Multichannel Scattering Theory

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General Nonlinear Schrödinger and other type Equations

Scattering!!

The beginning: Leonardo Da Vinci extensively examined the phenomenon of wave scattering off objects. In his summation of observations, he wrote:
"**The air that is between bodies is full of the intersections formed by the radiating images of these bodies.**"
Remarkably, this statement encapsulates the essence of the two fundamental meta-theorems of scattering theory.

Firstly, there is direct scattering: systems, however complicated, break into subsystems with simpler and independent dynamics as time approaches infinity. Secondly, all the information about the system is carried into the asymptotic states. Consider light arriving without any prior information from infinity. It interacts with the painting of the Mona Lisa. As time approaches positive infinity, the light becomes free, and the painting returns to a static state. The reflected light contains all the information about the painting extending to positive infinity in space.

Nonlinear Equations-New Beginning

Past works, see e.g. the detailed review to appear. Here we focus on large/general data Scattering. The state of a physical system at time t is given by:

$$\psi(t) \equiv U(t)\psi(0), \quad \psi(0) \in \mathcal{H} \quad \text{Hilbert space.}$$

$U(t)$ is assumed to be a unitary operator for simplicity, but NOT LINEAR in general.

Asymptotic behavior: Either blows up in finite time, or converges to independent channels:

(F) Free waves with dynamics $U_0(t)$.

(S) Solitons and other coherent Structures.

(W) Weakly localized waves, like self-similar sub-ballistic spread.

(C) Chaos ???

Nonlinear Equations- Large Data

NLS: Tao (2004-2014)

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi(t) - |\psi|^m\psi, \quad (4/n) < m < 4/(n-2).$$

$|\psi(t)|_{H^1} \lesssim 1$, **uniformly in t; spherically symmetric data.**

Then, $\psi(t) - e^{i\Delta t}\psi_+ - \psi_{WL}(t) \to 0$ in $H^1(\mathbb{R}^n)$, $n \geq 3$.

ψ_{WL} is smooth for cubic NLS, $n = 3$,

$\psi_{WL} =$ **localized** $+ o(\dot{H}^1)$.

NLW:Collot-Duyckaert-Kenig-Merle (2014-2023), The critical case dimension 6 (2024).

$u_{tt} - \Delta u = |u|^{4/(n-2)}u$, **data spherical symmetric, bounded energy**,

$$u(t) - U_0(t)u_+ - \sum_j \lambda_j(t)^{-n/2+1}S(x/\lambda_j(t)) + o(\dot{H}^1) \to 0.$$

$n \geq 3$. S is the unique ground state solution.

New Results

Time dependent interactions in any dimension

$$i\frac{\partial \psi}{\partial t} = H_0\psi + \mathcal{N}(|\psi|, x, t)\psi,$$

$$H_0 = \omega(p), \quad v = -\nabla_p \omega(p), p = -i\nabla_x.$$

$$\sup_t |\psi(t)|_{H^1_x} \lesssim 1.$$

$$\|J_f(x, t, p)e^{-iH_0 t}\mathcal{N}\psi(t)\|_2 \in L^1(dt),$$

Then, $\psi(t) - e^{i\omega(p)t}\psi_+ - \psi_{WL}(t) \to 0 \text{ in } H^1_x.$

In the case where the Interaction is localized in space, we also have that $\psi_{WL}(t) \in H^s_x$ for large s, depending on regularity and decay of $\mathcal{N}$.

$\psi_{WL}(t) = \psi_L(t) + \psi_{SS}(t)$, **i.e. localized plus "self similar".**

In three or more dimensions, we do not need localization of the interaction:

The main condition for the existence of the free channel is

$$\sup_t \|\psi(t)\|_{H^1_x} \lesssim 1. \quad \sup_t |\mathcal{N}(|\psi|, x, t)\psi| \in L^a_x.$$

Here a should be small enough, so that

$$\|J_f(x,p)e^{+iH_0 t}u\|_{L^{a'}} \lesssim t^{-1-0}\|u\|_{L^a}.$$

We know less about the weakly localized part in the **non-radial case**.

It's All About The J_f For The Free Channel

Let

$$J_C \equiv e^{-iH_0 t} J_f e^{+iH_0 t}$$

be a projection (smooth) on the **Propagation Set** of the Free Channel in the extended phase-space and time. Then the existence of the Free Channel Wave Operator follows if the following **strong limit** exists:

$$\Omega_f^+ \psi(0) \equiv \lim_{t \to \infty} U_0(-t) J_C U(t) \psi(0).$$

This result is very general. In three or more dimensions, it covers a large class of scattering problems, linear and nonlinear. Here are some examples: We get that all global solutions are asymptotic to a free wave plus a weakly localized part. **In 5 or more dimensions, the WL part is in fact localized, for V decaying fast enough in x. No conditions on the t dependence.**
The localized part maybe time-dependent, yet it is probably the best result has in this generality.
More subtle result is that for interactions which are quasi-periodic in time, with sufficient decay of the eigenstates of the Floquet operator, we have that Local Decay estimates hold on the subspace of scattering states. ($\omega = p^2$).

The linear Schrödinger type equations of the form

$$i\frac{\partial\psi}{\partial t} = (\omega(p) + V(x,t))\psi.$$

The time dependent Charge Transfer Potentials In this case the hamiltonian describes a Quantum particle interacting with a set of moving potentials at non collisional velocities. The potentials are localized but can be time dependent:
Charge transfer Hamiltonians: Let Assumption of global existence hold. When the space dimension $n \geq 3$, the charge transfer interaction $\mathcal{N}(x, t, \psi) = \sum_{j=1}^{N} V_j(x - tv_j, t)$, where $V_j(x, t) \in L_t^\infty L_x^2(\mathbb{R}^{n+1})$, $j = 1, \cdots, N$, with $v_j \neq v_l$ if $j \neq l$, satisfies our conditions.

Proposition

Let assumptions above be satisfied. Then for every $j = 1, \cdots, N$, $\epsilon \in (0, 1/2)$ and $\alpha \in (0, 1 - 2/n)$, $n \geq 5$, it follows that:

In the strong limit sense, and for all $\alpha, \alpha' \in (0, 1-2/n)$,

1. $$\psi_{j,+}(x) := \lim_{t\to\infty} e^{itH_0} F_c(\frac{|x-2tP|}{t^{\alpha}} \leq 1)\psi_j(t) \text{exists in} L^2_x(\mathbb{R}^n) \tag{1}$$

$$\psi_{j,+}(x) = \lim_{t\to\infty} e^{itH_0} F_c(\frac{|x-2tP|}{t^{\alpha'}} \leq 1)\psi_j(t); \tag{2}$$

2. there exist N moving weakly localized parts, $\psi_{wl,j}(t) \equiv \psi_{wl,j,\epsilon}(t)$ such that the equation

$$\lim_{t\to\infty} \|\psi_j(t) - e^{-itH_0}\psi_{j,+}(x) - \psi_{wl,j}(t)\|_{L^2_x(\mathbb{R}^n)} = 0 \tag{3}$$

holds true, and $\psi_{wl,j}(t), j = 1, \cdots, N$, are moving weakly localized parts around tv_j satisfying

$$(e^{itP\cdot v_j}\psi_{wl,j}(t), |x|e^{itP\cdot v_j}\psi_{wl,j}(t))_{L^2_x(\mathbb{R}^n)} \lesssim_{\epsilon} t^{1/2+\epsilon}, \qquad t \geq 1. \tag{4}$$

Asymptotic Completeness for 3-body Quasi-particles
In this case, the hamiltonian is of the following form:

$$H_0 = \omega(p_1) + \omega(p_2) + \omega(p_3).$$

$$H = H_0 + \sum_{i<j} V_{ij}(x_i - x_j)$$

Asymptotic Completeness in this case means that the asymptotic states converge (strongly) to a linear combination of all possible channels of scattering. The possibilities are $(1)(2)(3); (12)(3); (13)(2); (1)(23)$. The asymptotic Hamiltonian for the first channel is H_0, the Free Channel. For the two cluster decompositions the limiting hamiltonian is a bound state of the pair moving with a constant speed, and the third particle is moving away with a constant speed.

Self Similar Potentials

We start with a linear model

$$\begin{cases} i\partial_t\psi = H_0\psi + g(t)^{-2}V(\frac{x}{g(t)})\psi \\ \psi(x, g(t_0)) = e^{-iA\ln g(t_0)}\psi_b(x) \in L^2_x(\mathbb{R}^n) \end{cases}, \quad n \geq 3 \qquad (5)$$

for some $t_0 > 0$ (t_0 will be chosen later), $g(t) \in C^2(\mathbb{R})$ satisfying that there exists two positive constants $c_g \in (0,1), \epsilon \in (0,1/2)$ such that

$$\begin{cases} \inf_{t\in\mathbb{R}} g(t) \gtrsim 1, \\ g(t) \sim \langle t\rangle^{\epsilon} \text{ as } t \to \infty \\ g(t) \sim tg'(t) \sim t^2 g''(t) \text{ as } t \to \infty \\ \lim_{t\to\infty} \frac{g(t)-2tg'(t)}{g(t)} = c_g \end{cases}.$$

$V(x)$ and $H := H_0 + V(x)$ satisfying that H has a unique normalized eigenstate $\psi_b(x)$ with an eigenvalue $\lambda < 0$ and

$$\begin{cases} 0 \text{ is regular for } H \\ \langle x \rangle A\psi_b(x) \in L^2_x \\ \langle x \rangle V(x) \in L^\infty_x, V(x) \in L^2_x \end{cases}$$

where $P_x := -i\nabla_x$, $A := \frac{1}{2}(x \cdot P_x + P_x \cdot x)$ and P_c denotes the projection on the continuous spectrum of H. We refer to the system (5) as mass critical system(**MCS**).

Since $g(t)^{-2}V(\frac{x}{g(t)}) \in L_t^\infty L_x^2(\mathbb{R}^3 \times \mathbb{R})$ when $\inf_t g(t) > 0$, due to (SW20221), the channel wave operator

$$\Omega_\alpha^* := s\text{-}\lim_{t\to\infty} e^{itH_0} F_c(\frac{|x-2tP_x|}{t^\alpha} \leq 1) U(t,0) \tag{6}$$

exists from $L_x^2(\mathbb{R}^3)$ to $L_x^2(\mathbb{R}^3)$ for all $\alpha \in (0,1/3)$, where F_c denotes a smooth characteristic function.
Based on (5), we also consider a class of mixture models

$$\begin{cases} i\partial_t\psi = H_0\psi + W(x)\psi + g(t)^{-2}V(\frac{x}{g(t)})\psi \\ \psi(x,t_0) = \psi_d(x) + e^{-iA\ln(g(t_0))}\psi_b(x) \in \mathcal{H}_x^1(\mathbb{R}^n) \\ \sup_{t\in\mathbb{R}} \|\psi(t)\|_{H_x^1} \lesssim 1 \end{cases} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{7}$$

$W(x)$ satisfies that

$$\begin{cases} H_0 + W(x) \text{ has a normalized eigenvector } \psi_d(x) \text{ with an eigenvalue } \lambda_0 \\ W(x) \in L_x^2(\mathbb{R}^n) \end{cases} \tag{8}$$

We showed in (Sof-W3) that the weakly localized part, asymptotically, has at least two bubbles: a non-trivial self-similar part and a non-trivial localized part near the origin.

1. Let
$$\tilde{a}(t) := (\psi_b(x), e^{iA\ln(g(t))}\psi(t))_{L_x^2}. \tag{9}$$
$$\tilde{A}(\infty) := \lim_{t\to\infty} e^{i\lambda T(t)}\tilde{a}(t) \tag{10}$$
exists.
2. Furthermore, there exists $t_0 > 0$ such that with an initial condition
$$\psi(t_0) = e^{-iA\ln(g(t_0))}\psi_b(x), \tag{11}$$
$$|\tilde{A}(\infty)| > 0 \tag{12}$$
which implies
$$\liminf_{t\to\infty} |c(t)| \gtrsim 1. \tag{13}$$

Theorem

Let $\tilde{a}(t)$ be as in (9). If $W(x), V(x), H$ satisfy (7) and $g(t)$ satisfies (6), then when $n \geq 5$, $\epsilon \in (2/n, 1/2)$,
$$\tilde{A}(\infty) := \lim_{t\to\infty} e^{i\lambda T(t)}\tilde{a}(t) \tag{14}$$
exists.

Furthermore,

$$\psi_{w,l}(x,t) = c(t)e^{-iA\ln(g(t))}\psi_b(x) \oplus \psi_c(x,t) \tag{15}$$

$$c(t) := (e^{-iA\ln(g(t))}\psi_b(x), \psi_{w,l}(x,t))_{L^2_x}, \tag{16}$$

$$(e^{-iD\ln(g(t))}\psi_b(x), \psi_c(x,t))_{L^2_x} = 0, \tag{17}$$

where $|c(t)| \gtrsim 1$ and there exists $M > 1$ such that

$$\liminf_{t\to\infty} |(\psi_c(x,t), \psi_d(x))_{L^2_x}| \geq c. \tag{18}$$

Moreover, the $g(t)$-self-similar channel wave operator

$$\Omega_g^*\psi(0) := w\text{-}\lim_{s\to\infty} e^{isH}e^{iA\ln(g(T^{-1}(s)))}\psi(T^{-1}(s)) \tag{19}$$

exists in L^2_x and

$$\Omega_g^*\psi(0) = \tilde{A}(\infty)\psi_b(x). \tag{20}$$

These results extend to nonlinear perturbations (mass super-critical) of the model above.

Nonlinear Examples

Nonlinear Schrödinger Equations- Free Channel WO and weak localization in Radial case

$$\mathcal{N}(x,t,\psi) = V(x,t) + G(x,t)|\psi|^m + \frac{|\psi|^{m'}}{1+|\psi|^n}, \quad d \geq 3,$$

with e.g. $m = 1$, or m large, and for a solution that is global and uniformly bounded in H^1.

$$\mathcal{N}(x,t,\psi) = V(x,t) + G(x,t)|\psi|^m, \quad d = 1, 2.$$

In this case we need V, G to have some decay in x.
The same is true for the NLKG equation.

Nonlinear Schrödinger Equations- Properties of Weakly Localized part

For Localized in space Interactions (including radial symmetric Nonlinear terms):
We have, quite generally that $< |x| >_t \lesssim t^{1/2+0}$ and the part of the solution that spreads is self-similar(NLS).
Depending on the regularity of the potential, and analyticity of the purely nonlinear term, in 3 or higher dimensions, the localized part has high regularity.
The localized solutions are "Mikhlin" type functions for the above type Interactions: Quite Generally $A\psi$ is uniformly bounded in L^2_x. Here $2A = x \cdot p + p \cdot x$ is the generator of Dilations. In many cases this holds for all powers of A.
In 5 or more dimensions, we have the strongest result: In this case the solutions of NLS type equations, with interactions that decay fast enough in x, **the weakly localized part of the solution is localized**. The proof of this estimate is involved.

In one dimension, we cannot prove the existence of the free channel, except in one very recent work, where the nonlinear terms are defocusing, and of the right powers. In these cases the weakly localized part can be shown to spread slower than $t^{1/2}$ under further conditions. This proof is much more complicated than the standard situation.

Wave Equation In this case we can show the existence of the free channel wave operator for localized interactions, and furthermore the complete localization of the the weakly localized part.

The non-radial case is more complicated in three dimensions or less. For the WE, we have no boundedness of the L^2 norm. The proof of localization requires $d \geq 4$ and extra conditions on the structure of the nonlinear terms. Localized metric perturbations are allowed.

Existence of the free Channel decomposition Let us then consider a system with dynamics $U(t)$ which may be linear or not, acting on initial data $\psi(0)$, which leads to a global solution $\psi(t) = U(t)\psi(0)$.
In order to construct the free part of the solution at infinity, we now introduce the *free channel wave operator*

$$\Omega^*_{F_\pm}\psi(0) = s - \lim_{t\to\pm\infty} e^{iH_0 t} J_{free} U(t)\psi(0). \tag{21}$$

The key new idea is now to choose J_{free} the "right way". We would like to choose J_{free} as a smooth projection on the region of phase-space where the free solution concentrates.

A classical particle moves under the free flow according to

$$x(t) = x(0) + vt.$$

So, we are led to use $|x - v(k)t| \leq t^{\alpha}$. Then, we choose

$$J_{free} = F_c(\frac{|x - 2pt|}{t^{\alpha}} \leq 1). \tag{22}$$

$$\omega(k) = k^2, \quad v(k) = 2k, \quad v(p) = -2i\nabla_x, \tag{23}$$

$$0 < \alpha < 1. \tag{24}$$

In the short-range scattering problems α can be chosen small. But it is not possible in general; in particular in the long-range scattering case, α cannot be small (ifrim2022testing,lindblad2021asymptotics,lindblad2023modified,LS2015,L

Next, to prove the limit above exists, we use Cook's argument: Writing $(\Omega^*_{F_+} - I)\psi(0) = -\int_0^\infty e^{iH_0 t}[F_c\mathcal{N} + \tilde{F}']U(t)\psi(0)$, then, we need to prove that the integral converges absolutely. Now notice that by the Heisenberg formulation of QM, we have that:

$$e^{i\Delta t} x e^{-i\Delta t} = x + 2pt; \quad \partial_t\{e^{i\Delta t}(x - 2pt)e^{-i\Delta t}\} = 0.$$

Therefore,

$$D_H G(x - 2pt) = i[-\Delta, G] + \partial_t G = 0, G \text{ arbitrary}, \tag{25}$$

$$e^{-i\Delta t} G(x) e^{+i\Delta t} = G(x - 2pt), \tag{26}$$

$$e^{-i\Delta t} G(x - 2pt) U(t) = G(x) e^{-i\Delta t} U(t) \tag{27}$$

From these identities, we derive

$$e^{-iH_0 t}[F_c\mathcal{N} + \tilde{F}']U(t)\psi(0) = F_1(\frac{|x|}{t^\alpha} \leq 1)e^{+iH_0 t}[\mathcal{N} - \frac{\alpha}{t}F_1']U(t)\psi(0). \tag{28}$$

Therefore we need to prove the integrability (in norm) of the above expression. It consists of two terms. The first one, coming from the interaction term, is bounded by

$$\|F_1\|_{L^2_x}\|e^{+iH_0t}\mathcal{N}U(t)\psi(0)\|_{L^\infty_x} \lesssim t^{n\alpha/2}t^{-n/2}\|\mathcal{N}U(t)\psi(0)\|_{L^{1,s}}. \tag{29}$$

This estimate gives the main condition on the interaction $\mathcal{N}$. In its abstract form it is: $(t \geq 1)$

$$\|F_1(\frac{|x|}{t^\alpha} \leq 1)F_2(\sum_j |p-\tau_j| > t^{-\beta})U_0(-t)\mathcal{N}\psi(t)\|_{L^2_x} \lesssim t^{-1-\epsilon}.$$

Here the free dynamics is generated by $H_0 = \omega(p)$, with thresholds at τ_j. In three or more dimensions, with the standard Laplacian generating the free flow, we do not need to localize away from the thresholds. If the interaction term in localized in space (with sufficient decay), then one can get the needed estimate also in 1 and 2 dimensions. Moreover, since we assume the solution is uniformly bounded in H^1, we can allow one derivative in the interaction, on each side. Therefore terms like $-\nabla_i g_{ij}(x,t)\nabla_j$ can also be incorporated with a suitable martix g_{ij}.

This expression is integrable in time if α is sufficiently small, and the dimension $n \geq 3$. The number of derivatives s depends on dimension for the Wave equations of the Hyperbolic type, but is zero for the Schroëdinger type. Also note that the effective dimension n for some Hyperbolic equations is $n-1$.
For this to hold we only need to know that $\mathcal{N} U(t)\psi(0)$ is uniformly (in time) bounded in L^1, and in fact a weaker condition is sufficient for integrability. **This estimate is very general, and it only uses L^p_x estimates on the solution and interaction terms, but not point-wise decay of the interaction at infinity. Therefore it does not require the assumption of spherical symmetry.**

In fact, it is here that we need an a-priory estimate w.r.t. the **full,** interacting flow. This is now done by using the method described before, of proving Propagation Estimates by an appropriate choice/s of Propagation Observables. In this case the answer is very simple, we use as Propagation Observable the operator $F_c(\frac{|x-2pt|}{t^\alpha} \leq 1)$ itself.
To this end we compute:

$$\partial_t(U(t)\psi(0), e^{+i\Delta t}F_1(\frac{|x|}{t^\alpha} \leq 1)e^{-i\Delta t}U(t)\psi(0)) = \tag{30}$$

$$-(\Omega^*(t)\psi(0), \frac{\alpha}{t}F_1'\Omega^*(t)\psi(0)) + \tag{31}$$

$$2\Re(\Omega^*(t)\psi(0), F_1 e^{-i\Delta t}\mathcal{N}\psi(t)). \tag{32}$$

$$\Omega^*(t) \equiv e^{-i\Delta t}U(t). \tag{33}$$

The first term on the RHS is positive. The second term is integrable as we showed in the previous step. Since the integral of the LHS is uniformly bounded (by L^2 norms), it follows that the F_1' term is also integrable. This is then used to control the term we need for proving the existence of the free channel wave operator.

Weak Localization of the Non-Radiative Part

The key is to prove that outgoing waves, far away, can only be free waves! The fact that the incoming waves far away vanish is easier to verify.

$$\|e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0) - F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1)P^+F_1(\sqrt{t+1}|P| \geq 1) \tag{34}$$

$$\begin{aligned} &e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)} \\ \leq&\|F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1)P^+F_1(\sqrt{t+1}|P| < 1)e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)} \\ &+\|F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} < 1)P^+e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)} \\ &+\|P^-e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)} \end{aligned} \tag{35}$$

$$
\begin{aligned}
\leq & \|F_1(\sqrt{t+1}|P|<1)e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)}+ \\
& \|F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}}<1)P^+e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)} \\
& +\|P^-e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)}\to 0
\end{aligned}
$$

as $t\to\infty$.

We arrive at (as $t \to \infty$,

$$\|\psi(t) - \psi_{wl}(t) - e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)}$$

$$= \|\psi(t) - F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} < 1)\psi(t) - e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)}$$

$$\leq \|F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1)\psi(t) - e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)}$$

$$\leq \|\psi_1(t) - F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1)P^+F_1(\sqrt{t+1}|P| \geq 1)e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x}$$

$$+ \|e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0) - F_c(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1)P^+F_1(\sqrt{t+1}|P| \geq 1) \tag{36}$$

$$e^{-itH_0}\Omega^*_{\alpha,\beta}\psi(0)\|_{L^2_x(\mathbb{R}^n)}$$

$$+ \|\psi_2(t)\|_{L^2_x(\mathbb{R}^n)} + \|\psi_3(t)\|_{L^2_x(\mathbb{R}^n)} \to 0, \tag{37}$$

Local Decay in a New Way

We can prove Local Decay using the proved AC.
By AC the subspace of scattering states is identified by the range of the wave operators $\Omega_\pm$.
For $U(t,0)\Omega_+\phi$, using incoming/outgoing decomposition, we split $U(t,0)\Omega_+\phi$ into four pieces:

$$\begin{aligned} U(t,0)\Omega_+\phi =& P^+ e^{-itH_0}\Omega_+^*\Omega_+\phi + P^- e^{-itH_0}\Omega_-^*\Omega_+\phi + \\ & P^+(1-\Omega_{t,+}^*)U(t,0)\Omega_+\phi + P^-(1-\Omega_{t,-}^*)U(t,0)\Omega_+\phi \\ =:& \psi_1(t) + \psi_2(t) + C_1(t)U(t,0)\Omega_+\phi + C_2(t)U(t,0)\Omega_+\phi \end{aligned}$$

with

$$P_c(t) = s\text{-}\lim_{s\to\infty} U(t,t+s)F_c\left(\frac{|x-2sP|}{s^\alpha}\le 1\right)U(t+s,t), \quad \text{on } L^2_x(\mathbb{R}^5),$$

$$P^+\Omega_{t,+}^* := s\text{-}\lim_{a\to\infty} P^+ e^{iaH_0}U(t+a,t)P_c(t) \quad \text{on } L^2_x(\mathbb{R}^5),$$

$$P^-\Omega_{t,-}^* := s\text{-}\lim_{a\to-\infty} e^{iaH_0}U(t+a,t)P_c(t) \quad \text{on } L^2_x(\mathbb{R}^5)$$

and

$$C_1(t) := P^+(1 - \Omega^*_{t,+}), \quad C_2(t) := P^-(1 - \Omega^*_{t,-}).$$

It is not clear here whether $\Omega^*_{t,+}$ and $\Omega^*_{t,-}$ exist with $P_c(t)$ defined only in one direction($t \to \infty$), but with $P^\pm$, $P^\pm\Omega^*_{t,\pm}$ exist on $L^2_x(\mathbb{R}^5)$. Here we also use the following time-dependent intertwining property

$$\Omega^*_{t,\pm} U(t,0) = e^{-itH_0}\Omega^*_\pm \quad \text{on } L^2_x(\mathbb{R}^5). \tag{38}$$

Lemma

If V satisfies decay and regularity assumptions, then

$$C_j(t) = C_{jm}(t) + C_{jr}(t), \quad j = 1, 2 \tag{39}$$

for some operators $C_{jm}(s,u)$ and $C_{jr}(s,u)$ satisfying

$$\sup_{t\in\mathbb{R}} \| C_{jr}(t) \|_{L^2_x \to L^2_x} \leq 1/1000, \tag{40}$$

$$\sup_{t\in\mathbb{R}} \| \langle x \rangle^{-\eta} (1 - C_{1r}(t) - C_{2r}(t))^{-1} C_{jr}(t) \langle x \rangle^{\sigma} \|_{L^2_x \to L^2_x} \lesssim_\sigma 1 \tag{41}$$

for all $\eta > 5/2, \sigma \in (1, 101/100)$ and

$$\| C_{jm} U(t,0) \Omega_+ \phi(x) \|_{L^2_{x,t}} \lesssim \|\phi\|_{L^2_x(\mathbb{R}^5)}. \tag{42}$$

Then according to a similar argument as what we did for the time-independent system, we finish the proof.

The proofs of compactness and smallness are long. They are based on applying propagation estimates to the integral representation of the corresponding wave operators. Then, decomposition to small and large frequencies, and applying the various **Propagation estimates** for the free flow.

Soliton Resolution

It is **not true** that the asymptotic localized solutions are solitons, that is, time independent solutions up to gauge and other kinetic symmetries. However, it maybe true that solitons are the **generic solutions.** Suppose the solution is time periodic. Then, in the nonlinear case, via Floquet theory, it corresponds to an embedded eigenvalue in the continuous spectrum. It is possible to show quite generally, using time dependent resonance theory that such eigenvalues are non-generic. Same is true for quasi-periodic in time potentials, provided local decay estimates can be proven for such hamiltonians. We proved such estimates in 5 or more dimensions. So, one can consider solutions which are localized, smooth and are almost periodic in time. Such potentials can be approximated by quasi-periodic potentials, on any finite time interval. So, we are left with time dependence that is more general than (asymptotically) almost periodic potentials. Like Chaotic? The Petite Conjecture is the claim that such chaotic solutions do not exist for dispersive/hyperbolic equations on $\mathbb{R}^n$.

Open Problems

Breathers in Higher dimensions
Prove/disprove that equations (NLS, KG, WE or systems) with nonlinear Interactions of the form $\mathcal{N}(\psi, t)$ have no Breather solutions in dimension 2 or higher.

Quantum Ping-Pong
Prove/disprove that any interaction that is bounded, localized in space, **and non-negative** has no localized solutions.

Inverse Scattering
(a) How to determine bound solutions energies and resonances from a known nonlinear S-Matrix?
(b) How adding a nonlinear term to a linear equation can improve the solution of the inverse problem with partial data?

THE END