The Rank of the Normal Function of the Ceresa Cycle

Richard Hain

Duke University

ICERM
Brown University
May 16, 2024

The Ceresa cycle

Throughout C will be a smooth projective curve (over $\mathbb{C}$) of genus $g \geq 2$. For each $x \in C$ the Abel–Jacobi map

$$\alpha_x : C \to \operatorname{Jac} C$$

takes y to the divisor class of $y - x$. Denote its image by C_x. This is an algebraic 1-cycle in $\operatorname{Jac} C$. Let ι be the involution $u \mapsto -u$ of $\operatorname{Jac} C$. Set

$$C_x^- = \iota_* C_x.$$

Since ι acts as -1 on $H^1(\operatorname{Jac} C)$, it acts as $(-1)^k$ on $H^k(\operatorname{Jac} C)$. So the algebraic 1-cycle

$$C_x - C_x^-$$

on $\operatorname{Jac} C$ is homologous to 0. This is the Ceresa cycle of (C, x).

Detecting homologically trivial cycles via Hodge theory

Suppose that $Z = \sum_j n_j Z_j$ is an algebraic d-cycle on a smooth projective variety X. When $[Z] = 0$ in $H_{2d}(X)$, there is an extension

$$0 \longrightarrow H_{2d+1}(X)(-d) \longrightarrow E_Z \longrightarrow \mathbb{Z}(0) \longrightarrow 0$$

Pull back the LES of $(X, |Z|)$ along $cl_Z : \mathbb{Z} \to H_{2d}(|Z|)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{2d+1}(X) & \longrightarrow & H_{2d+1}(X,|Z|) & \longrightarrow & H_{2d}(|Z|) & \longrightarrow & H_{2d}(X) \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle cl_X} & & \vdots \\
0 & \longrightarrow & H_{2d+1}(X) & \longrightarrow & E_Z(d) & \longrightarrow & \mathbb{Z}(d) & \longrightarrow & 0
\end{array}$$

The extension E_Z is generated by $H_{2d}(X)$ and Γ, where $\partial\Gamma = Z$.

The group of 1-extensions

Suppose that V is a Hodge structure of negative weight, then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, V) \cong J(V) := V_{\mathbb{C}}/(V_{\mathbb{Z}} + F^0 V).$$

Given an extension of MHS

$$0 \to V \to E \to \mathbb{Z} \to 0$$

have exact sequences

$$0 \to V \to E \to \mathbb{Z} \to 0$$

and

$$0 \to F^0 V \to F^0 E \to \mathbb{C} \to 0.$$

Choose lifts $e_{\mathbb{Z}} \in E_{\mathbb{Z}}$ and $e_F \in F^0 E$ of 1. Their difference lies in $V_{\mathbb{C}}$ and is well defined mod $V_{\mathbb{Z}} + F^0 V$. The class of the extension is the image of $e_{\mathbb{Z}} - e_F$ in $J(V)$.

If V has weight -1, then $V_{\mathbb{C}} = F^0 V \oplus \overline{F^0 V}$, which implies that

$$V_{\mathbb{R}} \to V_{\mathbb{C}}/F^0 V$$

is an $\mathbb{R}$-linear isomorphism. It induces an isomorphism

$$J(V_{\mathbb{R}}) := V_{\mathbb{R}}/V_{\mathbb{Z}} \to J(V)$$

of tori. In particular, $J(V)$ is compact (but typically not algebraic).

The Griffiths invariant

A homologically trivial d-cycle on X thus determines a point[1]

$$\nu_Z \in J(H_{2d+1}(X)).$$

So the Ceresa cycle $C_x - C_x^-$ determines

$$\nu_{C,x} \in J(H_3(\operatorname{Jac} C)).$$

It is

$$\int_\Gamma \in \operatorname{Hom}(F^2H^3(\operatorname{Jac} C), \mathbb{C})/H_3(\operatorname{Jac} C; \mathbb{Z}) \cong J(H_3(\operatorname{Jac} C))$$

where $\partial\Gamma = C_x - C_x^-$. This intermediate jacobian is *not* algebraic.

---

[1] From now on I will suppress the Tate twist — always twist so that the odd weight Hodge structure V in $J(V)$ has weight -1.

Eliminating the base point

Set $H = H_1(C)$ and

$$\theta = \sum_{j=1}^{g} a_j \wedge b_j \in \Lambda^2 H$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ is a symplectic basis of H. The inclusion

$$H \xrightarrow{\wedge\theta} \Lambda^3 H \cong H_3(\operatorname{Jac} C).$$

induces an inclusion

$$\operatorname{Jac} C = J(H) \hookrightarrow J(\Lambda^3 H).$$

The primitive part of $H_3(\operatorname{Jac} C)$ is the quotient

$$\Lambda^3_0 H = (\Lambda^3 H)/(\theta \cdot H).$$

Its intermediate jacobian is

$$J(\Lambda^3_0 H) = J(\Lambda^3 H)/\operatorname{Jac} C.$$

Proposition (Pulte)

If $x, y \in C$, then

$$\nu_{C,x} - \nu_{C,y} = \text{the image of } 2([x] - [y]) \in \operatorname{Jac} C \subset J(\Lambda^3 H)$$

Consequently, the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ does not depend on $x \in C$. It vanishes when C is hyperelliptic.

Notation: Denote the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ by ν_C.

One can study ν_C by letting C vary and use variational methods.

Families of homologically trivial cycles

Suppose that $f : X \to S$ is a smooth projective morphism and that Z an algebraic cycle on X whose restriction to each fiber is homologically trivial and has codimension e and dimension d:

$$\begin{array}{ccccc} Z_s & \hookrightarrow & Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ \{s\} & \longrightarrow & S & = & S \end{array}$$

Set

$$\mathbb{V} = R^{2e-1} f_* \mathbb{Z}_X(e)$$

This has fiber $H_{2d+1}(X_s)(-d)$ over $s \in S$ and weight -1. We have the family

$$J(\mathbb{V}) \to S$$

of intermediate jacobians.

The normal function of a family of cycles

These data give rise to an extension over S of VMHS

$$0 \to \mathbb{V} \to \mathbb{E} \to \mathbb{Z}_S \to 0 \qquad (\dagger)$$

It corresponds to the section

$$\nu_Z : s \mapsto \nu_{Z_s} \in J(V_s)$$

of $J(\mathbb{V})$. It is holomorphic and satisfies Griffiths infinitesimal period relation:

$$\nabla\tilde{\nu} \in F^{-1}\mathcal{V} \otimes \Omega^1_S$$

where $(\mathcal{V}, \nabla)$ is the associated flat bundle $\mathbb{V} \otimes \mathcal{O}_S$ and $\tilde{\nu}$ is a local lift of ν to a section of $\mathcal{V}$.

It also satisfies strong conditions "at infinity" which correspond to the existence of a limit MHS on $\mathbb{E}$ at points of $\overline{S} - S$.

The relative Ceresa cycle: setup

Here $S = \mathcal{M}_{g,1}$, the moduli space of smooth pointed genus curves (C, x), and X is the universal jacobian $\mathcal{J}$ over it. Let

$$\mathscr{C} \underset{f}{\overset{x}{\leftrightarrows}} \mathcal{M}_{g,1}$$

be the universal curve over $\mathcal{M}_{g,1}$ with tautological section x.

We have the diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\mu_x} & \mathcal{J} \circlearrowleft \iota \\ x \uparrow\downarrow f & & \downarrow \\ \mathcal{M}_{g,1} & = & \mathcal{M}_{g,1} \end{array}$$

where μ_x is the relative Abel–Jacobi map.

The relative Ceresa cycle and its normal function

Set

$$\mathbb{H} = (R^1 f_* \mathbb{Z})^\vee \text{ and } \mathbb{V} = \Lambda^3_0 \mathbb{H}(-1).$$

The restriction of the algebraic cycle

$$Z = (\mu_x)_* \mathscr{C} - \iota_* (\mu_x)_* \mathscr{C} \subset \mathcal{J}$$

to the fiber $\operatorname{Jac} C$ of $\mathcal{J}$ over $[C, x]$ is $C_x - C_x^-$. It gives rise to the admissible normal function

$$J(\Lambda^3 \mathbb{H}) \underset{\nu_x}{\overset{\curvearrowleft}{\longrightarrow}} \mathcal{M}_{g,1}$$

which descends to the *Ceresa* normal function

$$J(\Lambda^3_0 \mathbb{H}) \underset{\nu}{\overset{\curvearrowleft}{\longrightarrow}} \mathcal{M}_g$$

It vanishes on the hyperelliptic locus and thus in genus 2.

The rank of a normal function

Suppose that ν is a normal function section of $J(\mathbb{V}) \to S$. The inclusion $\mathbb{V}_{\mathbb{R}} \hookrightarrow \mathbb{V}_{\mathbb{C}}$ induces a canonical isomorphism

$$\mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}} =: J(\mathbb{V}_{\mathbb{R}}) \to J(\mathbb{V}).$$

So ν corresponds to a section $\nu_{\mathbb{R}}$ of $J(\mathbb{V}_{\mathbb{R}})$ over S.

The bundle $J(\mathbb{V}_{\mathbb{R}})$ is a flat family of compact real tori. So locally $\nu_{\mathbb{R}}$ is a map

$$\tilde{\nu}_{\mathbb{R}} : U \to V_{\mathbb{R}}, \quad \text{where } U \subset S \text{ is contractible.}$$

The *rank* of $\nu_{\mathbb{R}}$ at s is defined to be the rank of $\nabla \tilde{\nu}_{\mathbb{R}}$ at s. Set

$$\operatorname{rk} \nu = \tfrac{1}{2} \max_{s \in S} \operatorname{rk}_s \nu_{\mathbb{R}}.$$

This is an integer. The rank of a torsion section is zero.

The main theorem

Theorem

The rank of the normal function of the Ceresa cycle has maximal rank $3g-3$ *for all* $g \geq 3$.

The theorem is false in genus 2 as, in that case, the Ceresa normal function is identically zero.

The proof is by induction. The base case is $g = 3$. I will discuss the proof there and sketch the inductive setup.

I understand that Ziyang Gao has also given a proof using Ax–Schanuel.

Zhang's application

The Gross–Schoen cycle $\mathrm{GS}_{C,\xi}$ of a curve C and a point $\xi \in \mathrm{Pic}^1 C$ is a homologically trivial algebraic 1-cycle in C^3. Its normal function is an integer multiple (6, I believe) of the Ceresa normal function of (C, ξ). Below ξ is a $(2g-2)$nd root of K_C.

Theorem (S.-W. Zhang)

For each $g \geq 3$, there is a non-empty Zariski open subset U of $\mathcal{M}_g/\mathbb{Q}$ such that:

- *Northcott property of Bloch–Beilinson height of the Gross–Schoen cycle: for $H, D \in \mathbb{R}_+$*

  $$\#\{[C] \in U(\overline{\mathbb{Q}}) : \deg[C] < D \text{ and } \langle \mathrm{GS}_{C,\xi}, \mathrm{GS}_{C,\xi}\rangle_{BB} < H\} < \infty;$$

- *for all $[C] \in U(\mathbb{C}) - U(\overline{\mathbb{Q}})$, $\mathrm{GS}_{C,\xi}$ has infinite order in $\mathrm{CH}^2(C^3)$.*

Technical tools

This is a summary of work of Griffiths, Green, Nori, with a few additions. Suppose that $\mathbb{V} \to S$ is a PVHS of weight -1. Let

$$\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_S \text{ and } \nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S$$

be the associated flat bundle and its connection. It satisfies Griffiths transversality

$$\nabla : F^p\mathcal{V} \to F^p(\mathcal{V} \otimes \Omega^1_S) = F^{p-1}\mathcal{V} \otimes \Omega^1_S.$$

A basic tool for studying a normal function $\nu : S \to J(\mathbb{V})$ is the complex $\mathcal{V} \otimes \Omega^\bullet_S$ and its Hodge graded quotients

$$\mathrm{Gr}^p_F(\mathcal{V} \otimes \Omega^\bullet_S) : 0 \to \mathrm{Gr}^p_F \mathcal{V} \to \mathrm{Gr}^{p-1}_F \mathcal{V} \otimes \Omega^1_S \to \mathrm{Gr}^{p-2}_F \mathcal{V} \otimes \Omega^2_S \to$$

Its differential $\overline{\nabla}$ is $\mathcal{O}_S$-linear. So this is a complex of holomorphic vector bundles.

The Green–Griffiths infinitesimal invariant

- Locally a normal function $\nu : S \to J(\mathbb{V})$ can be lifted to a holomorphic section $\tilde{\nu}$ of $\mathcal{V}$. It is well defined up to a section of $F^0\mathcal{V}$.
- The Griffiths infinitesimal invariant $\delta(\nu)$ of ν is the image of $\nabla\tilde{\nu}$ in
$$H^0(S, \mathcal{H}^1(F^0(\mathcal{V} \otimes \Omega_S^\bullet))).$$
- Green's variant — the *Green–Griffiths invariant* — is its image $\overline{\delta}(\nu)$ in $H^0(S, \mathcal{H}^1(\mathrm{Gr}_F^0(\mathcal{V} \otimes \Omega_S^\bullet)))$.

A canonical cocycle representative of $\overline{\delta}(\nu)$

- The $(1,0)$ component of the derivative $\nabla\nu_{\mathbb{R}}$ of a *real* lift of ν is an element $\nabla'\nu_{\mathbb{R}}$ of

$$H^0(S, \mathrm{Gr}_F^{-1}\mathcal{V} \otimes \Omega_S^1)$$

  that provides a *canonical* 1-cocycle that represents $\overline{\delta}(\nu)$.
- For each $s \in S$, it can be regarded as a $\mathbb{C}$ linear map

$$\nabla'\nu_{\mathbb{R}} : T_s S \to \mathrm{Gr}_F^{-1} V_s$$

  This has rank equal to the rank of ν at s.

Genus 3: introduction

Every non-hyperelliptic curve C of genus 3 is a plane quartic via its canonical embedding

$$C \to \mathbb{P}(H^0(\Omega^1_C))^\vee.$$

Collino and Pirola showed that away from the hyperelliptic locus

$$\mathcal{H}^1(\mathrm{Gr}^0_F(\mathcal{V} \otimes \Omega^\bullet))$$

is a vector bundle with fiber $S^4 H^0(\Omega^1_C) \otimes \det H^0(\Omega^1_C)^\vee$ over $[C]$.

Theorem (Collino–Pirola, 1995)

If C is a non-hyperelliptic curve of genus 3, the Green–Griffiths invariant $\overline{\nabla}\nu$ at C is a defining equation of the canonical image of C.

Linear algebra

- Suppose that C is a genus 3 curve. Set

$$A = F^0 H_1(C) \text{ and } B = \mathrm{Gr}_F^{-1} H_1(C) = H^0(\Omega_C^1).$$

  The intersection pairing induces an isomorphism $A \cong B^\vee$.
- If C is non-hyperelliptic, there are natural isomorphisms

$$T_{[C]}^\vee \mathcal{M}_3 \cong H^0(\Omega_C^{\otimes 2}) \cong S^2 H^0(\Omega_C^2) \cong T_{[\mathrm{Jac}\, C]}^\vee \mathcal{A}_3 \cong S^2 B.$$

- The fiber over $[C]$ of the complex $\mathrm{Gr}_F^0(\mathcal{V} \otimes \Omega_{\mathcal{M}_3}^\bullet)$ is

$$\frac{A \otimes \Lambda^2 B}{B} \to \frac{\Lambda^2 A \otimes B}{A} \otimes S^2 B \to \Lambda^3 A \otimes \Lambda^2 S^2 B.$$

  It is a complex of $\mathrm{GL}(B)$ modules.
- The differential $\overline{\nabla}$ is induced by $B \to A \otimes S^2 B$ adjoint to $B^{\otimes 2} \to S^2 B$.

- A little representation theory shows that the group of 1-cocycles is

$$S^2 S^2 B \otimes \det A = S^4 B \otimes \det A + S^2 A \otimes \det B.$$

  and the group of 1-coboundaries is $S^2 A \otimes \det B$.
- This gives the Collino–Pirola computation $H^1(\mathrm{Gr}_F^0(V \otimes \Omega^\bullet)) = S^4 B \otimes \det A$.
- The computation implies that, away from the hyperelliptic locus, $\nabla' \nu_{\mathbb{R}}$ is a symmetric bilinear form

$$S^2 A \otimes S^2 A \to \det A.$$

  Its rank is the rank of ν at C.

- The part coming from $S^4B \otimes \det A$ is

$$S^2A \otimes S^2A \to \mathbb{C}, \quad u \otimes v \mapsto f(uv), \quad u, v \in S^2A$$

  where $f \in S^4B$ is a quartic defining equation of C. This and its rank are easily computed from f.

Proposition

If C is the Klein quartic, then the coboundary component of $\nabla' \nu_{\mathbb{R}}$ vanishes and the other part has rank 6.

Corollary

The genus 3 Ceresa normal function has maximal rank on a dense open subset of $\mathcal{M}_3$.

Conjecture

If C is not hyperelliptic, then the component of $\nabla' \nu_{\mathbb{R}}$ in $S^2 A \otimes \det B$ vanishes.

- I believe I have a proof. It uses recent work of Cléry, Faber and van der Geer. If this component were not zero, it would have to be a Teichmüllar modular form of type $(0, 2, -1)$, of which it appears there are none.
- The component of $\nabla' \nu_{\mathbb{R}}$ with values in $S^4 B \otimes \det A$ is a non-zero multiple of the Teichmüller modular form $\chi_{4,0,-1}$ that plays a significant role in their paper. This should also yield a new proof of the Collino–Pirola theorem.
- If the conjecture is true, one can explicitly compute the rank of ν at all non-hyperelliptic curves.

Higher genus

The proof in the higher genus is not nearly as explicit. Here is a very brief sketch of the main ideas.

- The proof is by induction on g. The base case is $g = 3$.
- Suppose that $g > 3$. Denote by Δ_0 the boundary divisor of $\overline{\mathcal{M}}_g$ whose generic point is an irreducible nodal curve of genus g with one node.
- Denote the smooth locus of Δ_0 by Δ'. It is the quotient of $\mathcal{M}_{g-1,2}$ (the moduli space of smooth curves of genus $g-1$ and an ordered pair of distinct points) by the involution that swaps the two marked points.

- The idea is to regard ν as an admissible variation of MHS over $\mathcal{M}_g$ and and study its asymptotic properties along Δ_0 and normal to it.
- To do this, we approximate it by its family of limit MHS along the punctured (normalized) normal bundle L of Δ' in $\overline{\mathcal{M}}_g$. This is the quotient of the $\mathbb{C}^\times \times \mathbb{C}^\times$ bundle $\mathcal{M}_{g-1,\vec{2}} \to \mathcal{M}_{g-1,2}$ by the $\mathbb{C}^\times$ action

$$\lambda : [C, \vec{v}_p, \vec{v}_q] \mapsto [C; \lambda\vec{v}_p, \lambda^{-1}\vec{v}_q]$$

  and the involution $p \leftrightarrow q$. Denote its orbit by $[C; \vec{v}_p \otimes \vec{v}_q]$.
- The point $[C; \vec{v}_p \otimes \vec{v}_q]$ corresponds to a first order smoothing of $C/(p \sim q)$ and thus a point of L'.

- The variation $\mathbb{E}_g$ associated to the genus g Ceresa cycle "restricts" to a family $\mathbb{E}^{\mathrm{nil}}$ of nilpotent orbits of limit MHS over L'.
- It comes equipped with a second weight filtration $M_\bullet$, the *relative weight filtration*.
- Using it one constructs a *residual normal function* ν_0 over Δ' and one shows that

$$\operatorname{rk} \nu = \operatorname{rk} \nu_0 + \text{the "normal rank" of } \nu$$

- One computes the monodromy of the variation $\mathbb{E}^{\mathrm{nil}}$ using the homomorphism

$$\Gamma_{g-2,\vec{2}} \xhookrightarrow{\text{index 2}} \pi_1(L') \longrightarrow \Gamma_g \cong \pi_1(\mathcal{M}_g).$$

It forces the normal rank to be 1.

- Up to isogeny, which does not change the rank,

$$\nu_0 = \text{the genus } (g-1) \text{ Ceresa normal function } + \kappa$$

  where κ is the section

$$[C; p, q] \mapsto (g-1)(p+q) - K_C$$

  of the universal jacobian over $\mathcal{M}_{g-1,2}/C_2$.
- By induction, the genus $g-1$ Ceresa normal function has rank $3(g-1)-3 = 3g-6$. The normal function κ has rank 2. So

$$\operatorname{rk} \nu_0 = (3g-6) + 2 = 3g-4.$$

- Finally

$$\operatorname{rk} \nu = 1 + \operatorname{rk} \nu_0 = 3g-3.$$

Ceresa volume

- One can associate a $(1,1)$ form ϕ to a normal function $\nu : S \to J(\mathbb{V})$. If $\mathbb{V}$ is polarized, it is semi-positive. The rank of ν is the maximum r such that ϕ^r is non-zero.
- Since the genus g Ceresa normal function has maximal rank, ϕ^{3g-3} is a non-negative form which defines a positive measure on $\mathcal{M}_g$.
- At least in codimension 1, this form is locally L_1 on $\overline{\mathcal{M}}_g$. So it appears that the Ceresa cycle determines a positive measure on $\overline{\mathcal{M}}_g$.

References

- F. Cléry, C. Faber, G. van der Geer: *Concomitants of ternary quartics and vector-valued Siegel and Teichmüller modular forms of genus three*. Selecta Math. 26 (2020).
- R. Hain: *The Rank of the normal function of the Ceresa cycle*, notes (February, 2024) and manuscript (in preparation, 2024).
- S.-W. Zhang: *A Northcott property for Gross–Schoen cycles and Ceresa cycles*, manuscript, April, 2024.