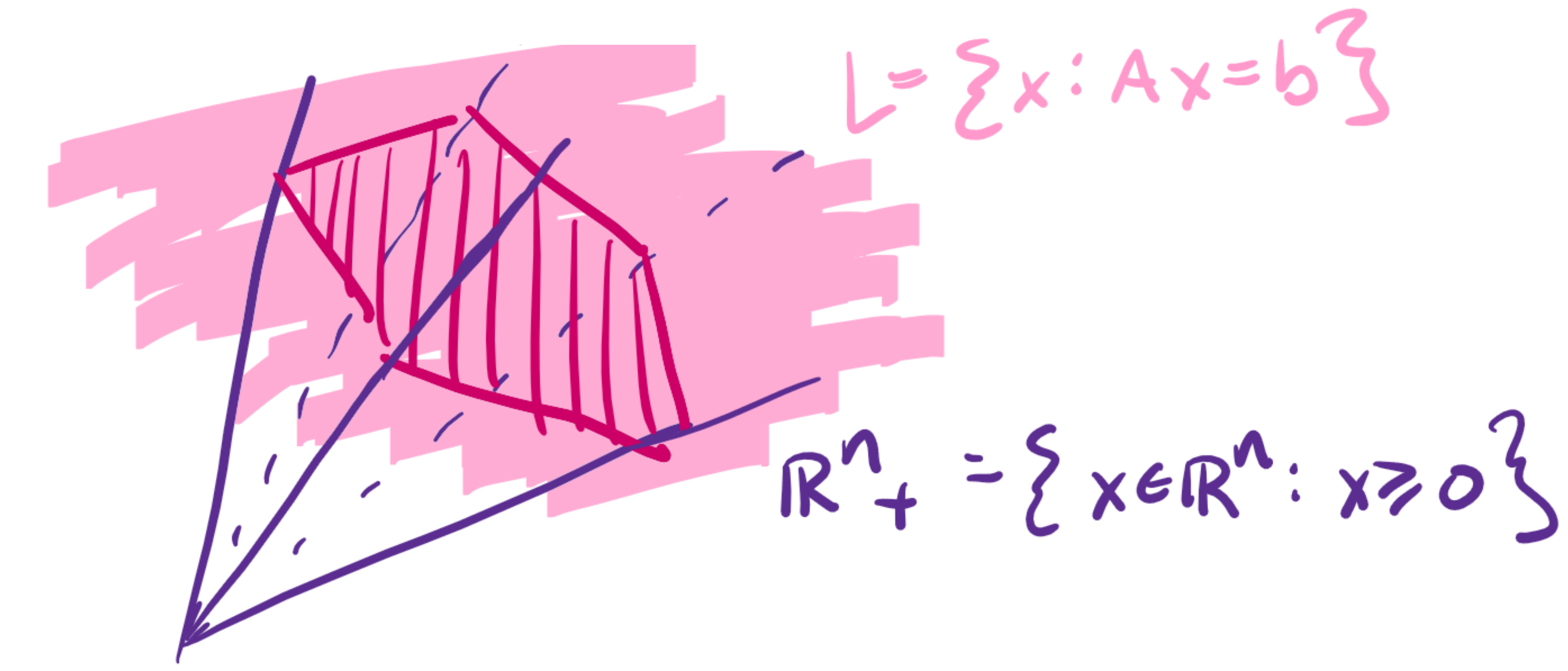
Polyhedron = cone of nonnegative vectors intersected with an affine space

= feasible set of a linear program

Spectrahedron = cone of positive semidefinite matrices intersected with an affine space

= feasible set of a semi-definite program





Spectrahedra

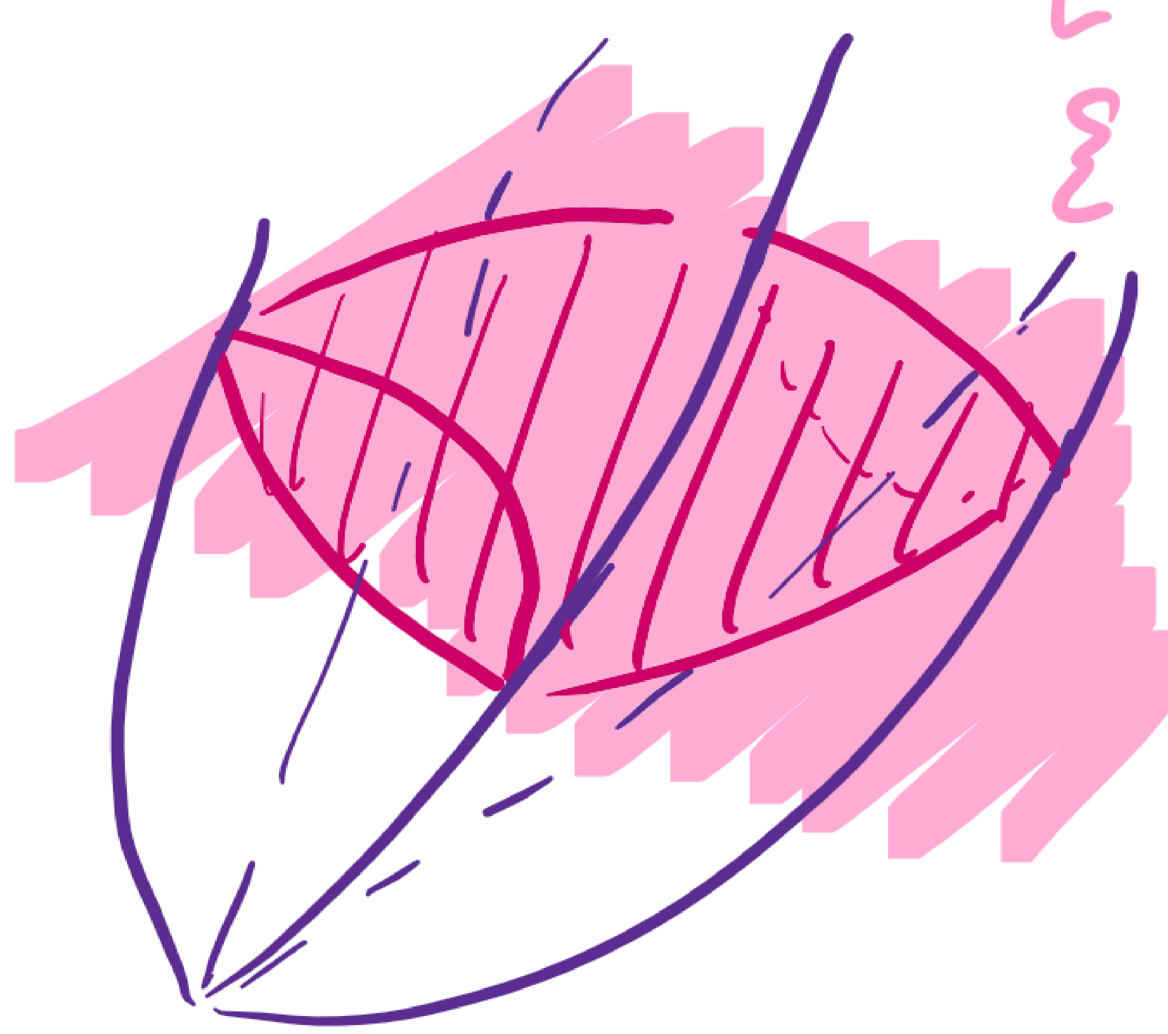
Polyhedron = cone of nonnegative vectors intersected with an affine space

= feasible set of a linear program

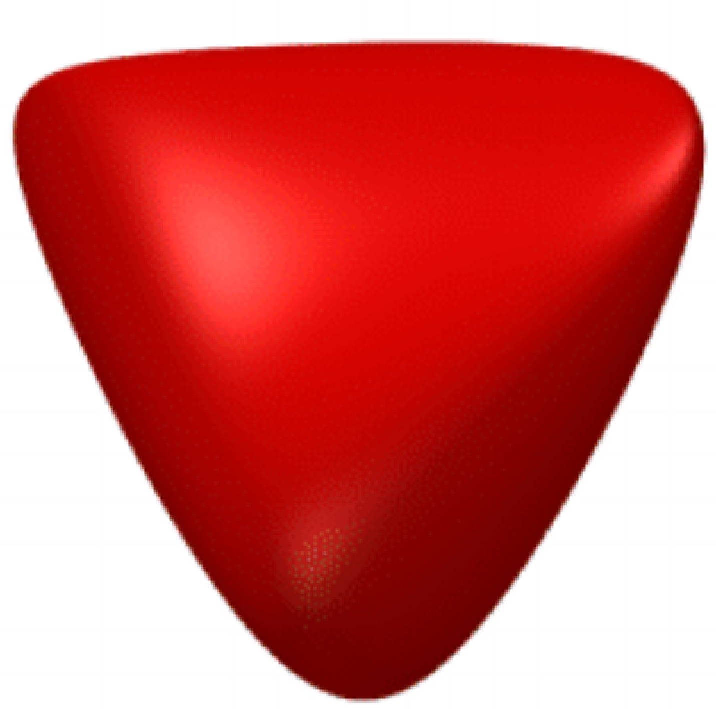
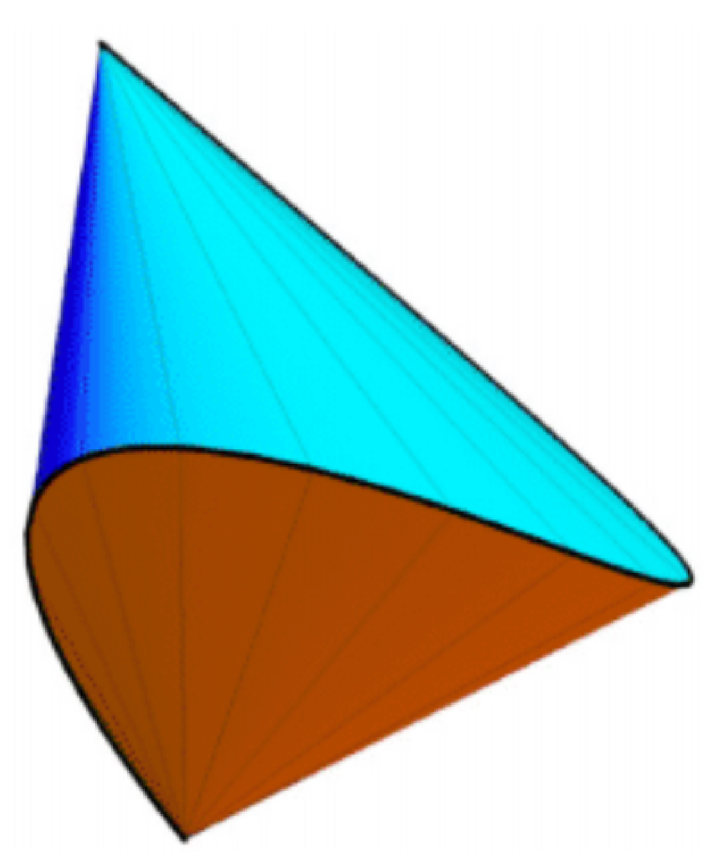
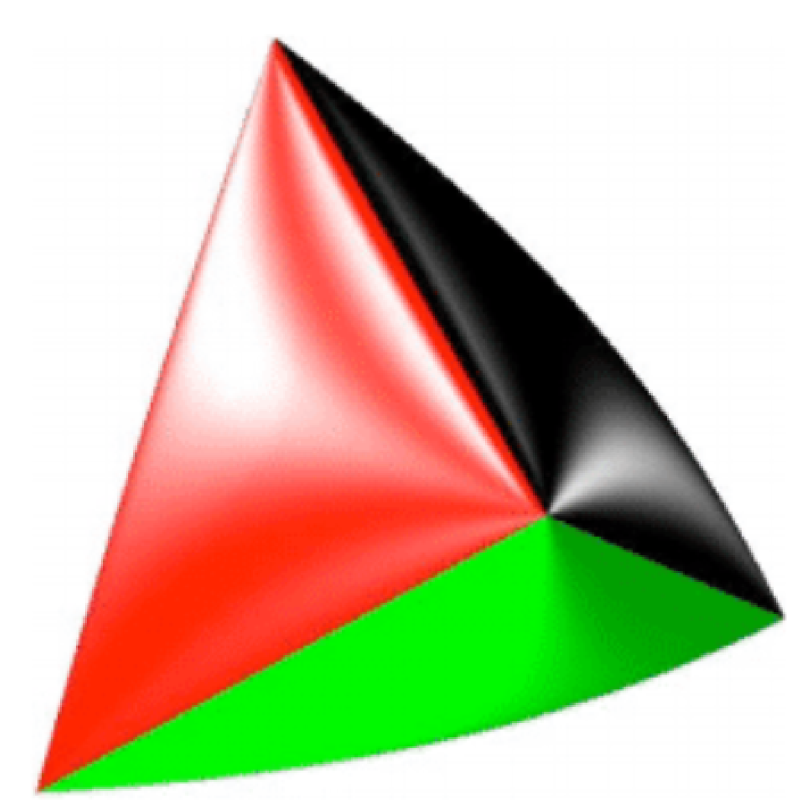
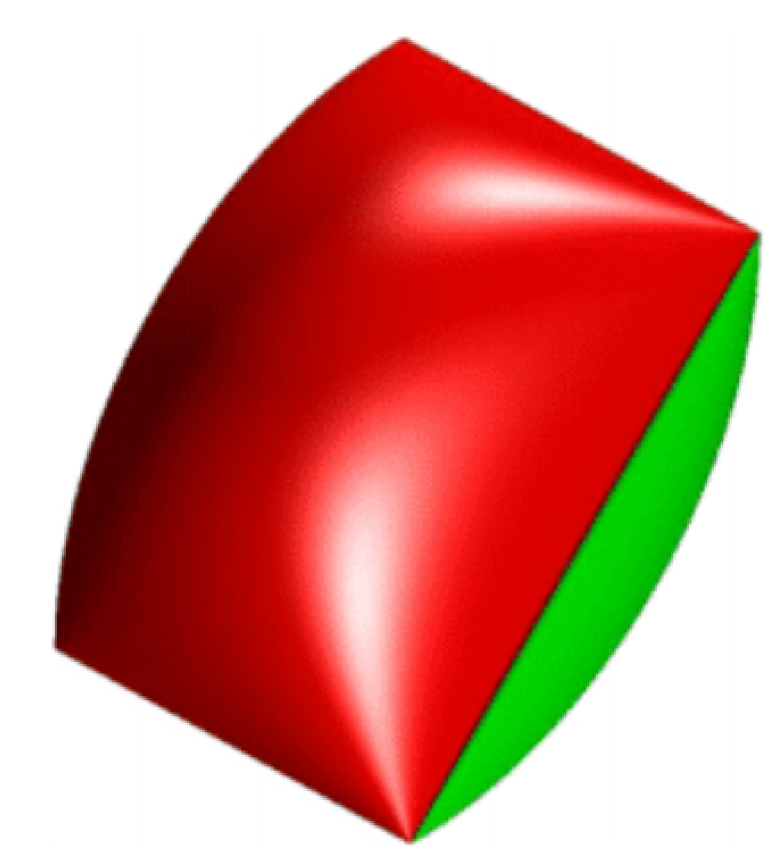
Spectrahedron = cone of positive semidefinite matrices intersected with an affine space

= feasible set of a semi-definite program

$L = \{ X : Tr(A_i^T X) = b_i \}$

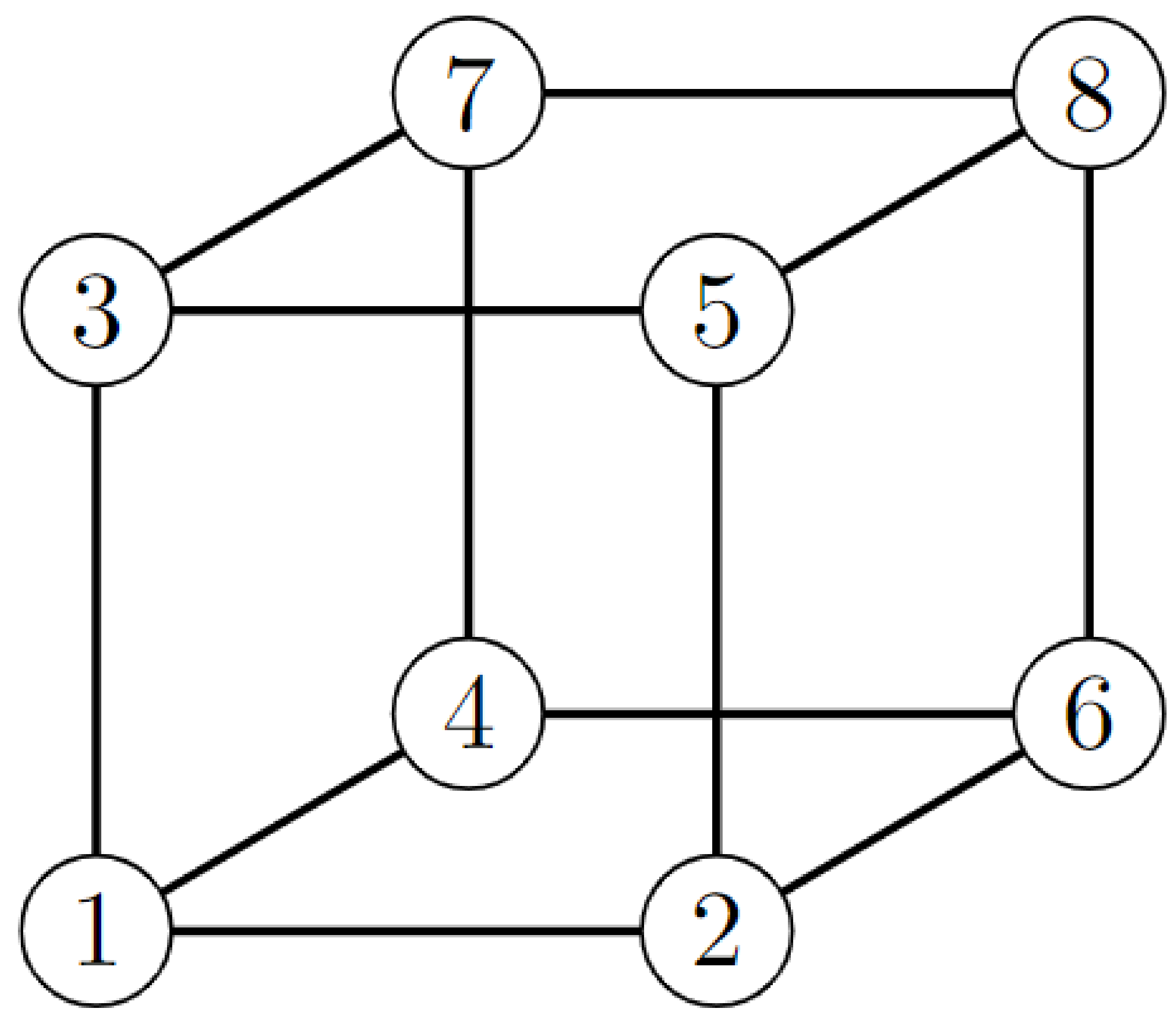


$$S_+^n = \{ X \in \mathbb{R}^{n \times n} : X \succeq 0 \} = \{ X \in \mathbb{R}^{n \times n} : v^T X v \geq 0 \ \forall v \in \mathbb{R}^n \}$$

Q: Why not just preserve the quadratic form?

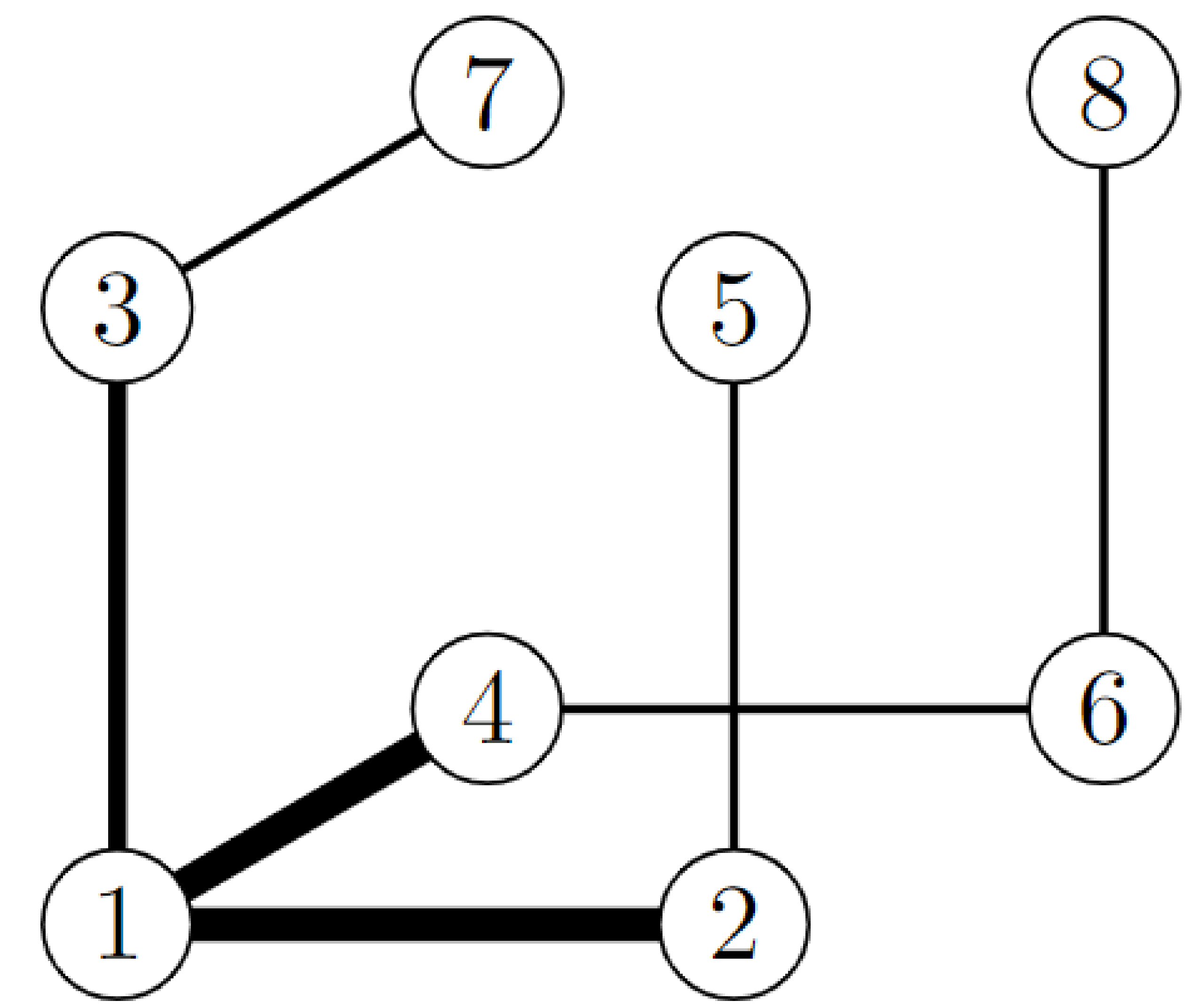
A: We tried that, and got absolute garbage



3-cube

eigenvalues:

$0^{(1)}, 2^{(3)}, 4^{(3)}, 6^{(1)}$



"4-sparsifier"

eigenvalues:

$$0, 0.3677, 0.6383, 1.3889,$$
$$2.4974, 3.6368, 4.3896, 11.0814$$

B., Steinerberger, Thomas '23

Sparsity Structure

polyhedron $P_G(k) := \left\{ (y_{ij}) \in \mathbf{R}^{\binom{n-k+1}{2}} \ : \ L_{st} \leq 0 \ \forall \ (s,t) \in E \right\}$

spectrahedron $S_G(k) := \left\{ (y_{ij}) \in \mathbf{R}^{\binom{n-k+1}{2}} \ : \ \begin{array}{c} L_{st} = 0 \ \forall \ (s,t) \notin E, s \neq t, \\ Y \succeq 0 \end{array} \right\}$

$$\mathrm{Sp}_G(k) = P_G(k) \cap S_G(k)$$ convex

$$\mathrm{Sp}_G(k) \subseteq \mathrm{Sp}_G(k-1)$$ nested

sparsity patterns $\leftrightarrow$ faces of $P_G(k)$ in $\mathrm{Sp}_G(k)$

An Example

$$K_5 \qquad k = 3$$

preserve $(0, \phi_1), (5, \phi_2), (5, \phi_3)$

$$\left(0, \phi_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right), \left(5, \phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}, \phi_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}, \phi_4 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\\0\end{bmatrix}, \phi_5 = \frac{1}{2\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix}\right)$$

An Example

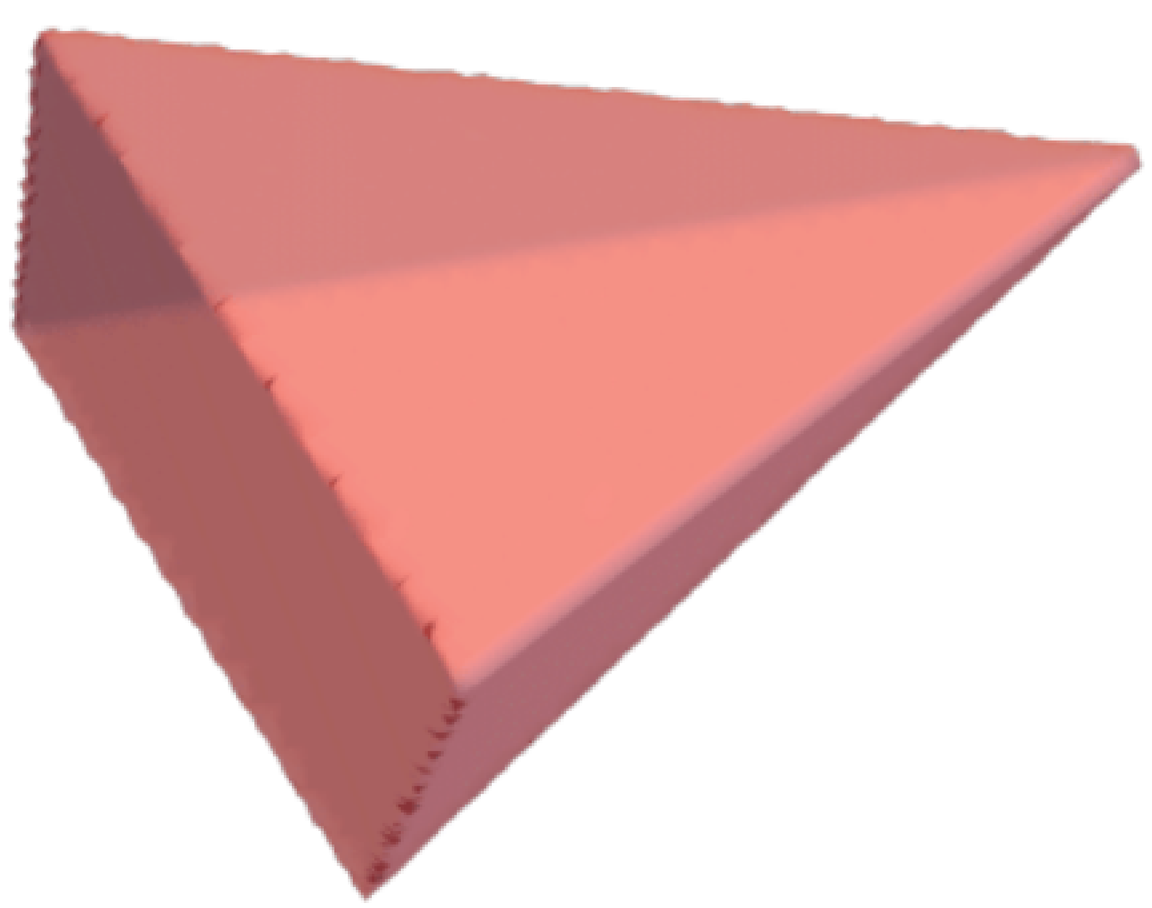
K_5

$k = 3$

preserve $(0, \phi_1), (5, \phi_2), (5, \phi_3)$

$$\begin{aligned}
5a + b + 2\sqrt{5}c &\leq 20 \quad (1,2) \\
5a + b - 2\sqrt{5}c &\leq 20 \quad (3,4) \\
-5a + b &\leq 20 \quad (1,3), (1,4), (2,3), (2,4) \\
-b - \sqrt{5}c &\leq 5 \quad (1,5), (2,5) \\
-b + \sqrt{5}c &\leq 5 \quad (3,5), (4,5)
\end{aligned}$$

$P_{K_5}(3)$



An Example

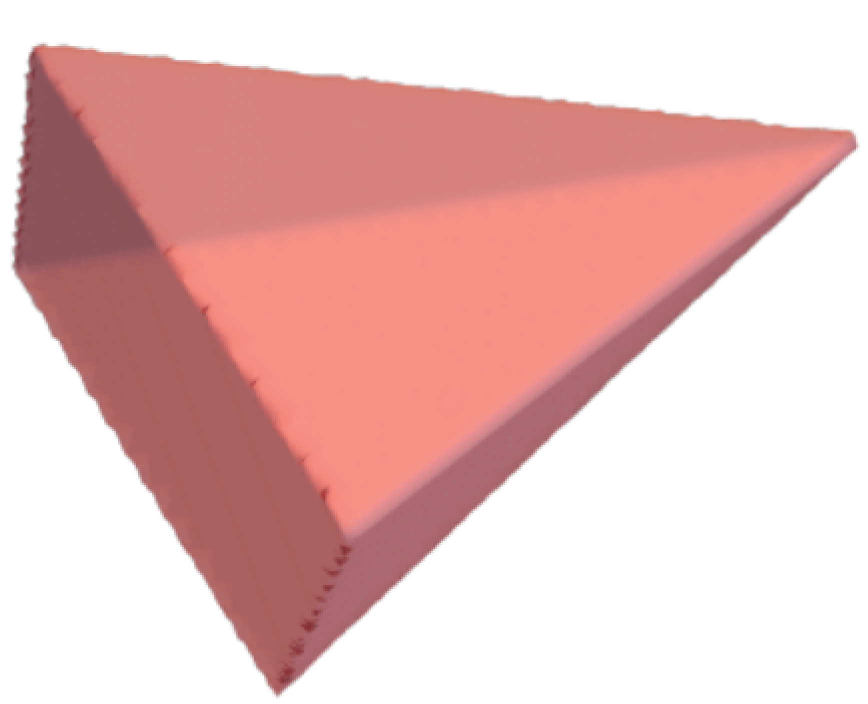
K_5

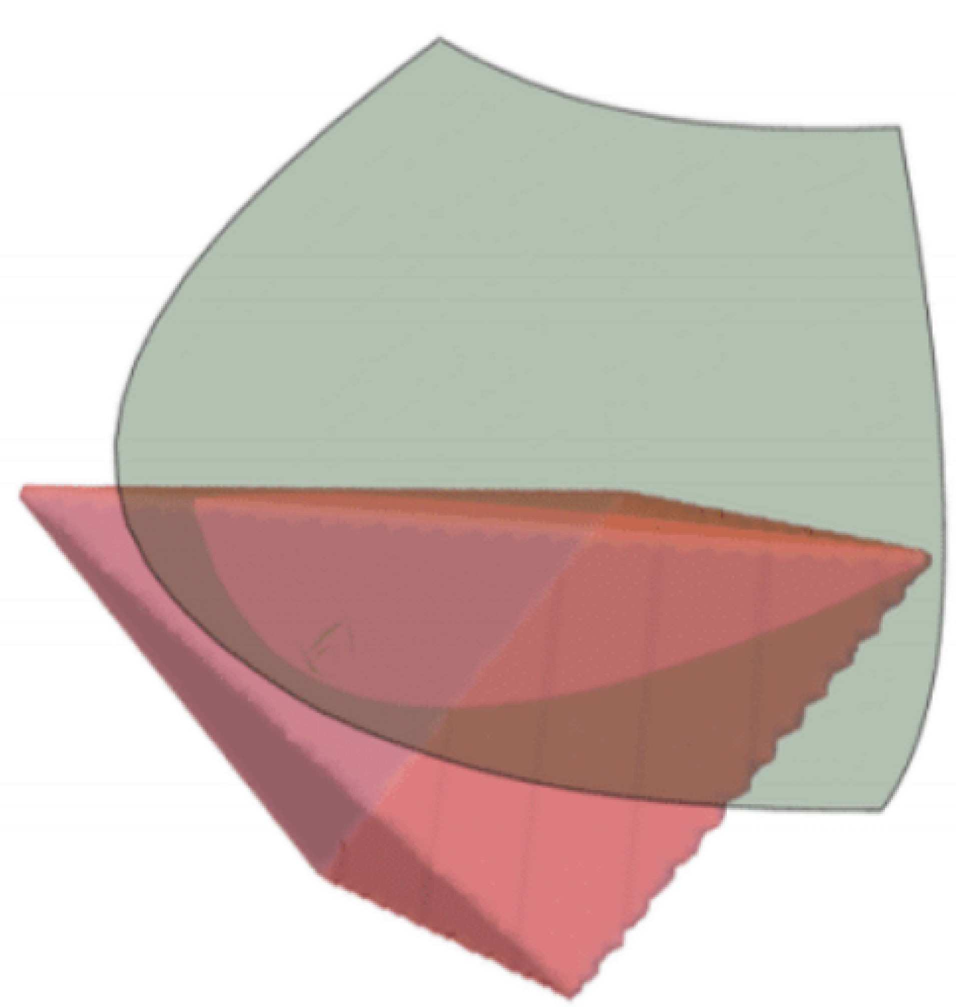
$k = 3$

preserve $(0, \phi_1), (5, \phi_2), (5, \phi_3)$

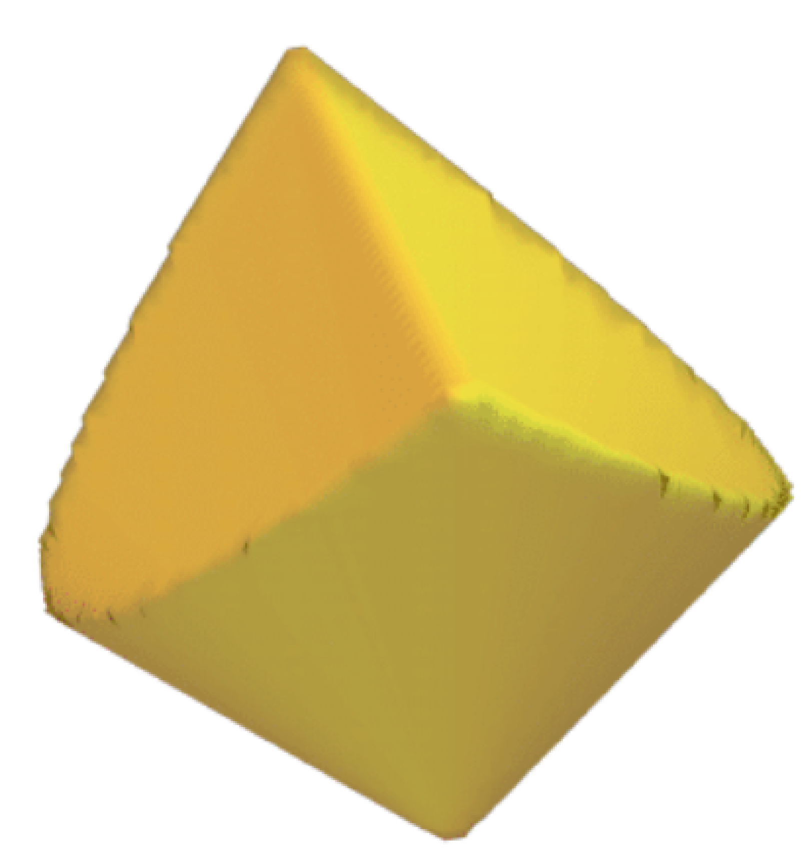
$$
\begin{array}{lll}
5a + b + 2\sqrt{5}c & \leq 20 & (1,2) \\
5a + b - 2\sqrt{5}c & \leq 20 & (3,4) \\
-5a + b & \leq 20 & (1,3),(1,4),(2,3),(2,4) \\
-b - \sqrt{5}c & \leq 5 & (1,5),(2,5) \\
-b + \sqrt{5}c & \leq 5 & (3,5),(4,5)
\end{array}
$$

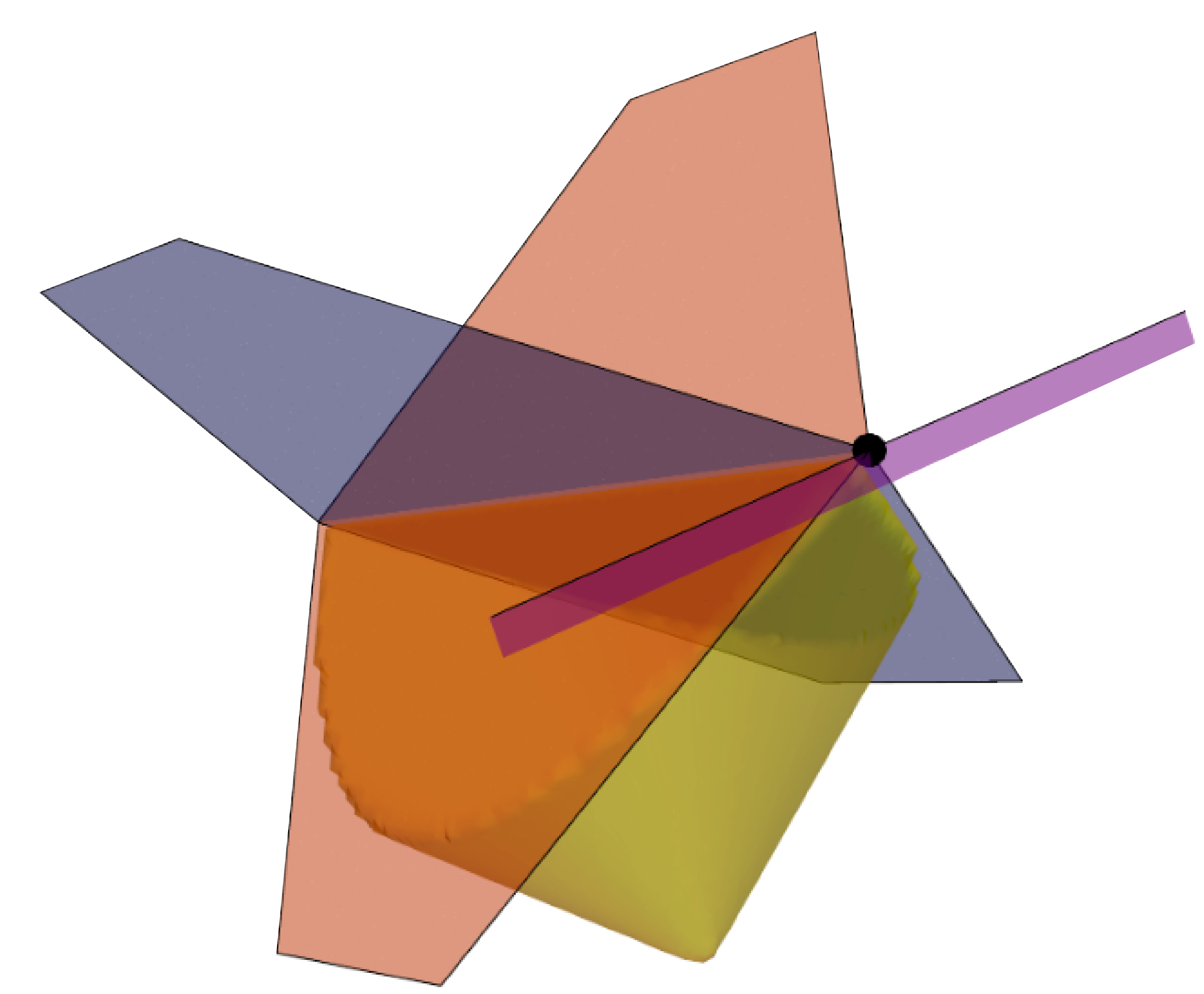
$P_{K_5}(3)$





$\mathrm{Sp}_{K_5}(3)$





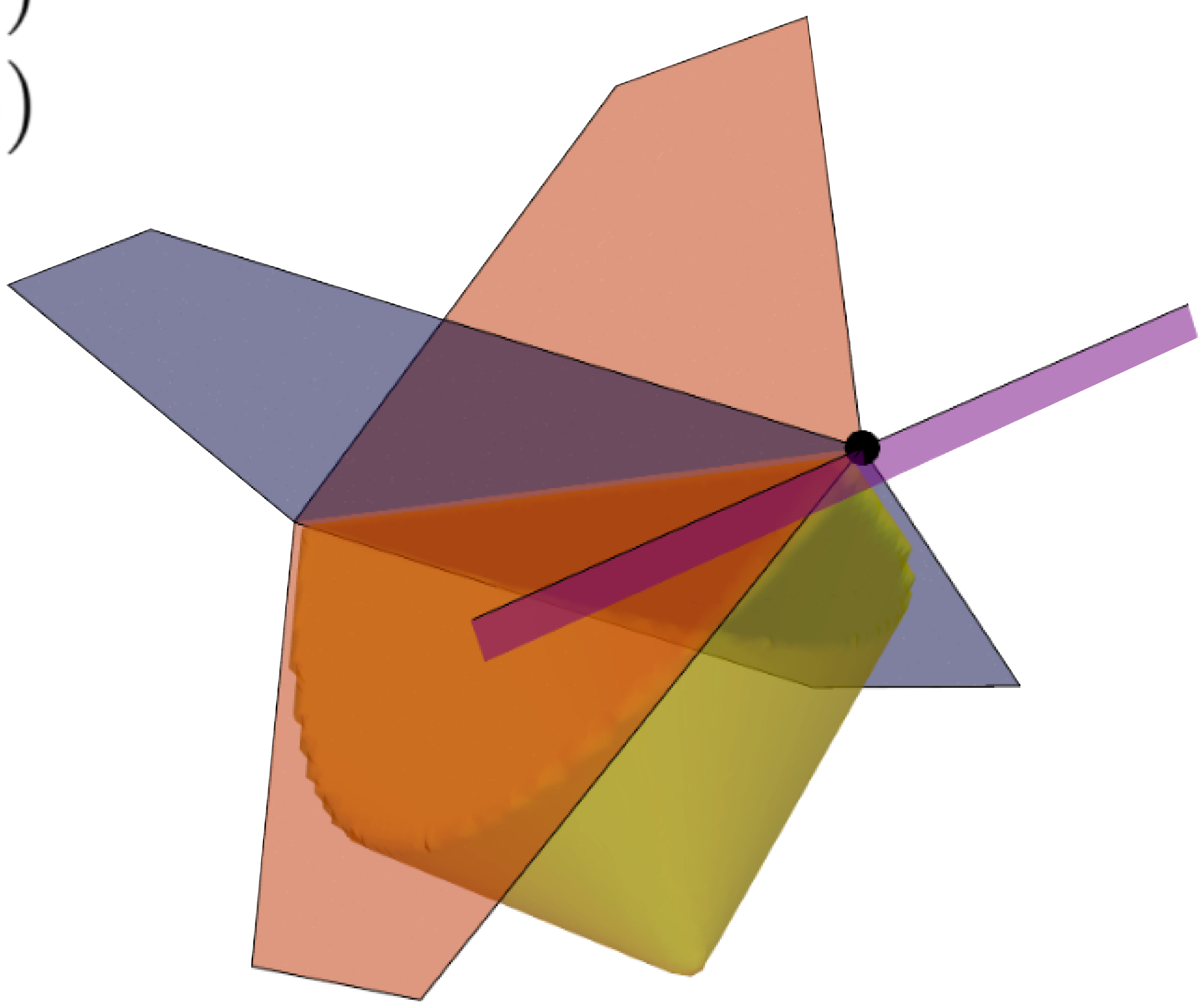
An Example

K_5

$k = 3$

preserve $(0, \phi_1), (5, \phi_2), (5, \phi_3)$

$$
\begin{array}{lll}
5a + b + 2\sqrt{5}c & \leq 20 & (1,2) \\
5a + b - 2\sqrt{5}c & \leq 20 & (3,4) \\
-5a + b & \leq 20 & (1,3), (1,4), (2,3), (2,4) \\
-b - \sqrt{5}c & \leq 5 & (1,5), (2,5) \\
-b + \sqrt{5}c & \leq 5 & (3,5), (4,5)
\end{array}
$$



An Example

$K_5 \quad k = 4$

Does the choice of basis matter?

$$\left(0, \phi_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right), \left(5, \phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}, \phi_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}, \phi_4 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\\0\end{bmatrix}, \phi_5 = \frac{1}{2\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix}\right)$$

An Example

$K_5 \quad k = 4$

Does the choice of basis matter?

$$\left(0, \phi_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right), \left(5, \phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}, \phi_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}, \phi_4 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\\0\end{bmatrix}, \phi_5 = \frac{1}{2\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix}\right)$$

$\Phi_{>}4 = \phi_5$ $\quad \mathrm{Sp}_4(K_5) = [0, 20]$ Spanning tree rooted at 5

$\Phi_{>}4 = \phi_4$ $\quad \mathrm{Sp}_4(K_5) = [0, 4]$ Missing (1,2), (3,4)

$\Phi_{>}4 = \phi_2$ $\quad \mathrm{Sp}_4(K_5) = [0, \infty]$ No sparsifiers

Thm: There is a choice of eigenbasis so that K_n has a spanning tree sparsifier for all $k < n$

An Example

$K_5 \quad k = 4$

Does the choice of basis matter?

$$\left(0, \phi_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right), \left(5, \phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}, \phi_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}, \phi_4 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\\0\end{bmatrix}, \phi_5 = \frac{1}{2\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix}\right)$$

$\Phi_{>}4 = \phi_5$ $\quad \mathrm{Sp}_4(K_5) = [0, 20]$ Spanning tree rooted at 5

$\Phi_{>}4 = \phi_4$ $\quad \mathrm{Sp}_4(K_5) = [0, 4]$ Missing (1,2), (3,4)

$\Phi_{>}4 = \phi_2$ $\quad \mathrm{Sp}_4(K_5) = [0, \infty]$ No sparsifiers

Thm: If you preserve whole eigenspaces, $\mathrm{Sp}_G(k)$ is independent of the choice of basis

The Linear Algebra Heuristic

B., Steinerberger, Thomas '23

Principle. *If* $G = ([n], E, w)$ *is a 'generic' graph and*

$$|E| \leq \binom{n}{2} - \binom{n-k+1}{2}$$

then, generically, the only $k-$*isospectral subgraph of* G *is* G *itself.*

The Linear Algebra Heuristic

B., Steinerberger, Thomas '23

Principle. *If* $G = ([n], E, w)$ *is a 'generic' graph and*

$$|E| \leq \binom{n}{2} - \binom{n-k+1}{2}$$

then, generically, the only $k-$*isospectral subgraph of* G *is* G *itself.*

typically true of Erdős-Renyi random graphs in experiments

The Linear Algebra Heuristic

B., Steinerberger, Thomas '23

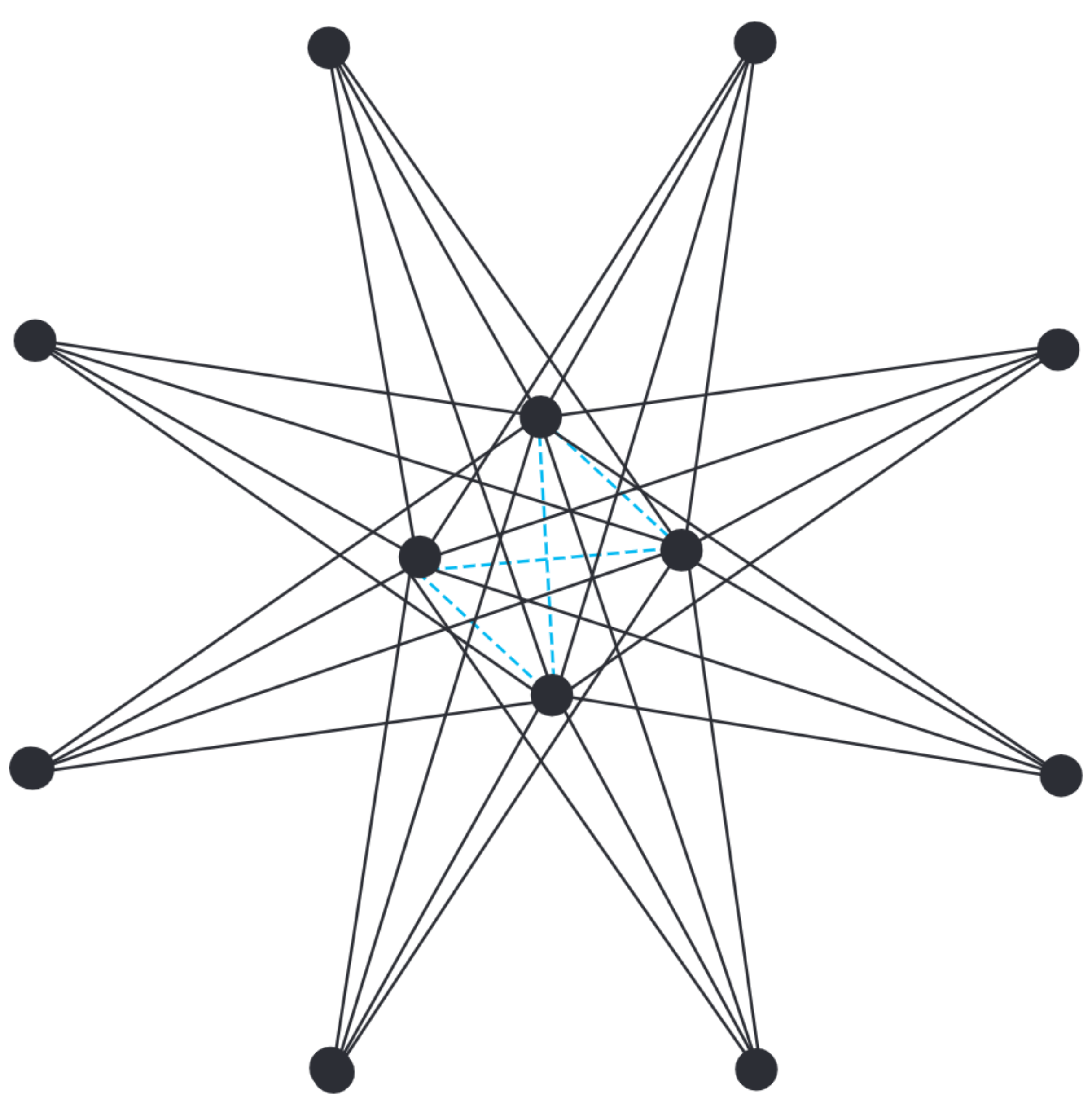
Principle. *If $G = ([n], E, w)$ is a 'generic' graph and*

$$|E| \leq \binom{n}{2} - \binom{n-k+1}{2}$$

then, generically, the only $k-$isospectral subgraph of G is G itself.

typically true of Erdős-Renyi random graphs in experiments

extremely not true in general



8-sparsifier, but the largest k for which the heuristic holds is 5

The Linear Algebra Heuristic

B., Steinerberger, Thomas '23

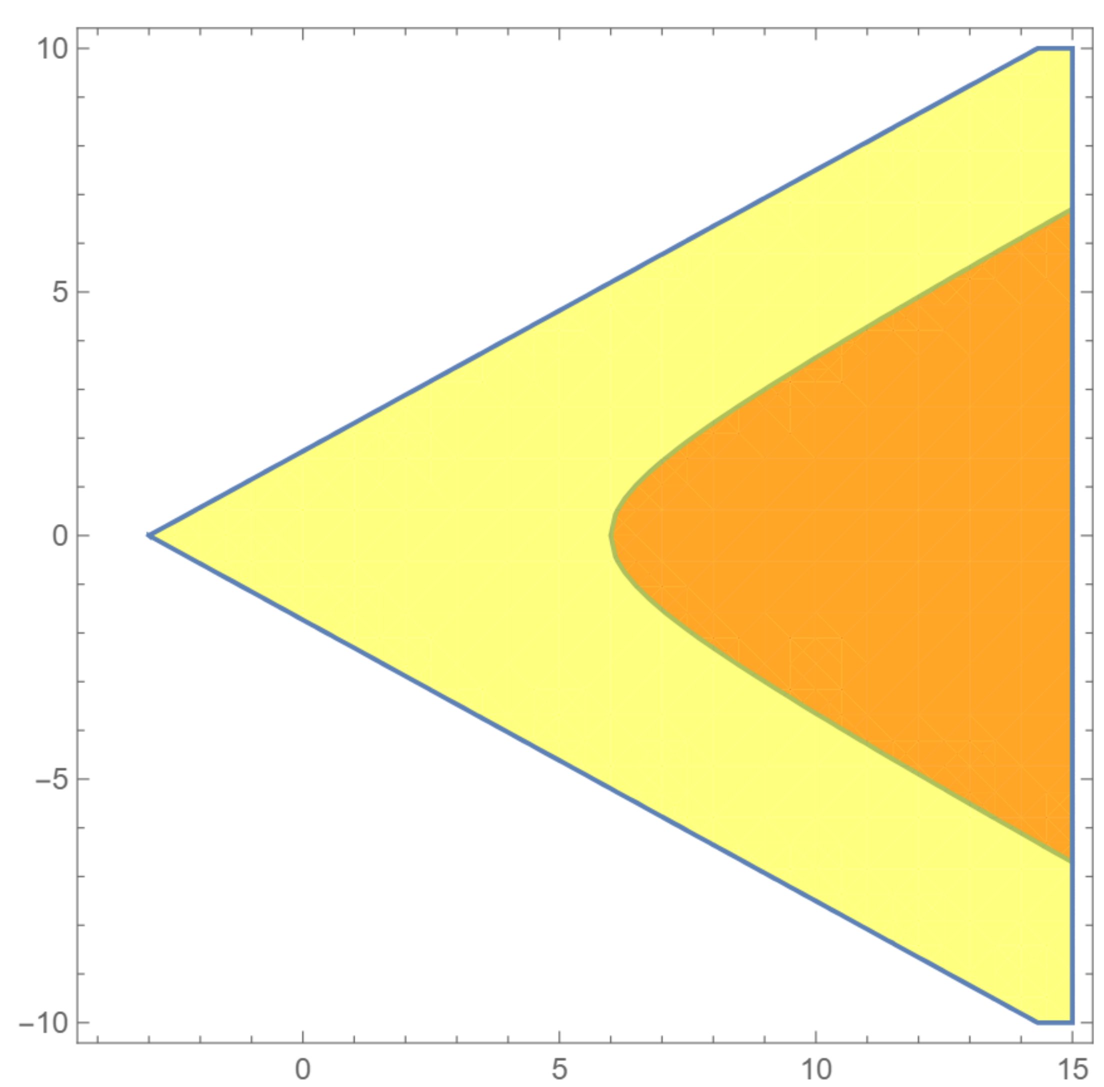
Principle. *If* $G = ([n], E, w)$ *is a 'generic' graph and*

$$|E| \leq \binom{n}{2} - \binom{n-k+1}{2}$$

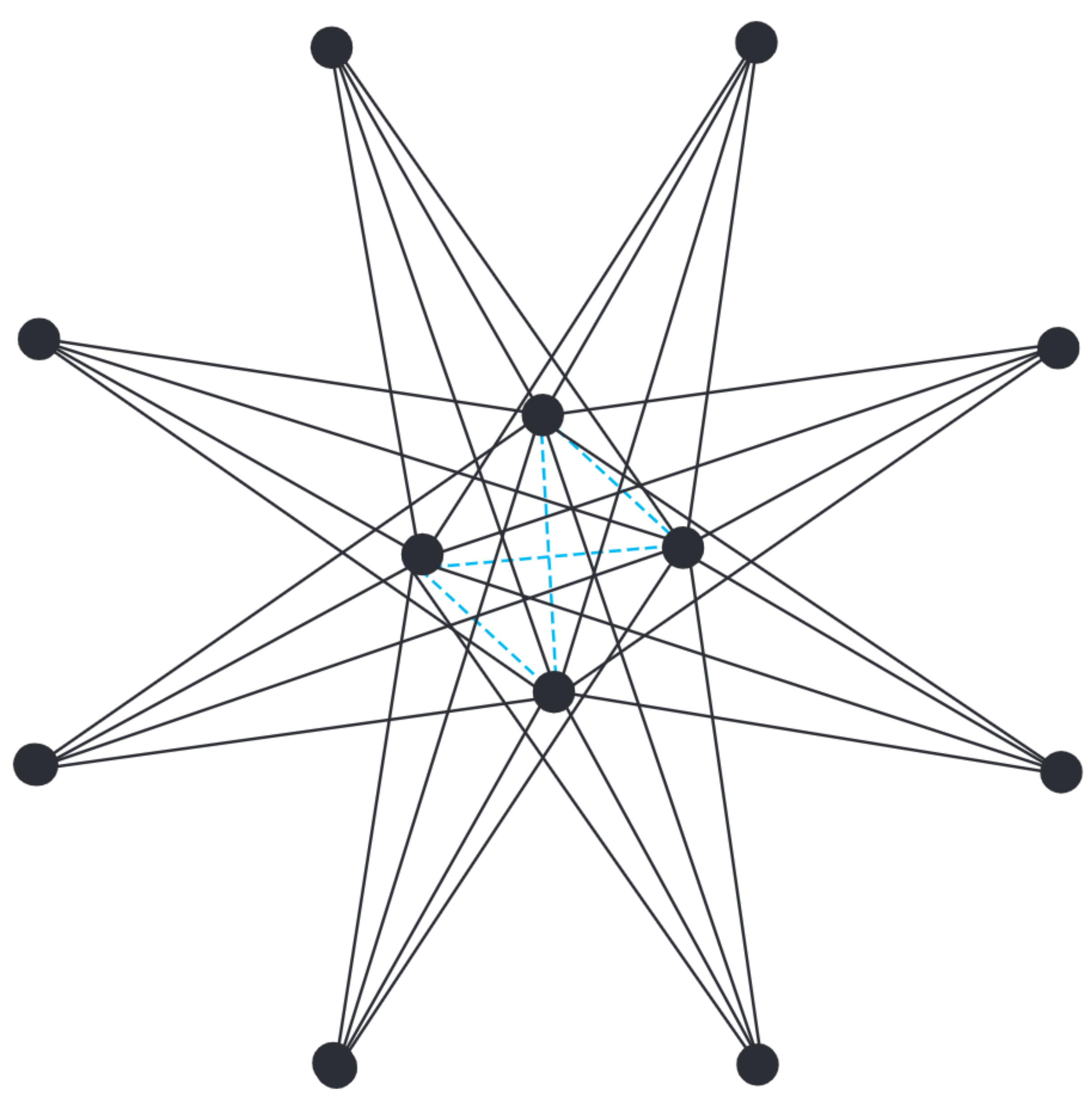
then, generically, the only $k-$*isospectral subgraph of* G *is* G *itself.*

typically true of Erdős-Renyi random graphs in experiments

extremely not true in general



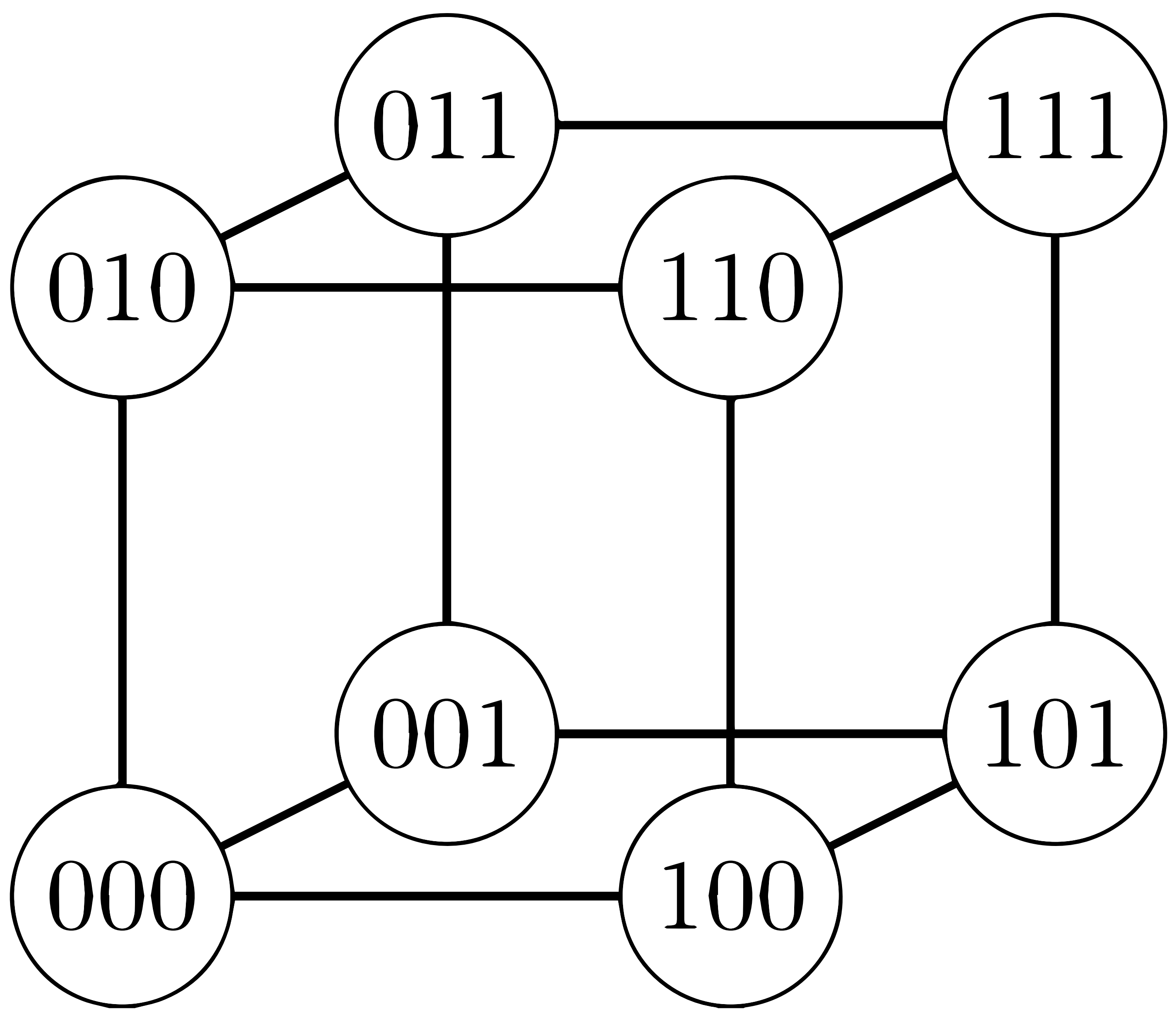
a family of graphs has few edges but wont sparsify



8-sparsifier, but the largest k for which the heuristic holds is 5

"The cube is already perfect just the way it is"

Theorem 4.5. *There is no $(d+1)$-sparsifier of the d-cube graph*



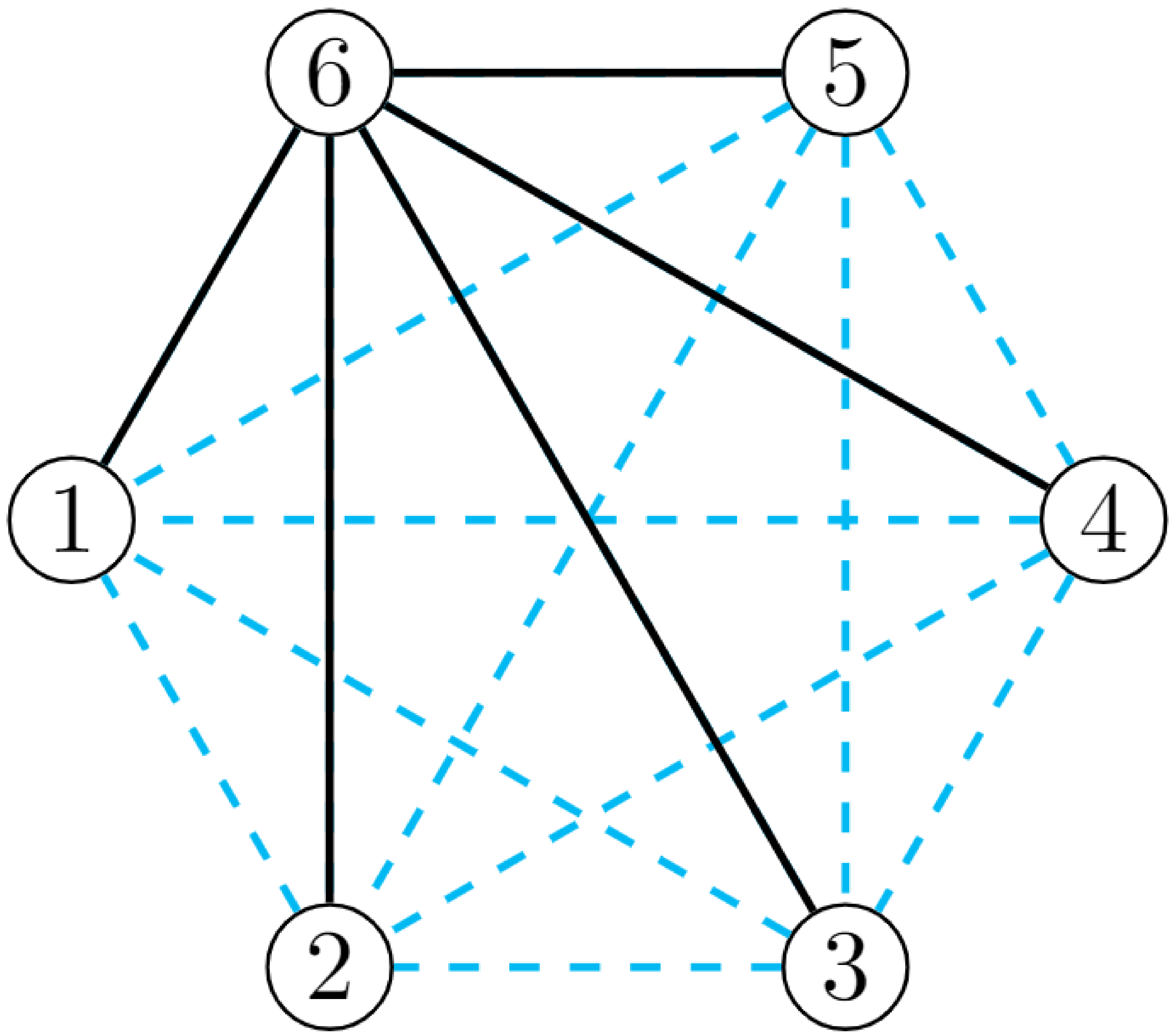
d-cube spectrum

$$(2i)^{\binom{d}{i}} \text{ for } i = 0, \ldots, d$$

B., Steinerberger, Thomas '23

Families

These families of graphs have spanning tree sparsifiers for the following values of k

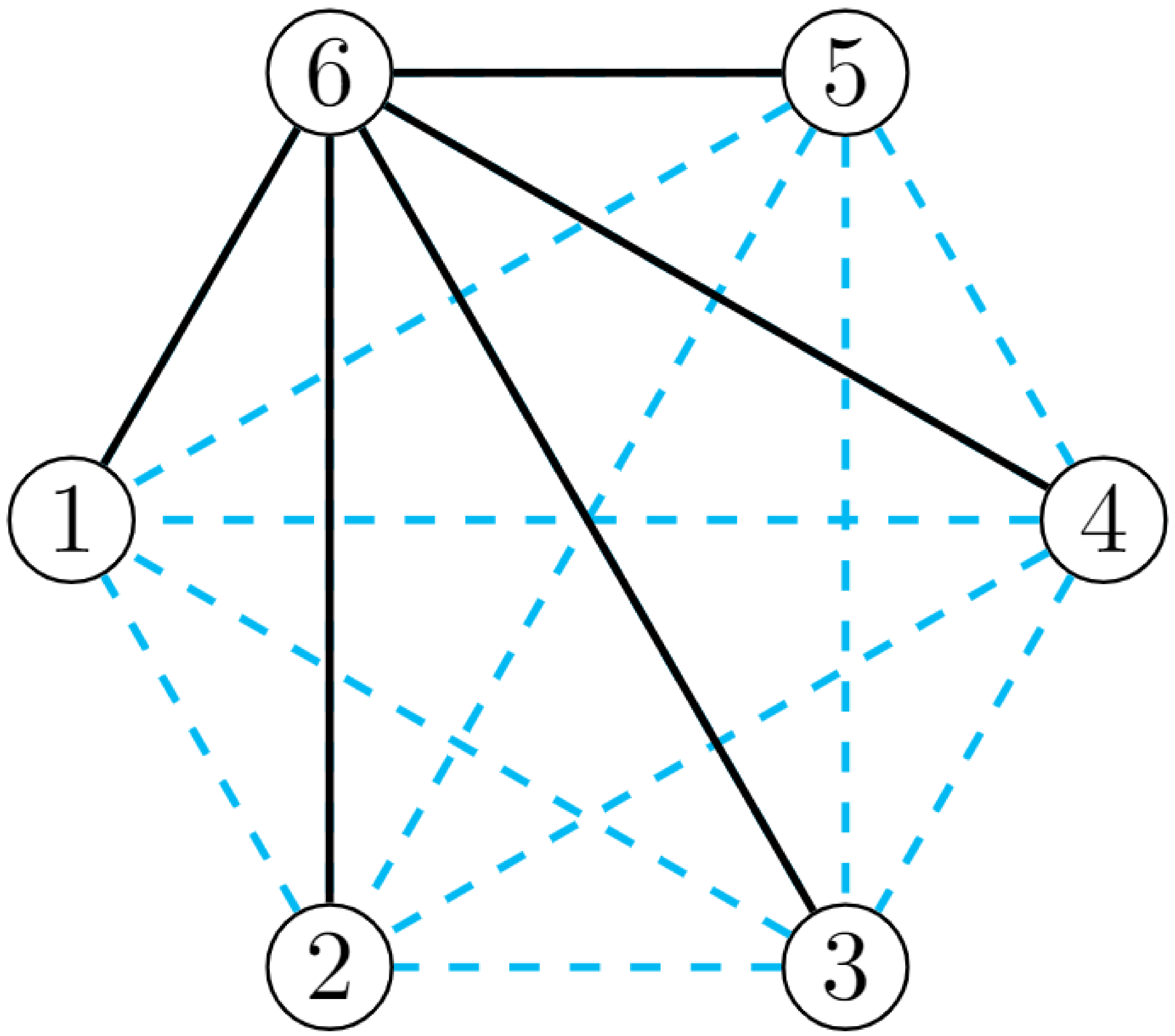


$$K_n : k = n - 1$$

B., Steinerberger, Thomas '23

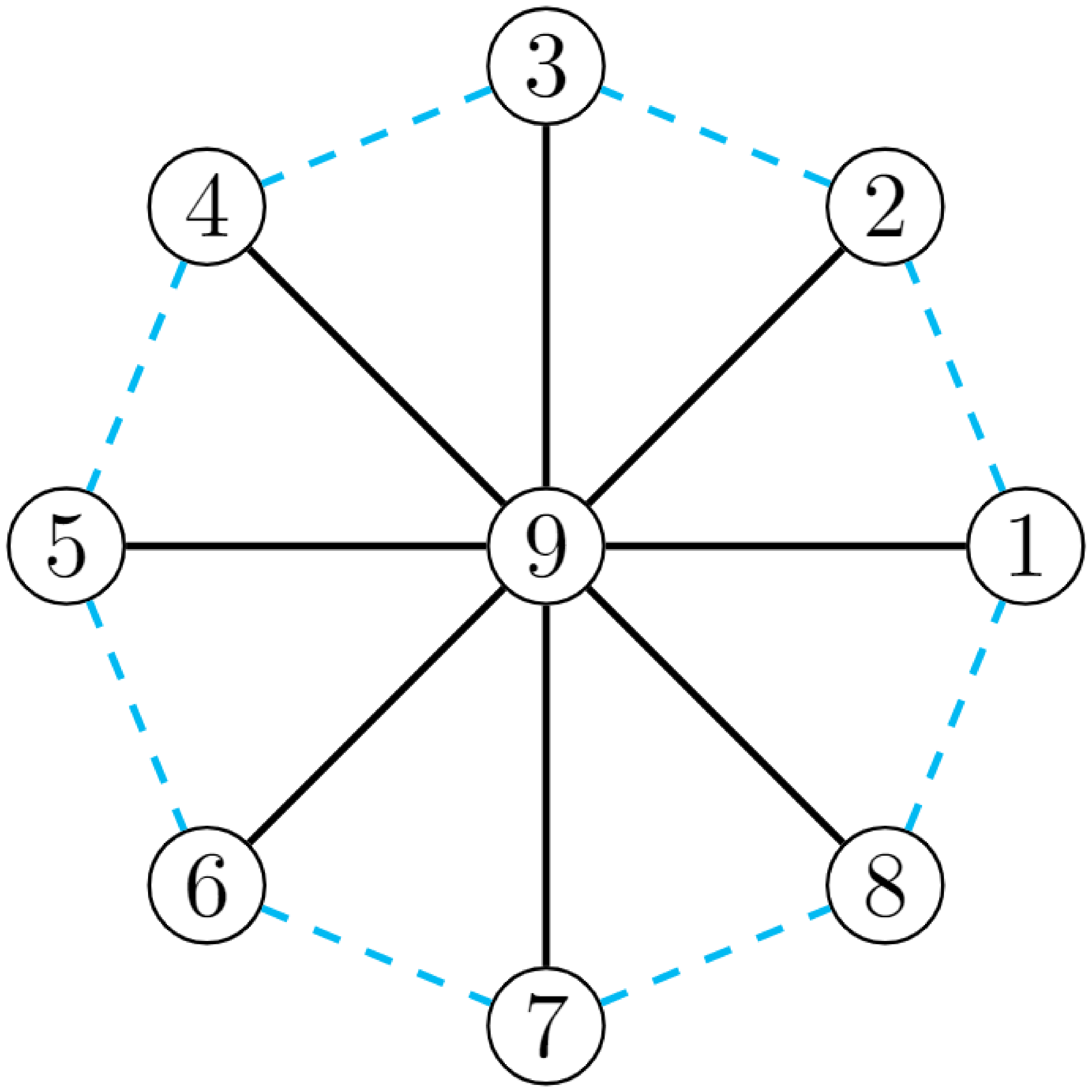
Families

These families of graphs have spanning tree sparsifiers for the following values of k



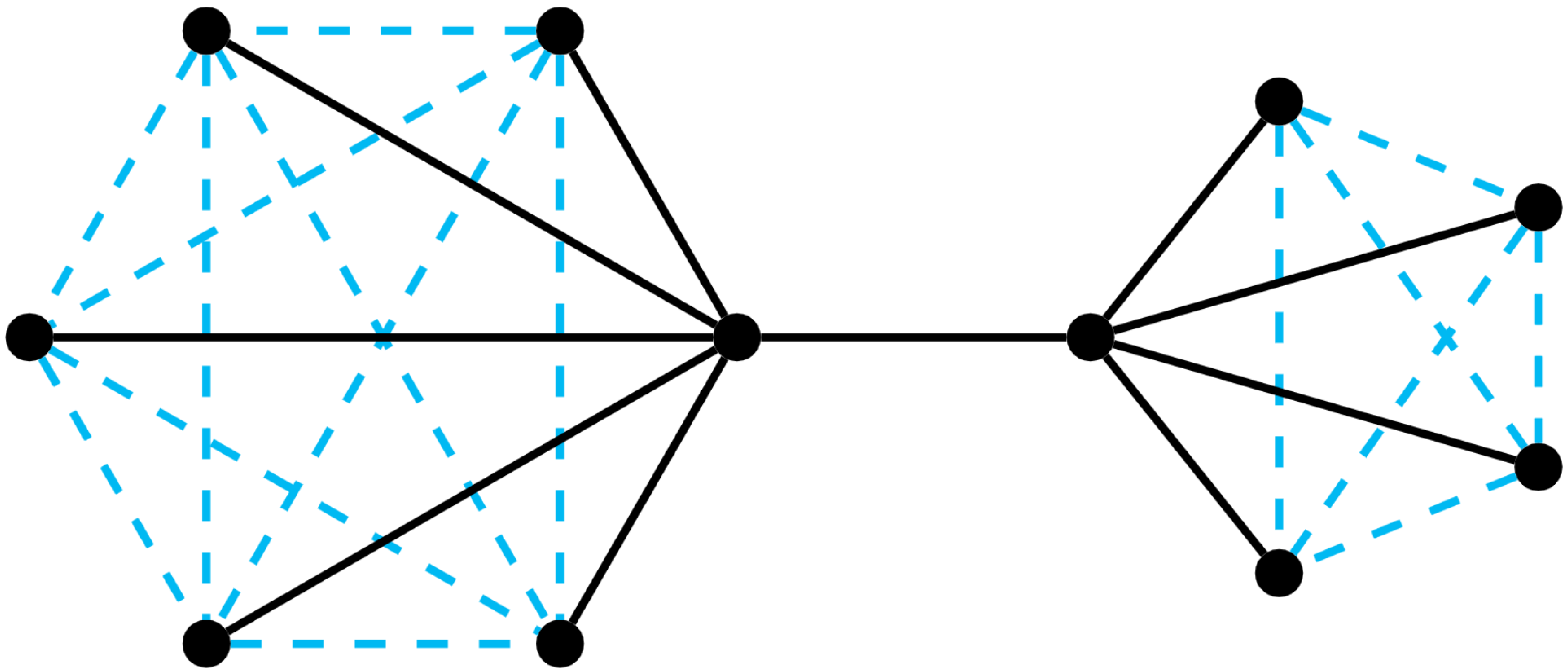
$$K_n : k = n - 1$$

$$W_n : k = 3$$



B., Steinerberger, Thomas '23

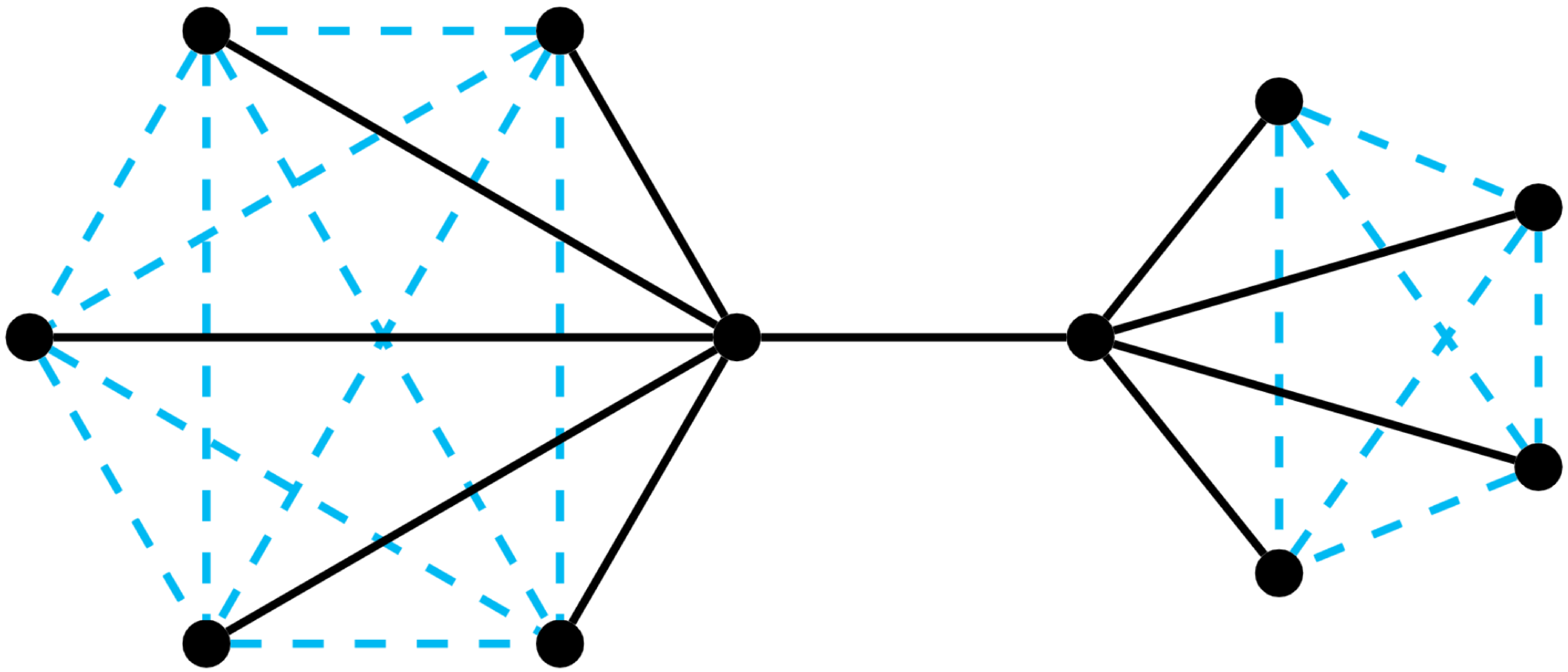
Families



$$B_{n,n} : k = 2$$

B., Steinerberger, Thomas '23

Families

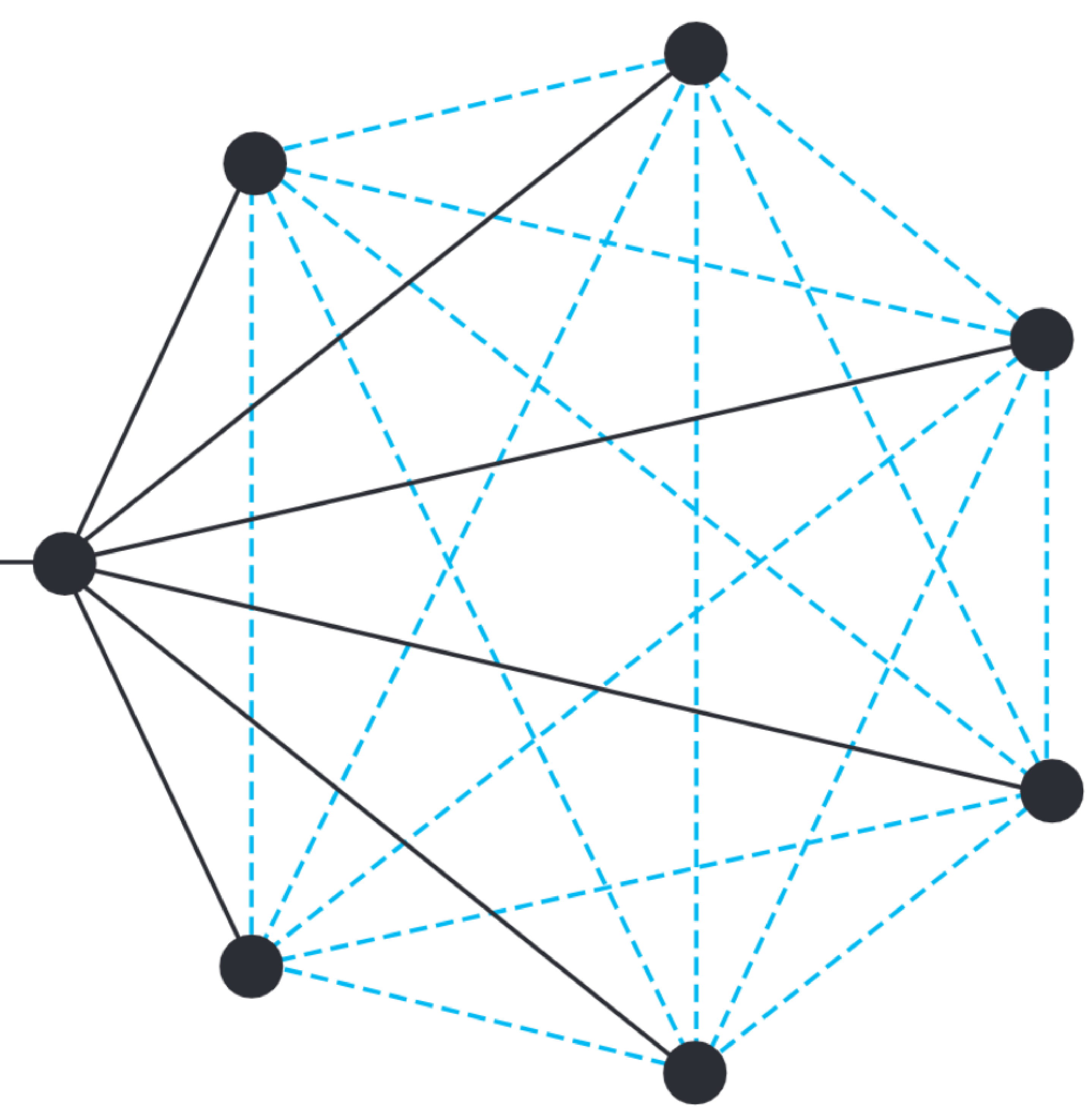


$B_{n,m} : k = 2$

Theorem 5.3. *Let $G = ([n], E, w)$ be a connected, weighted graph and let $T = ([n], E_T, w|_{E_T})$ be a spanning tree of G. Let $k \in [n]$ be arbitrary and let $\varphi_1, \ldots, \varphi_k$ be eigenvectors corresponding to the k smallest eigenvalues of the spanning tree T. Suppose that for all $(u, v) \in E$ either*

$$(u, v) \in E_T \qquad \text{or} \qquad \varphi_i(u) = \varphi_i(v) \qquad \text{for all } 1 \leq i \leq k,$$

then the spanning tree T is a $k-$sparsifier of G with respect to $\{\varphi_1, \ldots, \varphi_k\}$.



$L_{n,p} : k = 3$

B., Steinerberger, Thomas '23

Recap!

A new model of graph sparsification! motivated by things like…

Cheeger's Inequality!

HEAT DIFFUSION

Structural results, including…

spectrahedral geometry

well-posed-ness

some graphs shouldn't sparsifiy (the cube 😍)

A linear algebra heuristic…

that's not true but often useful

Explicit families of examples

that verify our construction is behaving sensibly

almost done…

Thank you!