Variation of pointed p-adic Ceresa classes

ongoing joint work with Wanlin Li

Which cycle classes are useful for detecting non-torsion Ceresa cycles?

Etale Ceresa classes

X/K a curve with a basepoint $x \in X(K)$.

descending central series

$$L = Lie\left(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x)\right)/\Gamma^3$$

$H_1^{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$

$$0 \to V_2 \to L \to V_1 \to 0$$

$\Lambda^2 V_1 / \cup^*$

ℓ-adic Ceresa class $Cer(X, x) \in H^1(G_K, V_1^* \otimes V_2)$.

Triviality of ℓ-adic Ceresa classes

Corey–Ellenberg–Li:
if $K = \mathbb{C}((t))$, then $Cer(X, x) = 0$.

Also holds for K a p-adic field with $\ell \neq p$.
In fact, $H^1(G_K, V_1^* \otimes V_2) = 0$.
Reason: weights ⇒ every extension of V_1 by V_2 splits uniquely.

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Generic non-triviality of p-adic Ceresa classes

$\pi: X \to S$ a smooth proper relative curve defined over a p-adic field K, $x: S \to X$ a section.

<u>Theorem (B.–Li, in progress):</u>
If $X \to S$ satisfies a big monodromy condition, then the set of $s \in S(K)$ such that $Cer\big(X_s, x(s)\big) \neq 0$ is open and dense in $S(K)$.

Hain–Matsumoto: if $X \to S$ satisfies a different big monodromy condition, then $Cer\left(X_\eta, x(\eta)\right) \neq 0$.

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Examples

- The universal curve over $\mathcal{M}_{g,1}$.
- The family of curves $y^3 = x^4 + 2tx^2 + 1$ over $\mathbb{A}^1 \setminus \{\pm 1\}$.
  - Laga–Snidman ⇒ X_t has non-torsion Ceresa *cycle* for all but finitely many $t \in \mathbb{Q}_p \setminus \{\pm 1\}$.
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How does L vary as (X, x) varies in a family?

Fontaine's functor D_{pst}

de Rham

If V is a nice representation of G_K, Fontaine attaches to it a vector space $D_{pst}(V)$ endowed with:

discrete (φ, N, G_K)-module

- some semilinear algebraic data;
- a filtration on $D_{pst}(V)_K$.

Hodge filtration

The original representation V can be reconstructed from these two extra structures on $D_{pst}(V)$.

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Reinterpreting $Cer(X, x) = 0$

(X, x) a pointed curve as before.

$$0 \to V_2 \to L \to V_1 \to 0$$

$$0 \to D_{pst}(V_2) \to D_{pst}(L) \to D_{pst}(V_1) \to 0$$

Weights $\Rightarrow$ unique splitting ψ compatible with semilinear algebraic data. (ψ is the *pst-splitting*.)

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Variation of semilinear algebraic data

Back to $\pi: X \to S$, $x: S \to X$ a family of pointed curves.
There is a sequence of $\mathbb{Q}_p$-local systems on S_{et}:

$$0 \to \mathbb{V}_2 \to \mathbb{L} \to \mathbb{V}_1 \to 0$$

where $\mathbb{L}_s$ is the Lie algebra attached to $(X_s, x(s))$.

Fact: $\mathbb{L}$ is de Rham.

For $s, s' \in S(K)$ sufficiently close, $D_{pst}(\mathbb{L}_s) \cong D_{pst}(\mathbb{L}_{s'})$ compatibly with semilinear-algebraic data.

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Variation of filtration

Have a similar sequence

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of filtered vector bundles with integrable connection on S.

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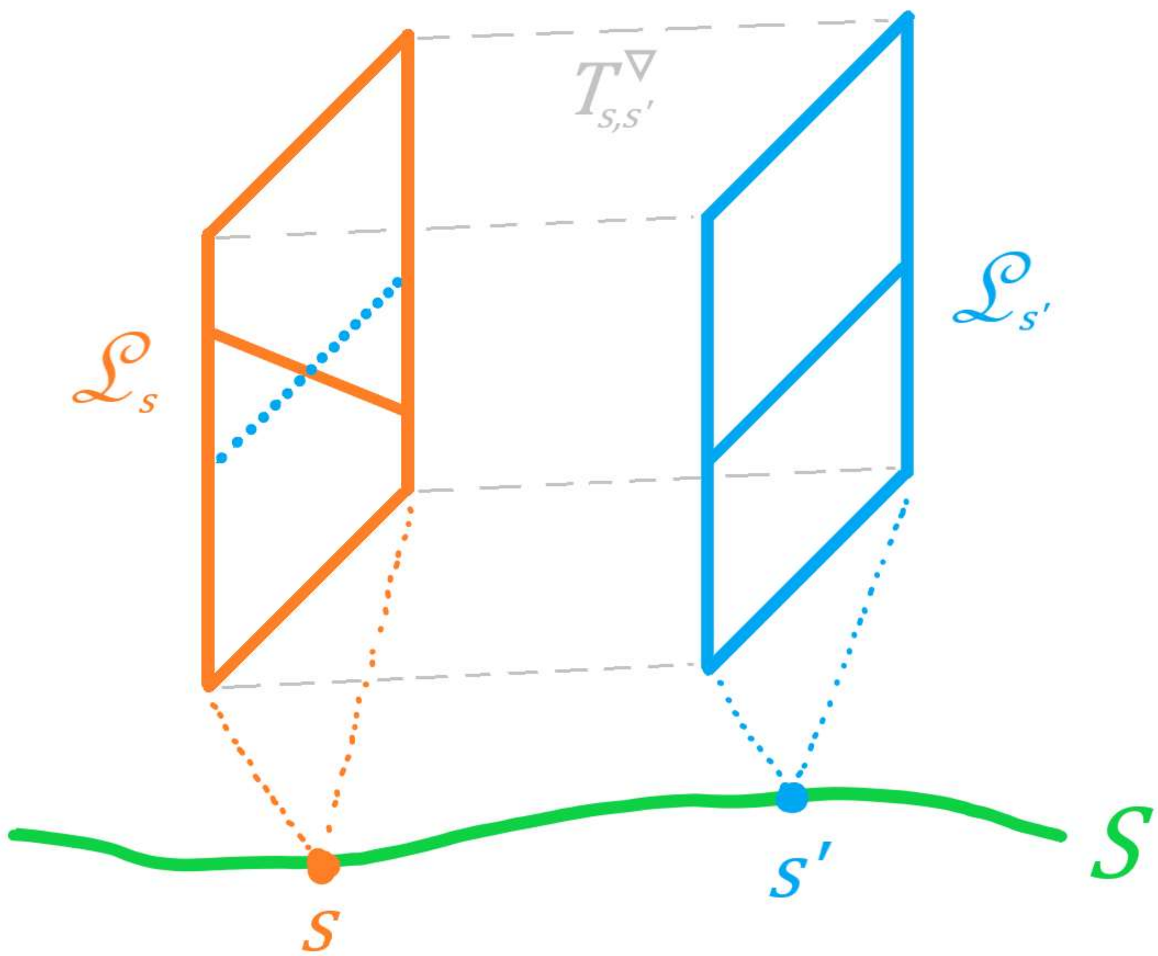
Parallel transport

For $s, s' \in S(K)$ sufficiently close, can identify the fibres of $\mathcal{L}$ via parallel transport

$$T^{\nabla}_{s,s'}: \mathcal{L}_s \xrightarrow{\cong} \mathcal{L}_{s'}$$

Let $\Phi(s')$ be the filtration on $\mathcal{L}_s$ corresponding to the given filtration on $\mathcal{L}_{s'}$.

So $D_{pst}(\mathbb{L}_{s'}) \cong D_{pst}(\mathbb{L}_s)$ compatibly with semilinear algebraic data, and its filtration corresponds to $\Phi(s')$.


$T^{\nabla}_{s,s'}$
$\mathscr{L}_s$
$\mathscr{L}_{s'}$
s
s'
S

p-adic period maps

Fix $s \in S(K)$ and a small neighbourhood $U_s \subseteq S(K)$. The map

$$\Phi : U_s \to \mathcal{G}_s \coloneqq \{\text{filtrations on } \mathcal{L}_s \text{ compatible with Lie bracket}\}$$

is called the *p-adic period map*. It is analytic.

Let $\mathcal{G}_s(\psi) \subseteq \mathcal{G}_s$ be the subset of filtrations compatible with the pst-splitting ψ of

$$0 \to D_{pst}(\mathbb{V}_{2,s}) \to D_{pst}(\mathbb{L}_s) \to D_{pst}(\mathbb{V}_{1,s}) \to 0$$

A period map criterion

Lemma:

For $s' \in U_s$ we have $\mathrm{Cer}\left(X_{s'}, x(s')\right) = 0$ iff $\Phi(s') \in \mathcal{G}_s(\psi)$.

Proof: the identifications of the sequences

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Large image of period maps

The set of $s' \in U_s$ with $Cer\left(X_{s'}, x(s')\right) \neq 0$ is either

$$\begin{cases} \text{open and dense in } U_s & \text{if } im(\Phi) \not\subseteq \mathcal{G}_s(\psi) \\ \emptyset & \text{else} \end{cases}$$

Big monodromy

Comparison with complex period maps

Embed K inside $\mathbb{C}$. Let $\Omega_s \subseteq S(\mathbb{C})$ be a contractible open neighbourhood of s. Using complex-analytic parallel transport in $\mathcal{L}_{\mathbb{C}}$, we obtain a complex analytic period map

$$\Phi_{\mathbb{C}}: \Omega_s \to \mathcal{G}_{s,\mathbb{C}}$$

Lemma:

$$im(\Phi_{\mathbb{C}})^{Zar} = (im(\Phi)^{Zar})_{\mathbb{C}}$$

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Monodromy

$\pi_1(S(\mathbb{C}); s)$ acts on $\mathcal{L}_{s,\mathbb{C}}$ by parallel transport, hence on $\mathcal{G}_{s,\mathbb{C}}$.

Lemma:
$im(\Phi_{\mathbb{C}})^{Zar}$ contains the $\pi_1(S(\mathbb{C}); s)$-orbit of $\Phi(s)$ (i.e. the Hodge filtration on $\mathcal{L}_{s,\mathbb{C}}$).

Reason: the complex period map extends to a universal covering of $S(\mathbb{C})$.

Big monodromy

The group of Lie algebra automorphisms of $\mathcal{L}_{s,\mathbb{C}}$ is an extension

$$1 \to \mathcal{V}_{1,s,\mathbb{C}}^{*} \otimes \mathcal{V}_{2,s,\mathbb{C}} \to Aut(\mathcal{L}_{s,\mathbb{C}}) \to GSp(\mathcal{V}_{1,s,\mathbb{C}}) \to 1$$

We let $G \subseteq Aut(\mathcal{L}_{s,\mathbb{C}})$ denote the Zariski-closure of the image of the monodromy representation $\pi_1(S(\mathbb{C}); s) \to Aut(\mathcal{L}_{s,\mathbb{C}})$, and let G' be the intersection of G with $\mathcal{V}_{1,s,\mathbb{C}}^{*} \otimes \mathcal{V}_{2,s,\mathbb{C}}$.

Definition:

$\pi: X \to S$ has *big monodromy* if $G' \nsubseteq F^0(\mathcal{V}_{1,s,\mathbb{C}}^{*} \otimes \mathcal{V}_{2,s,\mathbb{C}})$.

The main theorem

Theorem (B.–Li):
Suppose that $\pi: X \to S$ has big monodromy. Then the set of $s \in S(K)$ with $Cer\left(X_s, x(s)\right) \neq 0$ is open and dense in $S(K)$.

Proof sketch:

- Big monodromy $\Rightarrow \Phi(s)$ is not fixed by G'.
- $\Rightarrow G \cdot \Phi(s) \subseteq \mathcal{G}_{s,\mathbb{C}}$ contains two different filtrations on $\mathcal{L}_{s,\mathbb{C}}$ with the same induced filtration on $\mathcal{V}_{1,s,\mathbb{C}}$.
- At most one filtration can lie in $\mathcal{G}_s(\psi)_{\mathbb{C}}$.
- $\Rightarrow im(\Phi_{\mathbb{C}})^{Zar} \nsubseteq \mathcal{G}_s(\psi)_{\mathbb{C}}$, so $im(\Phi)^{Zar} \nsubseteq \mathcal{G}_s(\psi)$.

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- $\Rightarrow im(\Phi_{\mathbb{C}})^{Zar} \nsubseteq \mathcal{G}_s(\psi)_{\mathbb{C}}$, so $im(\Phi)^{Zar} \nsubseteq \mathcal{G}_s(\psi)$.

Future directions

- Generalise to unpointed Ceresa classes.
- Give explicit one-parameter families where the big monodromy condition is satisfied.
- Give explicit infinite families of curves over $\mathbb{Q}$ with non-trivial p-adic Ceresa class. (cf. Laga–Shnidman, Qiu–Zhang)