

Ceresa cycles of Fermat curves

(Based on joint work with Kumar Murty)

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Coarseness \longrightarrow

Three equivalence relations on alg. cycles: rational, algebraic, homological equivalence

rat. equivalent \implies alg. equivalent \implies hom. equivalent
 \Leftarrow (trivial) \Leftarrow (Thm of Griffiths)


Thm (Griffiths '69): If X is a general quintic in \mathbb{P}^4 , then

$$\frac{\mathcal{L}_1(X)^{hom}}{\mathcal{L}_1(X)^{alg}} \otimes \mathbb{Q} \neq 0.$$

Thm (Ceresa '83): If X is a general curve of genus > 2 , the Ceresa cycle of X is algebraically nontrivial.

$X =$ (smooth projective) curve / \mathbb{C} , genus $g > 0$

$Jac = Jac(X) =$ Jacobian of $X = \frac{Div^0(X)}{\text{princ. div's}}$

$e \in X$  $X \hookrightarrow Jac$ Image $:= X_e \in \mathcal{L}_1(Jac)$
 Base pt $x \mapsto x - e$ 1-dim'l alg. cycle on Jac

Ceresa cycle of X with base pt $e = Cer_e(X) := X_e - (-1)_* X_e \in \mathcal{L}_1^{hom}(Jac)$

$Cer_e(X)$ mod different equiv. relations:

Mod hom. \sim	Trivial
Mod alg. \sim	Independent of e
Mod rat. \sim	Depends on e

Ceresa's thm ('83): If $X =$ a **general** curve of genus > 2 , then $Cer(X)$ is alg. nontrivial.

Still at this point, no explicit example given!

Ceresa cycles of Fermat curves

Notation: $F_n =$ Fermat curve of degree n , given by $x^n + y^n = z^n$

Thm* (B. Harris, '83): $Cer(F_4)$ is **alg. nontrivial**.

← First explicit example of an alg. nontrivial hom. trivial cycle

Thm* (Bloch, '84): $Cer(F_4)$ is **alg. of infinite order**.

Further adaptations by Tadokoro, Otsubo, Kimura, ... to other Fermat curves and quotients. In particular:

Thm* (Kimura, 2000): $Cer(F_7)$ is alg. of infinite order.

← Adaptation of Bloch's

Thm* (Otsubo, 2012): $Cer(F_n)$ is alg. nonzero for all $4 < n \leq 1000$.

Thm* (Tadokoro, 2016): Ceresa cycle is alg. nonzero for some quotients of F_p for prime $p < 1000$ and $\equiv 1 \pmod{3}$.

← Adaptation of Harris'.

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- * Uses **Hodge theoretic** (Griffiths) Abel-Jacobi map.
 - * Uses **l-adic** Abel-Jacobi map.

Both methods have their limitations. Bloch's gives stronger results, but much harder to implement.

Harris' argument in a nutshell

- \forall variety Y : $\mathcal{L}_d(Y) / \sim_{rat} =: CH_d(Y) \supset CH_d^{hom}(Y)$ (Hom. trivial subgroup)
- Griffiths Abel-Jacobi map

$$AJ : CH_d^{hom}(Y) \rightarrow JH_{2d+1}(Y) := H_{2d+1}(Y, \mathbb{C}) / (F^{-d}H_{2d+1} + H_{2d+1}(Y, \mathbb{Z}))$$
$$\Gamma \mapsto \int_{\partial^{-1}\Gamma}$$

Griffiths intermediate Jacobian
(a compact complex torus)

Γ alg. trivial $\implies AJ(\Gamma)$ "vanishes" on $F^{d+2}H_{dR}^{2d+1}(Y)$

- With $Y = Jac(X)$, $d = 1$, $Z = Cer(X)$ ($X =$ a curve), get:

$Cer(X)$ alg. trivial $\implies AJ(Cer(X))$ "vanishes" on $H^{3,0}(Jac(X)) = \bigwedge^3 H^{1,0}(X)$

- In his "long paper" in '83, Harris calculates $AJ(Cer(X))$ in terms of iterated integrals.

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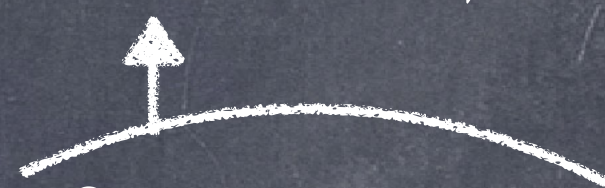
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1 dim'l for $X = F_4$



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In his "short paper" (also '83): Specializes to F_4 , deduces

Long paper + above +
Rohrlich on $H^{1,0}(F_n)$

$Cer(F_4)$ alg. trivial \implies Certain period integral is an integer.
 — " — torsion \implies — " — rational.

$AJ(\text{Cer}(F_4))$ alg. trivial \implies Certain period integral is an integer.

———— " ——— torsion \implies ——— " ——— rational.

- Can check nontriviality numerically.
- Adapted to other Fermat curves and some quotients by Tadokoro and Otsubo. There is a numerically verifiable sufficient condition for nontriviality mod alg. equiv. for a given F_n .

Limitations (due to its algorithmic nature and reliance on numerical approximations):

- 1) Can only check **nontriviality** (modulo algebraic equivalence)
- 2) Can only be used for **finitely*** many Fermat curves/quotients.

Modulo rational equivalence

Thm (E. - K. Murty, '21). For every prime $p > 7$ and every choice of base point, $|AJ(Cer_e(F_p))| = \infty$. In particular, $Cer_e(F_p)$ is of infinite order modulo rational equivalence.

Proof was a combination of:

- Harris' and Pulte's works on Ceresa cycles and Hodge theory of π_1
- Works of Kaenders and Darmon-Rotger-Sols on alg. cycles and Hodge theory of π_1
- Rohrlich's analogue of Manin-Drinfeld
- Gross-Rohrlich's work on nontorsion points on Jacobians of the F_n

We'll sketch a simplified version of the proof that doesn't directly refer to Hodge theory of π_1 .

Proof (simplified version)

Step 1) Reduction to the case where e is a cusp (i.e. satisfies $xyz=0$).

$$AJ(Cer_e(X)) = \underbrace{AJ(Cer_e(X)) \Big|_{H^3_{prim}(Jac)}}_{\text{Harris: Independent of } e} \oplus \underbrace{AJ(Cer_e(X)) \Big|_{H^1(X) \wedge cl(\Delta(X))}}_{\text{Pulte ('88): linear comb. of } e \text{ and the can. divisor (as a pt on Jac)}}$$

Harris: Independent of e .

Pulte ('88): linear comb. of e and the can. divisor (as a pt on Jac)

When $X = F_p$: By Rohrlich, if e is a cusp, this is torsion.

Step 2) Work with the modified diagonal cycle in X^3 instead

$$\Delta_{GKS,e}(X) := \{x, x, x\} - \{e, x, x\} - \{x, e, x\} - \{x, x, e\} + \{x, e, e\} + \{e, x, e\} + \{e, e, x\} \in CH_1^{hom}(X^3)$$

(Modified diagonal cycle of Gross, Kudla and Schoen)

Colombo and van Geeman ('93): $AJ(Cer_e(X)) \sim_{\mathbb{Q}^\times} AJ(\Delta_{GKS,e}(X))$

So we can instead show $|AJ(\Delta_{GKS,e}(X))| = \infty$.

Step 3) Relate to points on the Jacobian (Idea of Darmon-Rotger-Sols)

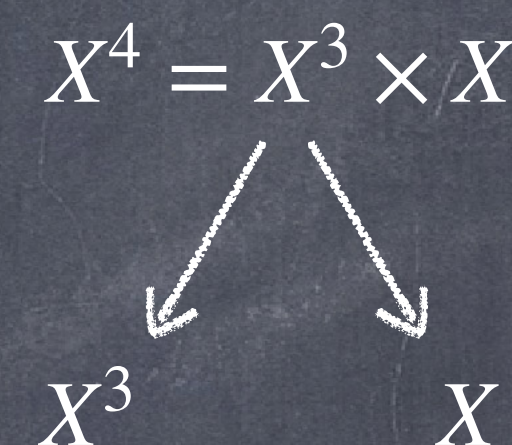
Let $\Gamma \in CH_1(X^2)$. The correspondence

$$\Gamma \times \Delta(X) \in CH_2(X^2 \times X^2) = CH_2(X^3 \times X)$$

gives a map

$$CH_1^{hom}(X^3) \rightarrow CH_0^{hom}(X) \stackrel{AJ}{=} Jac(X),$$

$$\Omega \mapsto (pr_4)_*(pr_{123}^*(\Omega) \cdot (\Gamma \times \Delta(X)))$$



Let $P_\Gamma :=$ the image of $\Delta_{GKS,e}(X)$.

If $|P_\Gamma| = \infty$ for some Γ , then $|AJ(\Delta_{GKS,e}(X))| = \infty$. (By functoriality of AJ.)

Step 4) Pick a suitable $\Gamma \in CH_1(F_p \times F_p)$.

Take Γ to be the graph of

$$\begin{aligned}\beta : F_p &\rightarrow F_p \\ (x, y, z) &\mapsto (-y, z, x)\end{aligned}$$

Set $Q = (\zeta_6, \bar{\zeta}_6, 1) \in F_p$. Then

$$P_\Gamma = (Q + \bar{Q} - 2e) + \underbrace{\text{a pt supp. on the cusps}}_{\text{Torsion by Rohrlich}}$$

Gross-Rohrlich ('78): If $p > 7$, then $|Q + \bar{Q} - 2e| = \infty$ (in $Jac(F_p)$).



Some questions

1) Fermat quotients

- Fix $p > 7$. For $1 \leq s \leq p-2$, consider

$$C_s : y^p = x(1-x)^s.$$

- The maps

$$F_p \rightarrow C_s \quad (x, y, 1) \mapsto (x^p, xy^s, 1)$$



$$\text{Jac}(J_p) \sim \prod_{1 \leq s \leq p-2} \text{Jac}(C_s)$$

(Non-hyperelliptic cases)

- **Gross-Rohrlich:** If $s \notin \{1, (p-1)/2, p-2\}$, the image of $Q + \bar{Q} - 2e$ in $\text{Jac}(C_s)$ is of infinite order.
- Question: Can the argument be adapted to these non-hyperelliptic quotients?

• Partial answer: (Nemoto, 2024) Suppose $s^2 + s + 1 \equiv 0 \pmod{p}$. Then $|AJ(Cer_e(C_s))| = \infty$ for every e .

"Pf": β descends to C_s . (In fact, this happens iff $s^2 + s + 1 \equiv 0 \pmod{p}$.)

What about other quotients?

2) Algebraic equivalence and connection to Beilinson-Bloch conj.

Supp. p, s as above with $|AJ(Cer_e(X))| = \infty$.

$$AJ(Cer_e(C_s)) \in Ext^1(\mathbb{Q}(-1), \underbrace{\bigwedge^3 H^1(C_s)}_{\text{Decomposes into many components}})$$

Decomposes into many components.

How does $Cer_e(C_s)$ (or its Abel-Jacobi image) decompose?

Beilinson-Bloch \rightsquigarrow $L(\bigwedge^3 H^1(X), s)$ should have a zero at 2.

Which factors have zeros?

3) Possible strategy for ...

Given $\Gamma' \in CH_n(X^{2n}) = CH_n(X^{2n-1} \times X)$



$$CH_{n-1}^{hom}(X^{2n-1}) \xrightarrow{\Gamma'_*} Jac(X)$$

$$\Delta_{GKS,e}(X) \times \Delta(X)^{n-2} \mapsto P_{\Gamma'}$$

Try to pick Γ' so that $P_{\Gamma'}$ is nontorsion.

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Thank you