Combinatorial Iterated Integrals and a unipotent Torelli theorem

joint w/ Raymond Cheng

Single integrals: Classical and Tropical

Let X be a Riemann surface. Let $\omega \in \Omega^1(X)$ be a holomorphic or closed 1-form, and γ be a path. We can define $\int_\gamma \omega$ by integrating $\gamma^*\omega$.

There's a combinatorial analogue introduced by Mikhalkin-Zharkov.

Γ - metric graph

η - tropical 1-form:

Data of $\eta(e) \in \mathbb{R}$ for each directed edge s.t.

$$\eta(\bar{e}) = -\eta(e) \qquad \eta(e)\,dt$$

reversed edges

such that for each v,

$$\sum_{\substack{e\text{- adjacent to } v \\ \text{directed from } v}} \eta(e) = 0$$

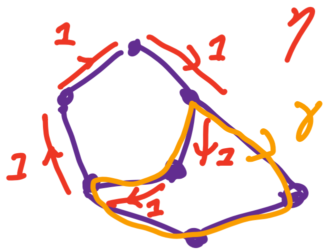
These are manifestly elements of $H_1(\Gamma)$.

For $\gamma = e_1 e_2 \cdots e_k$, $\int_\gamma \eta = \sum_{i=1}^{k} \eta(e_i)\ell(e_i)$

$\Omega^1(\Gamma)$ - tropical 1-form

$$\int^{\text{trop}} : H_1(\Gamma) \otimes \Omega^1(\Gamma) \longrightarrow \mathbb{R}$$

can be interpreted as $\langle\ ,\ \rangle$ on $H_1(\Gamma)$ taking a pair of cycles to the signed length of their intersection.



$$\int_\gamma \eta = -2.$$

Applications

1) Monodromy Pairing.

Let $\begin{matrix}\mathcal{X}\\ \downarrow\\ \mathbb{D}\end{matrix}$ be a semistable family of Riemann surfaces over a disc s/t for $t \neq 0$, $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ has nodal singularities.

ω_t - family of 1-forms
γ_t - family of paths

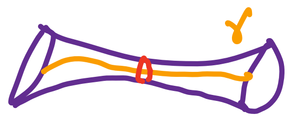
Asymptotics of $\int_{\gamma_t} \omega_t$:

When degenerating, Riemann surface forms the node $xy = t$

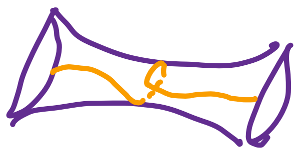
$$\omega \sim A\frac{dx}{x} = -A\frac{dy}{y} \text{ near the node}$$

contributes to $A\,\mathrm{Log}(t)$

Think: thick/thin



Corresponds to Dehn twist:



Tropicalize
$$\omega_t \longrightarrow \eta$$
$$\gamma_t \longrightarrow C$$

Coefficient of $\mathrm{Log}(t)$ in $\int_{\gamma_t}\omega_t$ is $\int_C \eta$.

Related to monodromy pairing.

2) p-adic integration

$\mathbb{K}/\mathbb{Q}_p$

$X/\mathbb{K}$ - alg. curve

$\omega \in \Omega^1(X)$

$p, q \in X(\mathbb{K})$

Two notions of p-adic integration: Abelian and Berkovich-Coleman integrals on bad reduction curves. They're the same on good reduction curves

Abelian integrals: ${}^{Ab}\int_p^q \omega \in \mathbb{K}$ (important for number theory)

path-independent!

Berkovich integrals:

${}^{BC}\int_{\gamma, p}^{q} \omega$ - path dependent, computable

Thm (K-Rabinoff-Zureick-Brown) ${}^{BC}\!\int - {}^{Ab}\!\int$ is given by a tropical integral determined by periods of ${}^{BC}\!\int$.

Lets you compute these integrals.

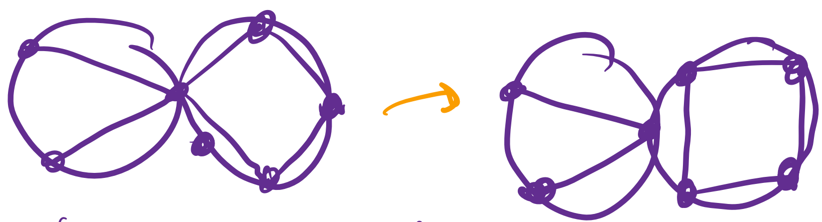
3) Tropical Torelli theorem (Caporaso-Viviani):

$$(H_1(\Gamma), \langle\ \rangle)$$

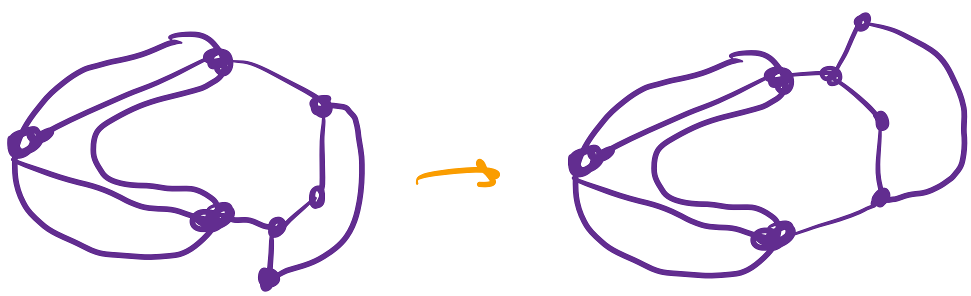
remembers bridgeless graphs up to 2-isomorphism

2-isomorphism means up to two moves

A) If you have a disconnecting vertex



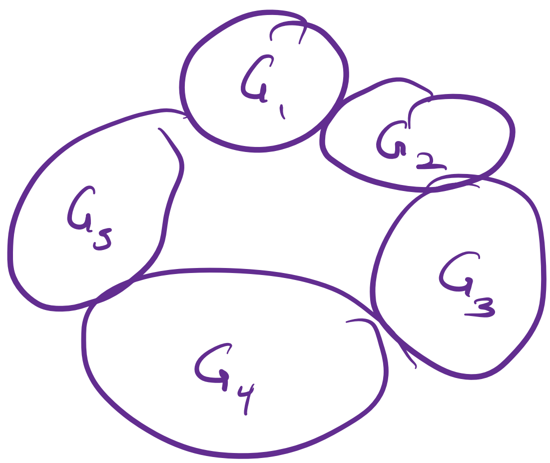
B) If you have a pair of disconnecting vertices



A graph is k-connected if it cannot be disconnected by removing fewer than k-vertices.

If a graph is 3-connected, then you can't apply those moves. Then, $(H_1(\Gamma), \langle\ ,\ \rangle)$ is a complete invariant.

If a graph is 2-connected but not 3-connected, it can be decomposed into a cycle of 3-connected blocks



Generalized Cycle Decomposition

Ambiguity about placement of those blocks.

Iterated Integrals due to Chen

X - Riemann surface, $x_0 \in X$
$\omega_1, \ldots, \omega_r \in \Omega^1(X)$

We'll define $\int_\gamma \omega_1 \ldots \omega_r$ recursively by primitives on $\tilde{X}$. Lift x_0 to $\tilde{x}_0 \in \tilde{X}$.

$$F_{\omega_1}(\tilde{x}) = \int_{\tilde{x}_0}^{\tilde{x}} \omega_1$$

$$F_{\omega_1 \omega_2}(\tilde{x}) = \int_{\tilde{x}_0}^{\tilde{x}} F_{\omega_1} \omega_2 \qquad \left(= \int \omega_1 \omega_2\right)$$

$$F_{\omega_1 \omega_2 \omega_3}(\tilde{x}) = \int_{\tilde{x}_0}^{\tilde{x}} F_{\omega_1 \omega_2} \omega_3 \qquad \left(= \int \omega_1 \omega_2 \omega_3\right)$$

$$\vdots$$

Depends on more than homology class of γ.
Interpolates between $H_1(X)$ and $\pi_1(X, x_0)$.

Properties:

1) Concatenation formula

$$\int_{\gamma_1 * \gamma_2} \omega_1 \omega_2 \dots \omega_r = \sum_{i=0}^{r} \left(\int_{\gamma_1} \omega_1 \dots \omega_i \right) \left(\int_{\gamma_2} \omega_{i+1} \dots \omega_r \right)$$

2) Nilpotence property

$$(\omega_i, \alpha_i) = \int_{\alpha_i} \omega_i$$

$$\int_{\underbrace{(\alpha_1 - 1) \dots (\alpha_r - 1)}_{\text{Multiply out and evaluate linearly}}} \omega_1 \omega_2 \dots \omega_k = \begin{cases} 0 & \text{if } r > k \\ (\omega_1, \alpha_1) \dots (\omega_k, \alpha_k) & \text{if } r = k \end{cases}$$

2) lets us encode iterated integrals algebraically

group ring

$$\mathbb{R}\pi_1(X, x_0) = \left\{ \sum a_\gamma [\gamma] : a_\gamma \in \mathbb{R} \right\}$$

Augmentation:
$$\varepsilon : \mathbb{R}\pi_1 \longrightarrow \mathbb{R}$$
$$\sum a_\gamma [\gamma] \longmapsto \sum a_\gamma$$
$$I = \ker(\varepsilon)$$

Integrals of length 1 are linear fnals on $\mathbb{R}\pi_1/I^2$

Integrals of length 2 $\left(\int \omega_i \omega_j\right)$ are linear fnals on $\mathbb{R}\pi_1/I^3$ which remember a tiny bit of non-Abelian data:

$$\begin{array}{ccccccccc} 0 & \to & I^2/I^3 & \longrightarrow & \mathbb{R}\pi_1/I^3 & \longrightarrow & \mathbb{R}\pi_1/I^2 & \to & 0 \\ & & \parallel & & & & & & \\ & & H^1(X)^{\otimes 2} & & & & & & \end{array}$$

Iterated integrals of closed 1-forms put a mixed Hodge structure on $\mathbb{R}\pi_1/I^3$ (Morgan and then Hain)

<u>Thm</u> (Hain, Pulte) MHS determines (X, x_0) up to finite ambiguity for basepoint.

(Unipotent Torelli theorem)

<u>Tropical Iterated Integrals</u>

Toy model for iterated integrals. Also used in geometric group theory.

$$^t\!\!\int : \pi_1(\Gamma, v_0) \otimes \Omega^1(\Gamma)^{\otimes \ell} \longrightarrow \mathbb{R}$$

$$\gamma \otimes (\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_\ell) \mapsto \int_\gamma \eta_1 \cdots \eta_\ell .$$

Define it for paths in Γ. For an edge $e \in E(\Gamma)$,

$$\int_e \eta_1 \eta_2 \cdots \eta_\ell = \frac{1}{\ell!} \eta_1(e)\eta_2(e)\cdots\eta_\ell(e)\,\ell(e)^\ell$$

For γ a word on the edges,

$$\int_{\gamma e} \eta_1 \eta_2 \cdots \eta_\ell = \sum_{i=0}^{\ell} \left(\int_\gamma \eta_1 \cdots \eta_i\right)\left(\int_e \eta_{i+1} \cdots \eta_\ell\right)$$

We have concatenation + nilpotence properties.

Lemma $\gamma \mapsto \int_\gamma \eta_1 \eta_2 \cdots \eta_\ell$ descends to a homomorphism

$$\mathbb{Z}[\pi_1]/I^{\ell+1} \longrightarrow \mathbb{Q}.$$

Applications

1) Asymptotics of iterated integrals

X_t - degenerating family of Rieman surfaces

$\int_{\gamma_t} \omega_1 \omega_2 \ldots \omega_k$ has leading asymptotics $(\log(t))^k$.

Coeffient is given by a tropical iterated integral.

There's a MHS on $\mathbb{R}\pi_1(X_t, x)/I^{\ell+1}$

Limit mixed Hodge structure. Lowest weight bit is given by tropical iterated integrals. Really is a tropicalization.

2) p-adic integration. $p, q \xrightarrow{\text{specialize}} v, w$

$BC\int_{\gamma, p}^{q} \omega_1 \ldots \omega_k$ defined by piecing integrals together

requires a path γ in Γ.

Single-valued integral due to Vologodsky making use of monodromy condition. Important in Chabauty method + regulators.

In work with Daniel Litt, we have given an explicit description of Vologodsky integration which can be turned into an algorithm.

3) Tropical Torelli theorem

Package iterated integrals as

$$\left(\mathbb{Z}\pi_1(\Gamma, v_0)/I^3, \int\right) \text{ - integration algebra.}$$

$\mathrm{Aut}_{cyc}\left(\mathbb{Z}\pi_1(\Gamma, v_0)/I^3, \int\right)$ - automorphisms of that induce identity on $H_1(\Gamma; \mathbb{F}_2)$.

$\cong (\mathbb{Z}/2\mathbb{Z})^m$ where $m = \#$ of 2-connected components of Γ.

<u>Theorem</u> (Cheng-K) Let Γ be a bridgeless metric graph with $g(\Gamma) \geq 2$. Let $v \in |\Gamma|$. Then cycle-respecting iso. type of $(\mathbb{Z}\pi_1/I^3, \int)$ determines v up to finite $|\mathrm{Aut}_{cyc}|$ choices.

(Cycle-respecting = induces iso. on $H_1(\Gamma; \mathbb{F}_2)$)

<u>Idea of Pf</u> Given v, w, there are only finitely many cycle-respecting iso.'s

$$\phi : \left(\mathbb{Z}\pi_1(\Gamma, v)/I^3\right) \to \left(\mathbb{Z}\pi_1(\Gamma, w)/I^3\right)$$

Once we pick an iso., knowing v determines w. This is proven, using the concatenation formula to get a handle on the tropical Abel-Jacobi map.

More might be true:

<u>Question</u> Does $\left(\mathbb{Z}\pi_1(\Gamma, v)/I^3, \int\right)$ determine (Γ, v)?

I think we can derive this from

<u>Conjecture?</u> Let Γ be a 2-connected graph. There is a non-trivial cycle preserving automorphism of $(\mathbb{Z}\pi_1(\Gamma, v)/I^3, \int)$ $\Leftrightarrow$ Γ is a hyperelliptic graph and v is a fixed point of the involution.

This is related to a conjecture of Corey-Ellenberg-Li.

Idea: The integration algebra should tell you how to put the graphs in the generalized cycle decomposition together. The data is too scrambled to easily see how to do that.