Scattering of Schrödinger Operators with Step Potentials

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NLS with a potential

We consider the cubic Schrödinger equation with a potential

\[ i \partial_t u = -\partial_x^2 u + Vu + |u|^2 u. \]

Denote

\[ H = -\partial_x^2 + V. \]

It is of great interest to study the global-in-time bounds and long-time asymptotic behaviors for small initial data.
Previous works

- V. S. Buslaev, G. S. Perelman, Scattering for the nonlinear Schrödinger equation: states close to a soliton, 1993
- J. Krieger and W. Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, 2006
- P. Germain, F. Pusateri and F. Rousset, The nonlinear Schrödinger equation with a potential in dimension 1, 2018
- G. Chen and F. Pusateri, The 1D NLS with a weighted $L^1$ potential, 2019; The 1D cubic NLS with a non-generic potential, 2022
Theorem (P. Germain, F. Pusateri and F. Rousset, 2018)

Assume that $V$ is decaying at a super-polynomial rate when $x \to \pm \infty$, $H$ has no endpoint resonances at 0, there exists $\epsilon_0 > 0$ such that, if $u_0$ satisfies $\|u_0\|_{H^3_x} + \|\tilde{u}_0\|_{H^1_\lambda} = \epsilon < \epsilon_0$, then a global solution exists, which decays pointwisely:

$$\|u(t)\|_{L^\infty_x} \lesssim \epsilon_0(1 + |t|)^{-1/2}$$

and is globally bounded in $L^2$ type spaces: for any time $t$,

$$\|\tilde{f}\|_{L^\infty_t([0,T];L^\infty_x)} + \|t^{-\alpha}\tilde{f}\|_{L^\infty_t([0,T];H^1_\lambda)} \lesssim \epsilon$$

The approach in this paper is based on the use of the distorted Fourier transform (Weyl-Kodaira-Titchmarsh theory), which will allow the application of some Fourier analytical techniques to nonlinear equations which involve external potentials.
Step potential: Scattering theory

We aim to extend this analysis to step-type potentials

\[ V(x) \to a_- \text{ as } x \to -\infty, \quad V(x) \to a_+ \text{ as } x \to +\infty \]

with \( a_\pm \) finite but \( a_- \neq a_+ \).
1D cubic NLS with step potential

Consider

\[ i \partial_t u = -\partial_x^2 u + Vu + \mathcal{N}(u) \]

where

\[ \mathcal{N}(u) = |u|^2 u \quad \text{and} \quad V = 1_+(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \]

That is, the 1D NLS with cubic nonlinearity and Heaviside potential.
Step potential: Scattering theory

We start by considering the standard Schrödinger operator

\[ H = -\partial_x^2 + V \]

as well as the equation

\[ Hf = Ef \]
Step potential: Scattering theory

The spectrum $\sigma(H)$ and resolvent set $\rho(H)$ are

$$\sigma(H) = [0, +\infty), \quad \rho(H) = \mathbb{C}\setminus\sigma(H)$$

To express the kernel of the resolvent operator $(H - E)^{-1}$ for $E \in \rho(H)$, we introduce the convention

- $\sqrt{E}$ in the upper half-plane
- $\sqrt{1 - E}$ in the right half-plane
Step potential: Scattering theory

For $E \in \rho(H)$, consider $e_{\pm}(x, E)$ solving

$$He_{\pm} = Ee_{\pm}$$

given by

$$e_{+}(x; E) = \begin{cases} 
  e^{i\sqrt{E} x} + R_{+}(E)e^{-i\sqrt{E} x} & \text{if } x < 0 \\
  T_{+}(E)e^{-\sqrt{1-E} x} & \text{if } x \geq 0
\end{cases}$$

where

$$T_{+}(E) = \frac{2\sqrt{E}}{\sqrt{E} + i\sqrt{1-E}}, \quad R_{+}(E) = \frac{\sqrt{E} - i\sqrt{1-E}}{\sqrt{E} + i\sqrt{1-E}}$$

and

$$e_{-}(x; E) = \begin{cases} 
  T_{-}(E)e^{-i\sqrt{E} x} & \text{if } x \leq 0 \\
  e^{\sqrt{1-E} x} + R_{-}(E)e^{-\sqrt{1-E} x} & \text{if } x > 0
\end{cases}$$

where

$$T_{-}(E) = \frac{2i\sqrt{1-E}}{\sqrt{E} + i\sqrt{1-E}}, \quad R_{-}(E) = \frac{-\sqrt{E} + i\sqrt{1-E}}{\sqrt{E} + i\sqrt{1-E}}$$
Step potential: Scattering theory

These satisfy, for $E \in \rho(H)$

$$e_{\pm}(x, E) \text{ exponentially decays as } x \to \pm \infty$$

Note that $T_+(E) \neq T_-(E)$, in fact

$$i\sqrt{E} T_-(E) = -\sqrt{1 - E} T_+(E)$$

so that

$$T_-(1) = 0, \quad T_+(0) = 0$$

Also the Wronskian $W(E)$ satisfies

$$\frac{W(E)}{T_-(E) T_+(E)} = -\sqrt{1 - E} + i\sqrt{E}$$

Since

$$\frac{W(E)}{T_-(E) T_+(E)} \neq 0 \quad \text{at} \quad E = 0, 1$$

there are no endpoint or embedded resonances. This is also called the generic case.
Step potential: Scattering theory

For $E \in \rho(H)$, the kernel of the resolvent $(H - E)^{-1}$ is

$$(H - E)^{-1}(x, y) = \frac{1}{W(E)} \begin{cases} e_-(x, E)e_+(y, E) & \text{if } x < y \\ e_-(y, E)e_+(x, E) & \text{if } y < x \end{cases}$$

By Stone’s formula

$$\frac{1}{2\pi i} \oint (H - E)^{-1}(x, y) \, dE = \delta(x - y)$$

where integration is around the contour
Step potential: Scattering theory

This gives

$$\frac{1}{2\pi i} \int_{E=0}^{+\infty} [(H-(E-i0))^{-1}(x, y)-(H-(E+i0))^{-1}(x, y)] \, dE = \delta(x-y)$$

With

$$E = k^2 = \mu^2 + 1$$

this jump in the resolvent kernel can be re-expressed to give

$$\delta(x-y) = \frac{1}{2\pi} \int_{k=0}^{+\infty} e_+(x, k)\overline{e_+(y, k)} \, dk + \frac{1}{2\pi} \int_{\mu=0}^{+\infty} e_-(x, \mu)\overline{e_-(y, \mu)} \, d\mu$$
Step potential: Distorted Fourier transform

This gives two branches of the distorted Fourier transform associated to the operator $H$:

$$(\mathcal{F}_+ f)(k) = \tilde{f}_+(k) = \frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{+\infty} e_+(y, k) f(y) \, dy$$

$$(\mathcal{F}_- f)(\mu) = \tilde{f}_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{+\infty} e_-(y, \mu) f(y) \, dy$$

$$\mathcal{F} f = \tilde{f} = (\tilde{f}_+(k), \tilde{f}_-(\mu))$$

The inverse is

$$(\mathcal{F}^{-1} g)(x) = \frac{1}{\sqrt{2\pi}} \left[ \int_{k=0}^{+\infty} e_+(x, k) g_+(k) \, dk + \int_{\mu=0}^{+\infty} e_-(x, \mu) g_-(\mu) \, d\mu \right]$$

for $g = (g_+(k), g_-(\mu))$.

The distorted Fourier transform diagonalizes $H$:

$$H = -\partial_x^2 + V = \mathcal{F}^{-1} \begin{pmatrix} k^2 & 0 \\ 0 & \mu_2 + 1 \end{pmatrix} \mathcal{F}. $$
Step potential: Scattering theory

Using the conversions

$$\mu = \begin{cases} \begin{align*}
    i\sqrt{1-k^2} & \quad \text{if } 0 < k < 1 \\
    \sqrt{k^2 - 1} & \quad \text{if } k > 1
\end{align*} \end{cases}, \quad k = \sqrt{1+\mu^2}$$

We can express, for $k \geq 0$

$$e_+(x, k) = \begin{cases} \begin{align*}
    e^{ikx} + R_+(k)e^{-ikx} & \quad \text{if } x < 0 \\
    T_+(k)e^{i\mu x} & \quad \text{if } x > 0
\end{align*} \end{cases}$$

where

$$T_+(k) = \frac{2k}{k+\mu}, \quad R_+(k) = \frac{k-\mu}{k+\mu}$$

and for $\mu \geq 0$

$$e_-(x, \mu) = \begin{cases} \begin{align*}
    T_-(\mu)e^{-ikx} & \quad \text{if } x < 0 \\
    e^{-i\mu x} + R_-(\mu)e^{i\mu x} & \quad \text{if } x > 0
\end{align*} \end{cases}$$

where

$$T_-(\mu) = \frac{2\mu}{k+\mu}, \quad R_-(\mu) = \frac{-k+\mu}{k+\mu}$$
For later reference, we can rewrite these into free and outgoing (but endpoint vanishing) pieces

\[
e_+(x, k) = (e^{ikx} - e^{-ikx})1_-(x) \quad \leftarrow \text{free}
\]
\[
+ T_+(k)(e^{-ikx}1_-(x) + e^{i\mu x}1_+(x)) \quad \leftarrow \text{outgoing}
\]
\[
e_-(x, k) = (e^{-i\mu x} - e^{i\mu x})1_+(x) \quad \leftarrow \text{free}
\]
\[
+ T_-(\mu)(e^{-ikx}1_-(x) + e^{i\mu x}1_+(x)) \quad \leftarrow \text{outgoing}
\]
Step potential: Decay estimate

Lemma (Pointwise decay)

*Under our assumptions on the potential, for* \( t \geq 1 \),

\[
\| e^{itH} h \|_{L^\infty} \lesssim \frac{1}{|t|^{1/2}} \| \tilde{h}_+ \|_{L^\infty} + \frac{1}{|t|^{1/2}} \| \tilde{h}_- \|_{L^\infty} \\
+ \frac{1}{|t|^{3/4}} \| \partial_k \tilde{h}_+ \|_{L^2} + \frac{1}{|t|^{3/4}} \| \partial_\mu \tilde{h}_- \|_{L^2}
\]

Lemma (Enhanced local decay)

*Under our assumptions on the potential, for* \( t \geq 1 \),

\[
\| \langle x \rangle^{-1} e^{itH} h \|_{L^\infty} \lesssim \frac{1}{|t|} \| \tilde{h}_+ \|_{H^1} + \frac{1}{|t|} \| \tilde{h}_- \|_{H^1}.
\]
Step potential: Cubic interaction

Recall the cubic NLS

\[ i \partial_t u = Hu + \mathcal{N}(u), \quad \mathcal{N}(u) = |u|^2 u \]

becomes, with \( f = e^{itH}u \),

\[ i \partial_t f = e^{itH} \mathcal{N}(e^{-itH} f) \]

Taking the distorted Fourier transform,

\[ i \partial_t \tilde{f}_+(k) = e^{itk^2} \mathcal{F}_+ [ \mathcal{N}(e^{-itH} f)](k) \]

\[ i \partial_t \tilde{f}_-(\mu) = e^{it(\mu^2 + 1)} \mathcal{F}_- [ \mathcal{N}(e^{-itH} f)](\mu) \]

Into the nonlinearity \( \mathcal{N}(\bullet) \) substitute

\[ [e^{-itH} f(\bullet, t)](x) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-itk^2} e_+(x, k) \tilde{f}_+(k) \, dk \]

\[ + \frac{1}{2\pi} \int_{0}^{\infty} e^{-it(\mu^2 + 1)} e_-(x, \mu) \tilde{f}_-(\mu) \, d\mu \]
Step potential: Cubic interaction

This gives the following representation for the solution, where we replace \((k, \mu)\) with \((\ell, \nu)\), \((m, \xi)\), and \((n, \sigma)\)

\[
\begin{align*}
\left( i \partial_t \tilde{f}_+(k) \right) & = \int \int \int \int_y \left( \begin{array}{c}
\frac{e^{ikt^2}}{e_+(y, k)}
\frac{e^{it(\mu^2+1)}}{e_-(y, \mu)}
\frac{e^{itm^2}}{e_+(y, m)}
\frac{e^{it(\xi^2+1)}}{e_-(y, \xi)}
\end{array} \right) \left( \begin{array}{c}
e^{-it\ell^2} e_+(y, \ell) \tilde{f}_+(\ell)
e^{-it(\nu^2+1)} e_-(y, \nu) \tilde{f}_-(\nu)
e^{-itn^2} e_+(y, n) \tilde{f}_+(n)
e^{-it(\sigma^2+1)} e_-(y, \sigma) \tilde{f}_-(\sigma)
\end{array} \right) dy \\
\left( i \partial_t \tilde{f}_-(\mu) \right) & = \int \int \int \int_y \left( \begin{array}{c}
\frac{e^{ikt^2}}{e_+(y, k)}
\frac{e^{it(\mu^2+1)}}{e_-(y, \mu)}
\frac{e^{itm^2}}{e_+(y, m)}
\frac{e^{it(\xi^2+1)}}{e_-(y, \xi)}
\end{array} \right) \left( \begin{array}{c}
e^{-it\ell^2} e_+(y, \ell) \tilde{f}_+(\ell)
e^{-it(\nu^2+1)} e_-(y, \nu) \tilde{f}_-(\nu)
e^{-itn^2} e_+(y, n) \tilde{f}_+(n)
e^{-it(\sigma^2+1)} e_-(y, \sigma) \tilde{f}_-(\sigma)
\end{array} \right) dy 
\end{align*}
\]

which actually represents 16 different terms, top/bottom in each of four places and accordingly, the triple integral is over \(\ell\) or \(\nu\), \(m\) or \(\xi\), \(n\) or \(\sigma\).
Step potential: Cubic interaction

Thus we obtain

\[ i \partial_t \tilde{f}_+(k) = \int \int \int e^{it(k^2 - \ell^2 + m^2 - n^2)} \mu_{+++-+}(k, \ell, m, n) \]
\[ \cdot \tilde{f}_+(\ell) \tilde{f}_+(m) \tilde{f}_+(n) \, d\ell \, dm \, dn \]
\[ + e^{-it} \int \int \int e^{it(k^2 - \nu^2 + m^2 - n^2)} \mu_{+++-+}(k, \nu, m, n) \]
\[ \cdot \tilde{f}_-(\nu) \tilde{f}_+(m) \tilde{f}_+(n) \, d\nu \, dm \, dn \]
\[ + \cdots \text{(6 more)} \]

where

\[ \mu_{+++-+}(k, \ell, m, n) = \int_{y=-\infty}^{+\infty} \overbrace{e_+(y, k)e_+(y, \ell)e_+(y, m)e_+(y, n)} \, dy \]

and similarly, for the other 15 terms.
Step potential: Cubic interaction

Using that

$$\hat{1}_\pm(\xi) = \pm \frac{1}{i\xi} + \pi\delta(\xi)$$

we can calculate each term. For example, when we split each $e_+$ into free and outgoing parts

$$e_+ = e_{+,F} + e_{+,O}, \quad e_{+,F}(y, k) = (e^{iyk} - e^{-iyk})1_-(y)$$

we obtain for the $e_{+,F}$ only part

$$\sum_{\alpha, \beta, \gamma, \epsilon \in \{-1, 1\}} \alpha \beta \gamma \epsilon \int_{-\infty}^{0} e^{-\alpha iky} e^{\beta ily} e^{-\gamma imy} e^{\epsilon iny} \, dy$$

$$= \sum_{\alpha, \beta, \gamma, \epsilon \in \{-1, 1\}} \alpha \beta \gamma \epsilon \left( \frac{1}{(\alpha k + \beta \ell - \gamma m + \epsilon n)i} + \pi\delta(\alpha k + \beta \ell - \gamma m + \epsilon n) \right)$$
Step potential: Cubic interaction

Since \((\alpha, \beta, \gamma, \epsilon) \to (-\alpha, -\beta, -\gamma, -\epsilon)\) keeps the sign of \(\alpha \beta \gamma \epsilon\) unchanged by flips the sign of the pv term,

\[
\pi \sum_{\alpha,\beta,\gamma,\epsilon \in \{-1,1\}} \alpha \beta \gamma \epsilon \delta(\alpha k + \beta \ell - \gamma m + \epsilon n)
\]

so that this term behaves like a product of four free (no potential) linear Schrödinger solutions.
1D cubic NLS with step potential

The bootstrap space is:

\[ \| f \|_X = \| \tilde{f}_+(k) \|_{L_{k>0}^\infty} + \| \tilde{f}_-(\mu) \|_{L_{\mu>0}^\infty} \]

\[ + \langle t \rangle^{-1/4+} \| |1 - k|^0+ \partial_k \tilde{f}_+(k) \|_{L_{k>0}^2} + \langle t \rangle^{-1/4+} \| \partial_\mu \tilde{f}_-(\mu) \|_{L_{\mu>0}^2} \]

\[ + \langle t \rangle^0- \| \langle k \rangle^3 \tilde{f}_+(k) \|_{L_{k>0}^2} + \langle t \rangle^0- \| \langle \mu \rangle^3 \tilde{f}_-(\mu) \|_{L_{\mu>0}^2} \]

The extra factor \(|1 - k|^0+\) is needed to deal with

\[ \mu = \begin{cases} i\sqrt{1 - k^2} & \text{if } k < 1 \\ \sqrt{k^2 - 1} & \text{if } k > 1 \end{cases} \]
1D cubic NLS with step potential

Theorem (J. Holmer and Z. Z., ongoing work)

Consider the 1D cubic Schrödinger equation with the Heaviside step potential $V(x)$ described above:

$$i\partial_t u = -\partial_x^2 u + Vu + |u|^2 u.$$ 

We have:

There exists $0 < \epsilon_0 \ll 1$ such that for all $\epsilon \leq \epsilon_0$ and $u_0$ with

$$\|u_0\|_{H^3} + \|xu_0\|_{L^2} \leq \epsilon,$$

then a global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ exists, with $u(0, x) = u_0(x)$; this solution satisfies the sharp decay rate

$$\|u(t)\|_{L^\infty} \lesssim \epsilon (1 + |t|)^{-1/2},$$

$$\|f\|_X \lesssim \epsilon.$$
Consider the 1D Gross-Pitaevskii equation

$$i \partial_t u + \partial_x^2 u + (1 - |u|^2)u = 0.$$ 

The black/dark solitons of the 1D GP equation are solitary wave with non-zero asymptotics when $|x| \to \infty$, which brings the challenge of non-decaying property of the linearized operator as $|x| \to \infty$:

$$U_c(x) := \frac{\sqrt{2 - c^2}}{2} \tanh(\frac{\sqrt{2 - c^2}}{2} x) + i \frac{c}{\sqrt{2}}.$$ 

It is called a grey soliton when $c \neq 0$, and a black soliton when $c = 0$. 
Black/dark solitons of the 1D Gross-Pitaevskii equation

- L. Di Menza and C. Gallo (2007), Linear stability of black solitons
- F. Béthuel, P. Gravejat, J.-C. Saut, and D. Smets (2008), Orbital stability of the black soliton
- F. Béthuel, P. Gravejat, and D. Smets (2015), Asymptotic stability in the energy space for dark solitons using hydrodynamics formulation, does not apply to black solitons
- P. Gravejat and D. Smets (2015), Asymptotic stability of the dark/black soliton, weak convergence in time in $H^1$
Black/dark solitons of the 1D Gross-Pitaevskii equation

Let $u = U_c + w$. Write

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} w \\ \overline{w} \end{pmatrix},$$

and denote the vector Schrödinger operator

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{pmatrix} \partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix} + \begin{pmatrix} V_+ & V_- \\ -V_- & -V_+ \end{pmatrix},$$

where

$$V_+ = 1 - 2|U_c|^2, \quad V_- = -U_c^2.$$

The linearized Gross-Pitaevskii equation around the dark/black soliton can be written as

$$i\partial_t W + \mathcal{H} W = \{\text{nonlinear}\}.$$

Compared to the linearization around the bright soliton, this linearized (matrix) operator $\mathcal{H}$ has non-zero and step-like asymptotes at $\pm \infty$. 
It is expected that the perturbation of size $O(\varepsilon)$ will evolve as a "spreading shelf" of length $O(t)$. This is indicated in numerical simulations in Ablowitz-Nixon-Horikis-Frantzeskakis (2011), where the perturbations of dark solitons under additional small forcing is observed via numerics. Hence we need to consider some $L^2$-based functional space that is truncated to be supported away from zero.
Black/dark solitons of the 1D Gross-Pitaevskii equation
Black/dark solitons of the 1D Gross-Pitaevskii equation

A linear toy model of linearizing around $\Psi(x) \equiv 1$: Taking $u(t, x) = \Psi(x) + w(t, x) = 1 + w(t, x)$, and writing $w(t, x) = f(t, x) + ig(t, x)$, we obtain the following linear part:

$$
\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 - 2 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.
$$
Taking the (flat) Fourier transform, we can see

\[ \partial_t^2 \hat{f} = -\xi^2 (\xi^2 + 2) \hat{f}, \]
\[ \partial_t^2 \hat{g} = -\xi^2 (\xi^2 + 2) \hat{g}. \]

In low frequency, the linear model has a wave-like behavior. We can solve out

\[ \hat{f}(t, \xi) = \cos(\xi \sqrt{\xi^2 + 2t}) \hat{f}(0, \xi) + \frac{\xi \sin(\xi \sqrt{\xi^2 + 2t})}{\sqrt{\xi^2 + 2}} \hat{g}(0, \xi), \]
\[ \hat{g}(t, \xi) = \cos(\xi \sqrt{\xi^2 + 2t}) \hat{g}(0, \xi) - \frac{\sqrt{\xi^2 + 2} \sin(\xi \sqrt{\xi^2 + 2t})}{\xi} \hat{f}(0, \xi). \]
When $|\xi| \ll |t|^{-1/2}$,

$$g(t, x) \sim (\cos(\sqrt{2}\xi t)\hat{g}(0, \xi))^{-1} \pm \frac{i}{2} \text{sgn}(x) \ast (\sqrt{2} \sin(\sqrt{2}\xi t)\hat{f}(0, \xi))^{-1}$$

$$\sim (\cos(\sqrt{2}\xi t)\hat{g}(0, \xi))^{-1} \pm \frac{\sqrt{2}}{4} \text{sgn}(x) \ast ((e^{\sqrt{2}i\xi t} - e^{-\sqrt{2}i\xi t})\hat{f}(0, \xi))^{-1}$$

The second term on the RHS indicates the existence of the "spreading shelf" of length $O(t)$ at the linear level.

$$\|g(t, x)\|_{L_x^2} = \|\hat{g}(t, \xi)\|_{L_\xi^2} \gtrsim |t|^{1/2}.$$
Thank you!