Two-solitons with logarithmic separation for 1D NLS with repulsive delta potential

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Two-solitons with logarithmic separation

Consider (NLS_γ) $\boxed{iu_t + u_{xx} - \gamma\delta u + |u|^{p-1}u = 0}$ $\gamma \geq 0$, $p > 2, p \neq 5$

- note (NLS_γ) is well-posed in $H^1(\mathbb{R})$.

First consider $\gamma = 0$. For (NLS_0), we have:

- *soliton* solitons: $u(t,x) = e^{it}Q(x)$, $Q(x) \sim c_p e^{-|x|}$ as $|x| \to \infty$
- can *boost*: $u(t,x) = e^{i(vx+\theta(t))}Q(x - vt)$ (also translate, scale, phase rotate)
- *multi-soliton* solutions ([Martel-Merle 06]; [Côte-Martel-Merle 11]): e.g.,

  $\| u(t,\cdot) - e^{i\theta}\left[Q(\cdot - \tfrac{z}{2}) + Q(\cdot + \tfrac{z}{2})\right] \|_{H^1} \to 0$, $z(t) \sim ct$ (as $t \to \infty$)
- [Nguyen 19]: (NLS_0) also admits such a solution with $\boxed{z(t) \sim 2\log t}$

  – the non-free asymptotic motion arises from nonlinear (attractive) interaction of the solitons through their tails

  – this construction also works in higher dimensions

  – such a solution was previously known in the integrable case $p = 3$ (*double-pole solution*)

Two-solitons with logarithmic separation

Consider (NLS_γ) $\boxed{iu_t + u_{xx} - \gamma\delta u + |u|^{p-1}u = 0}$ $\gamma \geq 0$, $p > 2, p \neq 5$

Thm [G-Inui]:
If $\gamma < \frac{3}{2}$, (NLS_γ) admits a solution satisfying (as $t \to \infty$)

- $\|u(t,\cdot) - e^{i\theta}\left[Q(\cdot - \frac{z}{2}) + Q(\cdot + \frac{z}{2})\right]\|_{H^1} \lesssim \frac{1}{t}$
- $|z(t) - 2\log t| \lesssim 1$

If $\gamma > 2$, (NLS_γ) admits no such solution.

Remarks:

- *feature*: the (repulsive) potential-soliton interaction is the same size as the (attractive) soliton-soliton interaction
- we expect such solutions should exist for all $\gamma < 2$
- *thresholds*: the "action" $S = E + M$ of such solution is $S(u) = 2S(Q)$. If $\gamma < 2$, (NLS_γ) has a (even) ground state Q_γ with $S(Q_\gamma) < 2S(Q)$
- our construction builds on [Nguyen 19] – here I will try to explain how the additional potential-soliton interaction enters the picture
- take $p < 5$ to avoid complication arising from soliton instability

Approximate solution

- 4 parameters:
  - scale $\lambda(s) \approx 1$: rescale $y = \frac{x}{\lambda}$, $\frac{ds}{dt} = \frac{1}{\lambda^2}$, $\Lambda = y\partial_y + \frac{2}{p-1}$
  - phase $\theta(s) \approx s$: with $u(t,x) = \lambda^{-\frac{2}{p-1}} e^{i\theta} w(s,y)$, (NLS_γ) is

$$\boxed{i\partial_s w + \partial_y^2 w - \lambda\gamma\delta w + |w|^{p-1}w - i\frac{\dot\lambda}{\lambda}\Lambda w + (1-\dot\theta)w = 0}$$

  - position $z(s) \gg 1$
  - velocity $|v(s)| \ll 1$
- approximate solution: $P = P(y; z(s), v(s)) = \chi\,[P_1 + P_2]$

$$= \chi(|y|)\left[e^{i\frac{v}{2}(y-\frac{z}{2})}Q(y-\tfrac{z}{2}) + e^{-i\frac{v}{2}(y+\frac{z}{2})}Q(y+\tfrac{z}{2})\right]$$

  where χ cuts off a neighbourhood of $y = 0$
- its error: $\mathcal{E}_P = i\partial_s P + \partial_y^2 P - \cancel{\lambda\gamma\delta P} + |P|^{p-1}P - i\frac{\dot\lambda}{\lambda}\Lambda P + (1-\dot\theta)P$

$$= e^{i\frac{v}{2}(y-\frac{z}{2})}\vec{m}\cdot\vec{M}Q(y-\tfrac{z}{2}) + \text{ reflection} \qquad \vec{m} \approx \begin{bmatrix} \dot\lambda/\lambda \\ \dot z - 2v \\ \dot\theta - 1 \\ \dot v \end{bmatrix}, \vec{M} = -\begin{bmatrix} i\Lambda \\ i\partial_y \\ 1 \\ y \end{bmatrix}$$

$$+|P|^{p-1}P - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2 \qquad \text{(nonlinear interaction)}$$

$$+ \text{ error terms from cut-off } \chi$$

Construction of true solution

- solve (NLS_γ) backwards from time $s = s_f$, with final data

  $w(s_f, y) = P(y; z(s_f), v(s_f))$,

  $\lambda(s_f) = 1, \quad \theta(s_f) = 0,$ with $z(s_f), v(s_f)$ to be chosen

- express solution as $\boxed{w(s,y) = P(y; z(s), v(s)) + \xi(y,s)}$

  with parameters $z(s), v(s), \lambda(s), \theta(s)$ to be chosen

- goal: establish uniform estimates

  $$\boxed{\|\xi\|_{H^1} \lesssim \frac{1}{s} \qquad |z - 2\log s| \lesssim 1 \qquad |v(s)| \lesssim \frac{1}{s}}$$

  for $s \in [s_0, s_f]$, s_0 fixed, $s_f \to \infty$

- then the existence of the desired solution follows from a compactness argument (à la [Martel-Merle 06], [Nguyen 19])

Parameter modulation

- express the PDE for the remainder term in the frame of a soliton: with $\xi(y,s) = w(y,s) - P(y; z(s), v(s)) = e^{i\frac{v}{2}(y-\frac{z}{2})}\eta(s, y-\frac{z}{2})$,

(NLS_γ) reads $\boxed{i\partial_s\eta - L_z^+\eta + NL(\eta) + \vec{m}\cdot\vec{M}\eta + \tilde{\mathcal{E}}_P + (1-\lambda)\gamma\delta_{-\frac{z}{2}}\eta = 0}$

with **linearized operator** $L_z^+ = -\partial_y^2 + 1 - pQ^{p-1}(y) + \gamma\delta_{-\frac{z}{2}}$

and the approximate solution error written as $\mathcal{E}_P = e^{i\frac{v}{2}(y-\frac{z}{2})}\tilde{\mathcal{E}}_P(s, y-\frac{z}{2})$

- dynamics of 3 parameters $\lambda(s), \theta(s), z(s)$ (but not yet $v(s)$!) are determined by imposing 3 **orthogonality conditions**:

$$\boxed{0 \equiv \langle\eta, Q\rangle \equiv \langle\eta, yQ\rangle \equiv \langle\eta, i\Lambda Q\rangle}$$

- **coercivity**: $\|\xi\|_{H^1}^2 \sim \|\eta\|_{H^1}^2 \lesssim$ "almost conserved" linearized energy + localized momentum $\lesssim \frac{1}{s^2}$

- **ODE estimates** from computing $\langle\partial_t\eta,\ \{Q, yQ, i\Lambda Q\}\rangle$:

$$\left|\frac{\dot\lambda}{\lambda}\right| + \left|\dot\theta - 1\right| \lesssim \frac{\log s}{s^2}, \qquad |\dot z - 2v| \lesssim \frac{\log s}{s^2} + |\langle\eta,\ iQ'\rangle|$$

- so we must still somehow control $|\langle\eta,\ iQ'\rangle|$. Need $|\langle\eta,\ iQ'\rangle| \lesssim \frac{1}{s\log s}$

Perturbed eigenfunction

- problem is to control $|\langle \eta, \; iQ' \rangle|$
- observation: Q' lies in the nullspace of the free linearized operator:
  $L^+ Q' = 0, \quad L^+ = -\partial_y^2 + 1 - pQ^{p-1}(y)$
- our *actual* linearized operator is perturbed by the potential:
  $$\boxed{L_z^+ = L^+ + \gamma \delta_{-\frac{z}{2}}}$$
- idea: **replace Q' with the perturbed eigenfunction** $\boxed{L_z^+ T_z = \nu_z T_z}$
- we need quite detailed properties:

- rough sizes:
  $0 < \nu_z \lesssim e^{-z} \qquad \|T_z - Q'\|_{H^1} \lesssim e^{-\frac{z}{2}} \qquad |T_z(-\tfrac{z}{2})| \lesssim e^{-\frac{z}{2}}$
- refined upper and lower bounds:
  $$\frac{2}{\gamma}\left[1 + \gamma - \sqrt{1+2\gamma}\right](1 - o(1)) \le \frac{\|Q'\|_2^2}{c_p^2 e^{-z}} \nu_z \le \frac{2\gamma}{\gamma+2}(1 + o(1))$$
  $$\frac{1}{\gamma}\left[1 + \gamma - \sqrt{1+2\gamma}\right](1 - o(1)) \le \left[1 - \frac{T_z(-\frac{z}{2})}{c_p e^{-\frac{z}{2}}}\right] \le \frac{\gamma}{\gamma+2}(1 + o(1))$$

– obtained by variational and ODE methods

Estimating $\langle \eta,\ iT_z \rangle$

- so we replace $\langle \eta,\ iQ' \rangle \mapsto \langle \eta,\ iT_z \rangle$; now we must control the latter
- using PDE $\quad i\partial_s \eta - L_z^+ \eta + NL(\eta) + \vec{m} \cdot \vec{M}\eta + \tilde{\mathcal{E}}_P + (1-\lambda)\gamma\delta_{-\frac{z}{2}}\eta = 0$,

$$\begin{aligned}\frac{d}{ds}\langle \eta,\ iT_z \rangle = & -\langle L_z^+ \eta,\ T_z \rangle \\ & + \tfrac{1}{2}\left[\langle \overline{\eta} \cdot F''(\tilde{P})\eta,\ Q' \rangle + (\dot{z} - 2v)\langle iQ',\ \partial_y \eta \rangle\right] \\ & + \langle \tilde{\mathcal{E}}_P,\ T_z \rangle \quad + O\left(\tfrac{1}{s^2 \log s}\right)\end{aligned}$$

- $\langle L_z^+ \eta,\ T_z \rangle = \langle \eta,\ L_z^+ T_z \rangle = \langle \eta,\ \nu_z T_z \rangle = O(\nu_z \|\eta\|_{H^1}) = O(e^{-z}\frac{1}{s}) = O(\frac{1}{s^3})$
  – this is exactly the reason for replacing $\langle \eta,\ iQ' \rangle \mapsto \langle \eta,\ iT_z \rangle$
- $\frac{1}{2}\left[\langle \overline{\eta} \cdot F''(\tilde{P})\eta,\ Q' \rangle + (\dot{z} - 2v)\langle iQ',\ \partial_y \eta \rangle\right]$
  $\approx \frac{d}{ds} \operatorname{Im} \int_{\mathbb{R}} \overline{\eta}(s,y)\partial_y \eta(s,y)(1-\chi)\left(\frac{|y|}{\log s}\right) dy = \frac{d}{ds} O(\frac{1}{s^2})$
- we try to **impose** $|\langle \tilde{\mathcal{E}}_P,\ T_z \rangle| \lesssim \frac{1}{s^3}$
- then time integration yields the desired estimate $|\langle \eta,\ iT_z \rangle| \lesssim \frac{1}{s \log s}$

Emergence of the soliton motion law

- so we want $|\langle \mathcal{E}_P,\ e^{i\frac{v}{2}(\cdot - \frac{z}{2})} T_z(\cdot - \frac{z}{2})\rangle| \lesssim \frac{1}{s^3}$
- recall: the approximate solution error is

$$\begin{aligned}\mathcal{E}_P \approx\ & e^{i\frac{v}{2}(y-\frac{z}{2})}\vec{m}\cdot\vec{M}Q(y-\tfrac{z}{2}) \\ & +|P|^{p-1}P - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2 \\ & +(2\partial_y\chi\partial_y + \partial_y^2\chi_\rho)(P_1+P_2)\end{aligned}$$

- **key computation**:

$$\langle \mathcal{E}_P,\ e^{-i\frac{v}{2}(\cdot+\frac{z}{2})}T_z(\cdot+\tfrac{z}{2})\rangle = C_1\left[\dot{v} + f(z)e^{-z}\right] + O(\tfrac{1}{s^3})$$

where $\boxed{f(z) = C_2\left[1 - \gamma\frac{T_z(-\frac{z}{2})}{c_p e^{-z}}\right]}$

captures the competing soliton-soliton and potential-soliton interactions

- for the desired estimate, impose **motion law** $\boxed{\dot{v}(s) = -f(z(s))e^{-z(s)}}$
- using our refined eigenfunction estimate:

$$\boxed{\begin{array}{llll}\gamma < \frac{3}{2} & \Longrightarrow & f(z) > 0 & \text{(net attraction)} \\ \gamma > 2 & \Longrightarrow & f(z) < 0 & \text{(net repulsion)}\end{array}}$$

Final step: land on the correct trajectory

- suppose $\gamma < \frac{3}{2}$, so $f(z) > 0$ (attractive)

- the final position $z(s_f)$ and velocity $v(s_f)$ must still be chosen to ensure the desired uniform estimates hold

- this argument proceeds as in [Nguyen 19]:

– our dynamical system $\left\{ \begin{array}{c} \dot{z} = 2v \\ \dot{v} = -f(z)e^{-z} \end{array} \right\}$ has conserved energy

$$E = v^2 - F(z), \qquad F(z) = \int_z^\infty f(z')e^{-z'}\,dz'$$

– our desired trajectory is the separatrix with $E = 0$

– so choose $v(s_f) = \sqrt{F(z(s_f))}$

– $z(s_f)$ is chosen by continuity argument to ensure $|z(s) - 2\log s| \lesssim 1$

Some concluding remarks

- **summary**: in this approach, the soliton-potential interaction is mediated through the eigenfunction of L^+ perturbed by the potential
- non-existence for $\gamma > 2$ (repulsive motion law) is proved using similar estimates
- $t \to -\infty$ behaviour of this solution?
- $\frac{3}{2} \leq \gamma \leq 2$? Attractive delta potential?
- other potentials and higher dimensions?