Scattering of Schrödinger Operators with Step Potentials

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NLS with a potential

We consider the cubic Schrödinger equation with a potential

$$i\partial_t u = -\partial_x^2 u + Vu + |u|^2 u.$$

Denote

$$H = -\partial_x^2 + V.$$

It is of great interest to study the global-in-time bounds and long-time asymptotic behaviors for small initial data.

Previous works

- V. S. Buslaev, G. S. Perelman, Scattering for the nonlinear Schrödinger equation: states close to a soliton, 1993
- J. Krieger and W. Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, 2006
- P. Germain, F. Pusateri and F. Rousset, The nonlinear Schrödinger equation with a potential in dimension 1, 2018
- G. Chen and F. Pusateri, The 1D NLS with a weighted L^1 potential, 2019; The 1D cubic NLS with a non-generic potential, 2022

1D cubic NLS with exponentially decaying potential

Theorem (P. Germain, F. Pusateri and F. Rousset, 2018)

Assume that V is decaying at a super-polynomial rate when $x \to \pm\infty$, H has no endpoint resonances at 0, there exists $\epsilon_0 > 0$ such that, if u_0 satisfies $\|u_0\|_{H^3_x} + \|\widetilde{u_0}\|_{H^1_\lambda} = \epsilon < \epsilon_0$, then a global solution exists, which decays pointwisely:

$$\|u(t)\|_{L^\infty_x} \lesssim \epsilon_0 (1 + |t|)^{-1/2}$$

and is globally bounded in L^2 type spaces: for any time t,

$$\|\widetilde{f}\|_{L^\infty_t([0,T];L^\infty_\lambda)} + \|t^{-\alpha}\widetilde{f}\|_{L^\infty_t([0,T];H^1_\lambda)} \lesssim \epsilon$$

The approach in this paper is based on the use of the distorted Fourier transform (Weyl-Kodaira-Titchmarsh theory), which will allow the application of some Fourier analytical techniques to nonlinear equations which involve external potentials.

Step potential: Scattering theory

We aim to extend this analysis to step-type potentials

$$V(x) \to a_- \quad \text{as} \quad x \to -\infty\,, \qquad V(x) \to a_+ \quad \text{as} \quad x \to +\infty$$

with $a_\pm$ finite but $a_- \neq a_+$.

1D cubic NLS with step potential

Consider

$$i\partial_t u = -\partial_x^2 u + Vu + \mathcal{N}(u)$$

where

$$\mathcal{N}(u) = |u|^2 u \quad \text{and} \quad V = \mathbf{1}_+(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

That is, the 1D NLS with cubic nonlinearity and Heaviside potential.

Step potential: Scattering theory

We start by considering the standard Schrödinger operator

$$H = -\partial_x^2 + V$$

as well as the equation

$$Hf = Ef$$

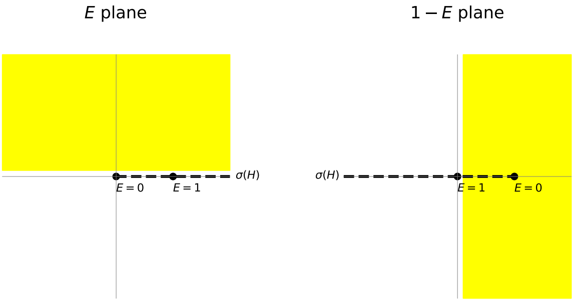
Step potential: Scattering theory

The spectrum $\sigma(H)$ and resolvent set $\rho(H)$ are

$$\sigma(H) = [0, +\infty)\,, \qquad \rho(H) = \mathbb{C}\backslash\sigma(H)$$

To express the kernel of the resolvent operator $(H - E)^{-1}$ for $E \in \rho(H)$, we introduce the convention

- $\sqrt{E}$ in the upper half-plane
- $\sqrt{1-E}$ in the right half-plane



Step potential: Scattering theory

For $E \in \rho(H)$, consider $e_\pm(x, E)$ solving

$$He_\pm = Ee_\pm$$

given by

$$e_+(x; E) = \begin{cases} e^{i\sqrt{E}\,x} + R_+(E)e^{-i\sqrt{E}\,x} & \text{if } x < 0 \\ T_+(E)e^{-\sqrt{1-E}\,x} & \text{if } x \geq 0 \end{cases}$$

where

$$T_+(E) = \frac{2\sqrt{E}}{\sqrt{E} + i\sqrt{1-E}}, \qquad R_+(E) = \frac{\sqrt{E} - i\sqrt{1-E}}{\sqrt{E} + i\sqrt{1-E}}$$

and

$$e_-(x; E) = \begin{cases} T_-(E)e^{-i\sqrt{E}\,x} & \text{if } x \leq 0 \\ e^{\sqrt{1-E}\,x} + R_-(E)e^{-\sqrt{1-E}\,x} & \text{if } x > 0 \end{cases}$$

where

$$T_-(E) = \frac{2i\sqrt{1-E}}{\sqrt{E} + i\sqrt{1-E}}, \qquad R_-(E) = \frac{-\sqrt{E} + i\sqrt{1-E}}{\sqrt{E} + i\sqrt{1-E}}$$

Step potential: Scattering theory

These satisfy, for $E \in \rho(H)$

$$e_\pm(x, E) \text{ exponentially decays as } x \to \pm\infty$$

Note that $T_+(E) \neq T_-(E)$, in fact

$$i\sqrt{E}\, T_-(E) = -\sqrt{1-E}\, T_+(E)$$

so that

$$T_-(1) = 0\,, \qquad T_+(0) = 0$$

Also the Wronskian $W(E)$ satisfies

$$\frac{W(E)}{T_-(E)T_+(E)} = -\sqrt{1-E} + i\sqrt{E}$$

Since

$$\frac{W(E)}{T_-(E)T_+(E)} \neq 0 \quad \text{at} \quad E = 0, 1$$

there are no endpoint or embedded resonances. This is also called the generic case.

Step potential: Scattering theory

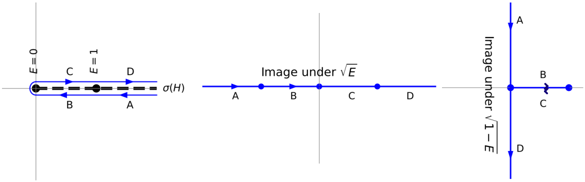
For $E \in \rho(H)$, the kernel of the resolvent $(H-E)^{-1}$ is

$$(H-E)^{-1}(x,y) = \frac{1}{W(E)} \begin{cases} e_-(x,E)e_+(y,E) & \text{if } x<y \\ e_-(y,E)e_+(x,E) & \text{if } y<x \end{cases}$$

By Stone's formula

$$\frac{1}{2\pi i}\oint (H-E)^{-1}(x,y)\,dE = \delta(x-y)$$

where integration is around the contour



Step potential: Scattering theory

This gives

$$\frac{1}{2\pi i}\int_{E=0}^{+\infty}[(H-(E-i0))^{-1}(x,y)-(H-(E+i0))^{-1}(x,y)]\,dE=\delta(x-y)$$

With

$$E=k^2=\mu^2+1$$

this jump in the resolvent kernel can be re-expressed to give

$$\delta(x-y)=\frac{1}{2\pi}\int_{k=0}^{+\infty}e_+(x,k)\overline{e_+(y,k)}\,dk+\frac{1}{2\pi}\int_{\mu=0}^{+\infty}e_-(x,\mu)\overline{e_-(y,\mu)}\,d\mu$$

Step potential: Distorted Fourier transform

This gives two branches of the distorted Fourier transform associated to the operator H:

$$(\mathcal{F}_+ f)(k) = \tilde{f}_+(k) = \frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{+\infty} \overline{e_+(y,k)}\, f(y)\, dy$$

$$(\mathcal{F}_- f)(\mu) = \tilde{f}_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{+\infty} \overline{e_-(y,\mu)}\, f(y)\, dy$$

$$\mathcal{F} f = \tilde{f} = (\tilde{f}_+(k), \tilde{f}_-(\mu))$$

The inverse is

$$(\mathcal{F}^{-1} g)(x) = \frac{1}{\sqrt{2\pi}} \Big[\int_{k=0}^{+\infty} e_+(x,k) g_+(k) dk + \int_{\mu=0}^{+\infty} e_-(x,\mu) g_-(\mu) d\mu \Big]$$

for $g = (g_+(k), g_-(\mu))$.

The distorted Fourier transform diagonalizes H:

$$H = -\partial_x^2 + V = \mathcal{F}^{-1} \begin{pmatrix} k^2 & 0 \\ 0 & \mu^2 + 1 \end{pmatrix} \mathcal{F}.$$

Step potential: Scattering theory

Using the conversions

$$\mu = \begin{cases} i\sqrt{1-k^2} & \text{if } 0 < k < 1 \\ \sqrt{k^2-1} & \text{if } k > 1 \end{cases}, \qquad k = \sqrt{1+\mu^2}$$

We can express, for $k \geq 0$

$$e_+(x,k) = \begin{cases} e^{ikx} + R_+(k)e^{-ikx} & \text{if } x < 0 \\ T_+(k)e^{i\mu x} & \text{if } x > 0 \end{cases}$$

where

$$T_+(k) = \frac{2k}{k+\mu}, \quad R_+(k) = \frac{k-\mu}{k+\mu}$$

and for $\mu \geq 0$

$$e_-(x,\mu) = \begin{cases} T_-(\mu)e^{-ikx} & \text{if } x < 0 \\ e^{-i\mu x} + R_-(\mu)e^{i\mu x} & \text{if } x > 0 \end{cases}$$

where

$$T_-(\mu) = \frac{2\mu}{k+\mu}, \quad R_-(\mu) = \frac{-k+\mu}{k+\mu}$$

Step potential: Scattering theory

For later reference, we can rewrite these into free and outgoing (but endpoint vanishing) pieces

$$\begin{aligned} e_+(x,k) =& (e^{ikx} - e^{-ikx})\mathbf{1}_-(x) && \leftarrow \text{free} \\ &+ T_+(k)(e^{-ikx}\mathbf{1}_-(x) + e^{i\mu x}\mathbf{1}_+(x)) && \leftarrow \text{outgoing} \end{aligned}$$

$$\begin{aligned} e_-(x,k) =& (e^{-i\mu x} - e^{i\mu x})\mathbf{1}_+(x) && \leftarrow \text{free} \\ &+ T_-(\mu)(e^{-ikx}\mathbf{1}_-(x) + e^{i\mu x}\mathbf{1}_+(x)) && \leftarrow \text{outgoing} \end{aligned}$$

Step potential: Decay estimate

Lemma (Pointwise decay)

Under our assumptions on the potential, for $t \geq 1$,

$$\|e^{itH}h\|_{L^\infty} \lesssim \frac{1}{|t|^{1/2}}\|\tilde{h}_+\|_{L^\infty} + \frac{1}{|t|^{1/2}}\|\tilde{h}_-\|_{L^\infty} + \frac{1}{|t|^{3/4}}\|\partial_k \tilde{h}_+\|_{L^2} + \frac{1}{|t|^{3/4}}\|\partial_\mu \tilde{h}_-\|_{L^2}$$

Lemma (Enhanced local decay)

Under our assumptions on the potential, for $t \geq 1$,

$$\|\langle x\rangle^{-1}e^{itH}h\|_{L^\infty_x} \lesssim \frac{1}{|t|}\|\tilde{h}_+\|_{H^1} + \frac{1}{|t|}\|\tilde{h}_-\|_{H^1}.$$

Step potential: Cubic interaction

Recall the cubic NLS

$$i\partial_t u = Hu + \mathcal{N}(u), \quad \mathcal{N}(u) = |u|^2 u$$

becomes, with $f = e^{itH}u$,

$$i\partial_t f = e^{itH}\mathcal{N}(e^{-itH}f)$$

Taking the distorted Fourier transform,

$$i\partial_t \tilde{f}_+(k) = e^{itk^2}\mathcal{F}_+[\mathcal{N}(e^{-itH}f)](k)$$

$$i\partial_t \tilde{f}_-(\mu) = e^{it(\mu^2+1)}\mathcal{F}_-[\mathcal{N}(e^{-itH}f)](\mu)$$

Into the nonlinearity $\mathcal{N}(\bullet)$ substitute

$$[e^{-itH}f(\bullet,t)](x) = \frac{1}{2\pi}\int_0^\infty e^{-itk^2}e_+(x,k)\tilde{f}_+(k)\,dk + \frac{1}{2\pi}\int_0^\infty e^{-it(\mu^2+1)}e_-(x,\mu)\tilde{f}_-(\mu)\,d\mu$$

Step potential: Cubic interaction

This gives the following representation for the solution, where we replace (k, μ) with (ℓ, ν), (m, ξ), and (n, σ)

$$\begin{pmatrix} i\partial_t \tilde{f}_+(k) \\ i\partial_t \tilde{f}_-(\mu) \end{pmatrix} = \iiint \int_y \begin{pmatrix} e^{itk^2}\overline{e_+(y,k)} \\ e^{it(\mu^2+1)}\overline{e_-(y,\mu)} \end{pmatrix} \begin{pmatrix} e^{-it\ell^2} e_+(y,\ell)\tilde{f}_+(\ell) \\ e^{-it(\nu^2+1)} e_-(y,\nu)\tilde{f}_-(\nu) \end{pmatrix}$$

$$\begin{pmatrix} e^{itm^2}\overline{e_+(y,m)}\,\overline{\tilde{f}_+(m)} \\ e^{it(\xi^2+1)}\overline{e_-(y,\xi)}\,\overline{\tilde{f}_-(\xi)} \end{pmatrix} \begin{pmatrix} e^{-itn^2} e_+(y,n)\tilde{f}_+(n) \\ e^{-it(\sigma^2+1)} e_-(y,\sigma)\overline{\tilde{f}_-(\sigma)} \end{pmatrix} dy$$

which actually represents 16 different terms, top/bottom in each of four places and accordingly, the triple integral is over ℓ or ν, m or ξ, n or σ.

Step potential: Cubic interaction

Thus we obtain

$$
\begin{aligned}
i\partial_t \tilde{f}_+(k) = &\iiint e^{it(k^2-\ell^2+m^2-n^2)} \mu_{++++}(k,\ell,m,n) \\
&\cdot \overline{\tilde{f}_+(\ell)} \tilde{f}_+(m) \overline{\tilde{f}_+(n)} \, d\ell \, dm \, dn \\
&+ e^{-it} \iiint e^{it(k^2-\nu^2+m^2-n^2)} \mu_{+-++}(k,\nu,m,n) \\
&\cdot \overline{\tilde{f}_-(\nu)} \tilde{f}_+(m) \overline{\tilde{f}_+(n)} \, d\nu \, dm \, dn \\
&+ \cdots (6 \text{ more})
\end{aligned}
$$

where

$$
\mu_{++++}(k,\ell,m,n) = \int_{y=-\infty}^{+\infty} \overline{e_+(y,k)} e_+(y,\ell) \overline{e_+(y,m)} e_+(y,n) \, dy
$$

and similarly, for the other 15 terms.

Step potential: Cubic interaction

Using that

$$\widehat{\mathbf{1}_{\pm}}(\xi) = \pm \frac{1}{i\xi} + \pi\delta(\xi)$$

we can calculate each term. For example, when we split each e_+ into free and outgoing parts

$$e_+ = e_{+,F} + e_{+,O}\,, \qquad e_{+,F}(y,k) = (e^{iyk} - e^{-iyk})\mathbf{1}_-(y)$$

we obtain for the $e_{+,F}$ only part

$$\sum_{\alpha,\beta,\gamma,\epsilon\in\{-1,1\}} \alpha\beta\gamma\epsilon \int_{-\infty}^{0} e^{-\alpha iky} e^{\beta i\ell y} e^{-\gamma imy} e^{\epsilon iny}\, dy$$

$$= \sum_{\alpha,\beta,\gamma,\epsilon\in\{-1,1\}} \alpha\beta\gamma\epsilon \left(\frac{1}{(\alpha k + \beta\ell - \gamma m + \epsilon n)i} + \pi\delta(\alpha k + \beta\ell - \gamma m + \epsilon n) \right)$$

Step potential: Cubic interaction

Since $(\alpha, \beta, \gamma, \epsilon) \to (-\alpha, -\beta, -\gamma, -\epsilon)$ keeps the sign of $\alpha\beta\gamma\epsilon$ unchanged by flips the sign of the pv term,

$$= \pi \sum_{\alpha,\beta,\gamma,\epsilon \in \{-1,1\}} \alpha\beta\gamma\epsilon \, \delta(\alpha k + \beta \ell - \gamma m + \epsilon n)$$

so that this term behaves like a product of four free (no potential) linear Schrödinger solutions.

1D cubic NLS with step potential

The bootstrap space is:

$$
\begin{aligned}
\|f\|_X = & \|\tilde{f}_+(k)\|_{L^\infty_{k>0}} + \|\tilde{f}_-(\mu)\|_{L^\infty_{\mu>0}} \\
& + \langle t \rangle^{-1/4+} \| |1-k|^{0+} \partial_k \tilde{f}_+(k) \|_{L^2_{k>0}} + \langle t \rangle^{-1/4+} \|\partial_\mu \tilde{f}_-(\mu)\|_{L^2_{\mu>0}} \\
& + \langle t \rangle^{0-} \| \langle k \rangle^3 \tilde{f}_+(k) \|_{L^2_{k>0}} + \langle t \rangle^{0-} \| \langle \mu \rangle^3 \tilde{f}_-(\mu) \|_{L^2_{\mu>0}}
\end{aligned}
$$

The extra factor $|1-k|^{0+}$ is needed to deal with

$$
\mu = \begin{cases} i\sqrt{1-k^2} & \text{if } k < 1 \\ \sqrt{k^2-1} & \text{if } k > 1 \end{cases}
$$

1D cubic NLS with step potential

Theorem (J. Holmer and Z. Z., ongoing work)

Consider the 1D cubic Schrödinger equation with the Heaviside step potential $V(x)$ *described above:*

$$i\partial_t u = -\partial_x^2 u + Vu + |u|^2 u.$$

We have:
There exists $0 < \epsilon_0 \ll 1$ *such that for all* $\epsilon \leq \epsilon_0$ *and* u_0 *with*

$$\|u_0\|_{H^3} + \|xu_0\|_{L^2} \leq \epsilon,$$

then a global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *exists, with* $u(0,x) = u_0(x)$*; this solution satisfies the sharp decay rate*

$$\|u(t)\|_{L_x^\infty} \lesssim \epsilon(1+|t|)^{-1/2},$$

$$\|f\|_X \lesssim \epsilon.$$

Black/dark solitons of the 1D Gross-Pitaevskii equation

Consider the 1D Gross-Pitaevskii equation

$$i\partial_t u + \partial_x^2 u + (1 - |u|^2)u = 0.$$

The *black/dark solitons* of the 1D GP equation are solitary wave with non-zero asymptotics when $|x| \to \infty$, which brings the challenge of non-decaying property of the linearized operator as $|x| \to \infty$:

$$U_c(x) := \frac{\sqrt{2-c^2}}{2} \tanh(\frac{\sqrt{2-c^2}}{2} x) + i \frac{c}{\sqrt{2}}.$$

It is called a grey soliton when $c \neq 0$, and a black soliton when $c = 0$.

Black/dark solitons of the 1D Gross-Pitaevskii equation

- L. Di Menza and C. Gallo (2007), Linear stability of black solitons
- F. Béthuel, P. Gravejat, J.-C. Saut, and D. Smets (2008), Orbital stability of the black soliton
- F. Béthuel, P. Gravejat, and D. Smets (2015), Asymptotic stability in the energy space for dark solitons using hydrodynamics formulation, does not apply to black solitons
- P. Gravejat and D. Smets (2015), Asymptotic stability of the dark/black soliton, weak convergence in time in H^1

Black/dark solitons of the 1D Gross-Pitaevskii equation

Let $u = U_c + w$. Write

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} w \\ \overline{w} \end{pmatrix},$$

and denote the vector Schrödinger operator

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{pmatrix} \partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix} + \begin{pmatrix} V_+ & V_- \\ -V_- & -V_+ \end{pmatrix},$$

where

$$V_+ = 1 - 2|U_c|^2, \; V_- = -U_c^2.$$

The linearized Gross-Pitaevskii equation around the dark/black soliton can be written as

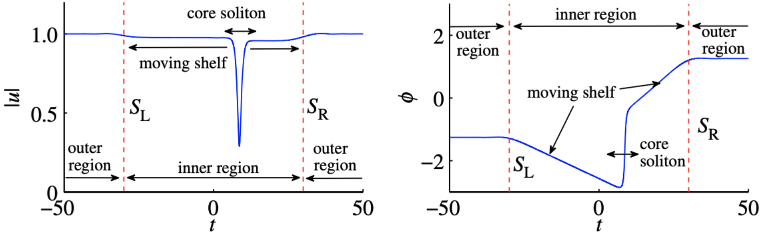
$$i\partial_t W + \mathcal{H} W = \{nonlinear\}.$$

Compared to the linearization around the bright soliton, this linearized (matrix) operator $\mathcal{H}$ has non-zero and step-like asymptotes at $\pm\infty$.

Black/dark solitons of the 1D Gross-Pitaevskii equation

It is expected that the perturbation of size $O(\epsilon)$ will evolve as a "spreading shelf" of length $O(t)$. This is indicated in numerical simulations in Ablowitz-Nixon-Horikis-Frantzeskakis (2011), where the perturbations of dark solitons under additional small forcing is observed via numerics. Hence we need to consider some L^2-based functional space that is truncated to be supported away from zero.

Black/dark solitons of the 1D Gross-Pitaevskii equation



Black/dark solitons of the 1D Gross-Pitaevskii equation

A linear toy model of linearizing around $\Psi(x) \equiv 1$: Taking $u(t,x) = \Psi(x) + w(t,x) = 1 + w(t,x)$, and writing $w(t,x) = f(t,x) + ig(t,x)$, we obtain the following linear part:

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 - 2 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

Black/dark solitons of the 1D Gross-Pitaevskii equation

Taking the (flat) Fourier transform, we can see

$$\partial_t^2 \widehat{f} = -\xi^2(\xi^2 + 2)\widehat{f},$$
$$\partial_t^2 \widehat{g} = -\xi^2(\xi^2 + 2)\widehat{g}.$$

In low frequency, the linear model has a wave-like behavior. We can solve out

$$\widehat{f}(t,\xi) = \cos(\xi\sqrt{\xi^2 + 2}t)\widehat{f}(0,\xi) + \frac{\xi \sin(\xi\sqrt{\xi^2 + 2}t)}{\sqrt{\xi^2 + 2}}\widehat{g}(0,\xi),$$
$$\widehat{g}(t,\xi) = \cos(\xi\sqrt{\xi^2 + 2}t)\widehat{g}(0,\xi) - \frac{\sqrt{\xi^2 + 2}\sin(\xi\sqrt{\xi^2 + 2}t)}{\xi}\widehat{f}(0,\xi).$$

Black/dark solitons of the 1D Gross-Pitaevskii equation

When $|\xi| \ll |t|^{-1/2}$,

$$\begin{aligned} g(t,x) &\sim (\cos(\sqrt{2}\xi t)\widehat{g}(0,\xi))^{\vee} \\ &\quad - \frac{i}{2}\operatorname{sgn}(x) \star (\sqrt{2}\sin(\sqrt{2}\xi t)\widehat{f}(0,\xi))^{\vee} \\ &\sim (\cos(\sqrt{2}\xi t)\widehat{g}(0,\xi))^{\vee} \\ &\quad - \frac{\sqrt{2}}{4}\operatorname{sgn}(x) \star ((e^{\sqrt{2}i\xi t} - e^{-\sqrt{2}i\xi t})\widehat{f}(0,\xi))^{\vee} \end{aligned}$$

The second term on the RHS indicates the existence of the "spreading shelf" of length $O(t)$ at the linear level.

$$\|g(t,x)\|_{L^2_x} = \|\widehat{g}(t,\xi)\|_{L^2_\xi} \gtrsim |t|^{1/2}.$$

Thank you!