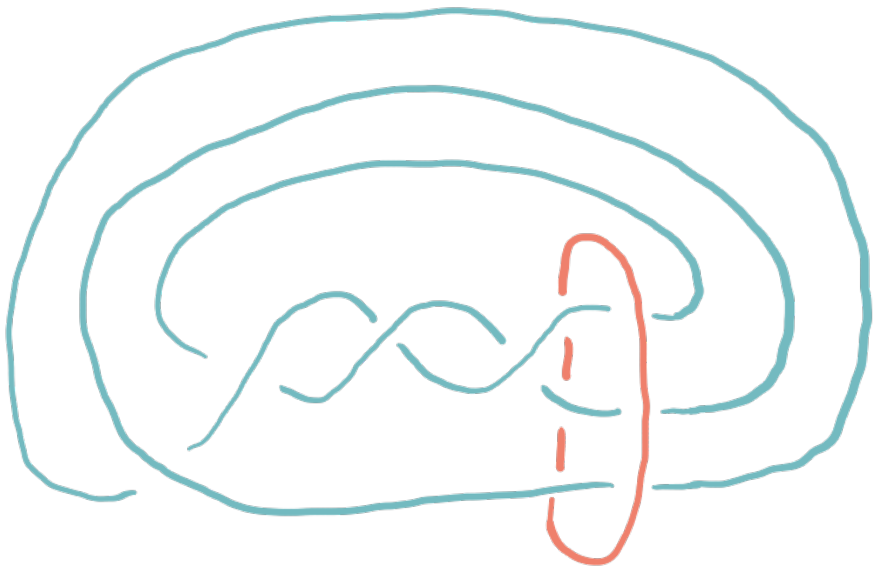
(AN ALGORITHM FOR NOT)

BI-ORDERING LINKS VIA BRAIDS

joint work in progress with J. Johnson & N. Scherich



ORDERING GROUPS

Def A left-ordering on a group is a strict total order which is preserved under left multiplication.

$(\mathbb{Z},+)$ $(\mathbb{R},+)$

Non ex: $(\mathbb{Z}/2,+)$

$0 < 1 \rightsquigarrow 1 < 2 = 0 \quad \rightarrow\leftarrow$

ORDERING GROUPS

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Def A bi-ordering on a group is a strict total order which is preserved under left and right multiplication.

ORDERING GROUPS

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BO

NOT BO

Q: Why (bi-)order 3-manifold groups?

EX $\pi_1(S^3 - L) := \pi_1(L) \twoheadrightarrow \mathbb{Z}$

(Boyer - Rolfsen - Wiest) $\Rightarrow$ $\pi_1(L)$ is LO but not necessarily BO

Notation L is BO if $\pi_1(L)$ is BO

ORDERING GROUPS

BO

NOT BO

Notation L is BO if $\pi_1(L)$ is BO

Problem: Give a (topological) characterization of biorderable knots/links.

How-to:

1. To show G is BO, find an order and check invariance under R&L multiplication.

2. To show G is not BO, show that no STO is invariant under R&L multiplication

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★ **Algorithm** If a braided link L is <u>not</u> BO our program returns a definitive "no" and a proof that L is not BO.

[based on Calegari-Dunfield algorithm for left-ordering]

★ **Thm** The braided links obtained as the closure of the braid $\sigma_1\sigma_2^{2k+1}$ together with the braid axis are <u>not</u> BO.

BRAIDED LINKS

Def A braided link is a link $L = \hat{\beta} \cup a$ where β is a braid, $\hat{\beta}$ denotes its closure in S^3 and a is the braid axis of β.

$\beta = \sigma_2 \sigma_1$

$\hat{\beta}$

$L = \hat{\beta} \cup a$

EX: $T(2,6)$ is a braided link.

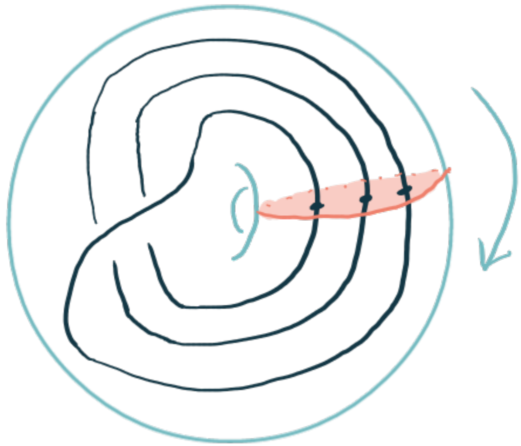
BRAIDED LINKS

Fact: Braided link complements are fibered.

★ $S^3 - a - L$ fibers over S^1 with fibers n-punctured disks

$S^3 - \nu(a) \cong$

$S^3 - \nu(a) - L \cong$



D_3

$$1 \longrightarrow \underset{\displaystyle F_n}{\underset{\cong}{\pi_1(D_n)}} \xrightarrow{i} \pi_1(L) \xrightarrow{\pi} \underset{\displaystyle \mathbb{Z}}{\underset{\cong}{\pi_1(S^1)}} \longrightarrow 1$$

BRAIDED LINKS

$$1 \longrightarrow \pi_1(D_n) \xrightarrow{i} \pi_1(L) \xrightarrow{\pi} \pi_1(S^1) \longrightarrow 1$$

$$\pi_1(D_n) \cong F_n \qquad \pi_1(S^1) \cong \mathbb{Z}$$

$\pi_1(L)$ is a semi-direct product (x, k) $\quad x \in \pi_1(D_n) \cong F_n \quad k \in \mathbb{Z}$

Q: How should we order a product?

A: Lexicographically! $\quad (x_1, k_1) < (x_2, k_2)$ if either

- $k_1 < k_2$
- OR $k_1 = k_2$ and $x_1 < x_2$

$$(x_1, k_1)(x, k) = (x_1 \beta^{k_1}(x), k_1 + k)$$
$$(x_2, k_2)(x, k) = (x_2 \beta^{k_2}(x), k_2 + k)$$

is right invariant

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is right invariant

Is this left invariant (and hence $\pi_1(L)$ is BO?)

$$(x, k)(x_1, k_1) = (x \beta^k(x_1), k + k_1)$$
$$(x, k)(x_2, k_2) = (x \beta^k(x_2), k + k_2)$$

is $\beta^k(x_1) < \beta^k(x_2)$?

enough to check:

is $\beta(x_1) < \beta(x_2)$?

if $x_1 < x_2$?

BRAIDED LINKS

Def An n-braid is order-preserving if there is a biordering of F_n which is preserved by β i.e. if $x_1 < x_2$ then $\beta(x_1) < \beta(x_2)$

Thm (Kin-Rolfsen) A braided link $L = \hat{\beta} \cup a$ is biorderable iff β is order-preserving.

POSITIVE CONES

Def A <u>positive cone</u> for F_n is a subset $P \subset F_n$ s.t.:

LO:

① $P \sqcup P^{-1} \sqcup \{1\} = F_n$ $\Rightarrow$ left-inv STO $\quad f < g$ iff $\quad f^{-1}g \in P$

② $P \cdot P \subset P$ $\Rightarrow$ transitivity $\quad 1 < f^{-1}g$

BO:

A positive cone is <u>conjugation invariant</u> if additionally:

③ $gPg^{-1} \subset P$ for all $g \in F_n$ $\Rightarrow$ R&L invariance

OP:

P is <u>β-invariant</u> if

④ $\beta(P) \subset P$

From now on I will most likely be considering positive cones which are conjugation invariant and β-invariant

POSITIVE CONES

Def A pre-cone of length K for F_n is a subset $P_k \subset F_n$ s.t.:

$$\mathrm{OP}\begin{cases}\mathrm{BO}\begin{cases}\mathrm{LO}\begin{cases}(1)\ P_k \sqcup P_k^{-1} \sqcup \{1\} = W_k := \{\text{words of length at most } k \in F_n)\\ (2)\ (P_k \cdot P_k) \cap W_k \subset P_k\end{cases}\\ (3)\ (gP_kg^{-1}) \cap W_k \subset P \quad \text{for all } g \in P_k\end{cases}\\ (4)\ \beta(P_k) \cap W_k \subset P_k\end{cases}$$

Note: If P is a positive cone (conj- & β-invariant) then $P \cap W_k$ is a length K pre cone.

POSITIVE CONES

Note: If P is a positive cone (conj- & β-invariant) then $P \cap W_k$ is a length k pre cone.

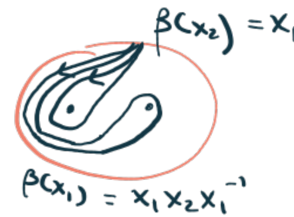
☆ **Algorithm** If a braided link L is <u>not</u> BO our program returns a definitive "no" and a proof that L is not BO.

Strategy: start building all possible k-precones which are conj- and β-invariant; find some $k \in \mathbb{Z}$ so that no such precones exist.

Prop: If a k-precone exists for all $k \in \mathbb{Z}$ then a positive cone P invariant under conj & β also exists.

THE ALGORITHM

Ex: $\beta = \sigma_1$ is not OP.



Need to show that no biorder of F_n is preserved by β

Suppose $x_1 < x_2$ (can get something similar if $x_2 < x_1$) and for contradiction that β preserves $<$. Then

$$\beta(x_1) < \beta(x_2)$$

$$\Rightarrow \underbrace{x_1 x_2 x_1^{-1}}_{=\beta(x_1)} < \underbrace{x_1}_{=\beta(x_2)}$$

$$\overset{\text{conj}}{\Rightarrow} x_2 < x_1 \quad \rightarrow\leftarrow$$

no biorder of F_n is preserved by β

THE ALGORITHM

$\beta(x_2) = x_1$
$\beta(x_1) = x_1 x_2 x_1^{-1}$

Ex: $\beta = \sigma_1$

Computer input a length K, and attemp to build all K-precones $K=3$

WLOG: $x_1 \in P_K$, now if P_K is a precone closed undr

(1) products of things already in P_K (add x_1^2) (add $x_1 x_2$)

(2) conj of things in P_K by generators (add $x_2 x_1 x_2^{-1}$) (add x_2)

(3) the image under β of things already in P_K (add $x_1 x_2 x_1^{-1}$) ...

repeat until no new words added

If the trivial element is added we've reached a contradiction

THE ALGORITHM

$\beta(x_2) = x_1$
$\beta(x_1) = x_1 x_2 x_1^{-1}$

Ex: $\beta = \sigma_1$

(1) products of things already in P_k $\quad$ (add x_1^2) $\quad$ (add $x_1 x_2$)

(2) conj of things in P_k by generators $\quad$ (add $x_2 x_1 x_2^{-1}$) $\quad$ (add x_2)

(3) the image under β of things already in P_k (add $x_1 x_2 x_1^{-1}$) $\quad$ ...

No contradictions yet...

$$x_1 \in P_k$$

two choices

$$x_1 x_2^{-1} \in P_k \qquad x_2 x_1^{-1} \in P_k$$

THE ALGORITHM

$\beta(x_2) = x_1$
$\beta(x_1) = x_1 x_2 x_1^{-1}$

Ex: $\beta = \sigma_1$

$x_1 \in P$

$x_1 x_2^{-1} \in P_k \qquad x_2 x_1^{-1} \in P_k$

two choices

Add $x_1 x_2^{-1}$ to P_k. Then apply (1)-(3) again:

$\Rightarrow x_2^{-1} x_1 \in P_k$ (conjugation)

$\Rightarrow \beta(x_2^{-1} x_1) = x_2 x_1^{-1}$ (apply β)

$\Rightarrow (x_1 x_2^{-1})(x_2 x_1^{-1}) = \mathrm{id}$ (mult.) $\rightarrow\leftarrow$

Similar if $x_2 x_1^{-1}$ instead

3 - BRAIDS

Murasugi's classification of 3-braids

Any 3-braid is conjugate to exactly one of the following:

(1) $\Delta^{2k}\sigma_1^m\sigma_2^{-1}$ with $m=-1,-2$, or -3

(2) $\Delta^{2k}\sigma_2^m$ with $m\in\mathbb{Z}$

P.A. (3) $\Delta^{2k}\sigma_1\sigma_2^{-a_1}\sigma_1\sigma_2^{-a_2}\cdots\sigma_1\sigma_2^{-a_n}$ $a_i\geq 0$ some $a_i>0$

$\sigma_1\sigma_2^{-1}$ is not OP (Kin-Rolfsen)

$\sigma_1\sigma_2^{-2}$ is OP (Cai-Clay)

$\sigma_1\sigma_2^{-3}$ is not OP (Johnson-Scherich-T) $\sigma_1\sigma_2^{-2k+1}$

$\sigma_1\sigma_2^{-4}$??

Thanks for listening!