Questions on Finite Graphs Motivated by Infinite Graphs

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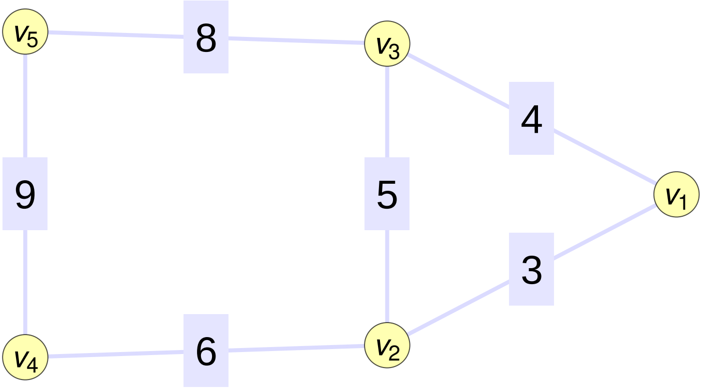
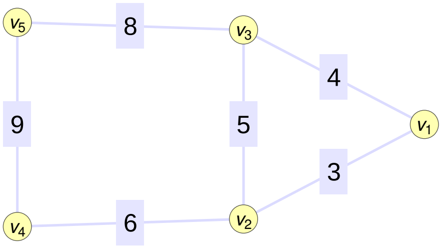


Let (G, r) be a finite graph with $r\colon \mathsf{E}(G) \to [0, \infty)$. Continuous-time random walk crosses an incident edge e at rate $r(e)$. It thus leaves $x \in \mathsf{V}(G)$ at rate $r(x) := \sum_{e \sim x} r(e)$.



$$\Delta = \begin{bmatrix} 7 & -3 & -4 & 0 & 0 \\ -3 & 14 & -5 & -6 & 0 \\ -4 & -5 & 17 & 0 & -8 \\ 0 & -6 & 0 & 15 & -9 \\ 0 & 0 & -8 & -9 & 17 \end{bmatrix}$$

The Laplacian matrix $\Delta_{(G,r)}$ has entries

$$\Delta(x,y) := \begin{cases} -r(x,y) & \text{if } x \neq y \text{ and } x \sim y, \\ 0 & \text{if } x \neq y \text{ and } x \not\sim y, \\ r(x) & \text{if } x = y. \end{cases}$$

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The infinitesimal generator is $-\Delta$, meaning that the transition probability $p_t(x, y)$ is the (x, y)-entry of $e^{-t\Delta}$, which equals $\langle e^{-t\Delta}\mathbf{1}_y, \mathbf{1}_x \rangle$. The stationary distribution is uniform

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Observation

The Laplacian $\Delta_{(G,r)}$ is positive semidefinite (written $\Delta \geq 0$).

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Each edge e corresponds to $r(e) \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq 0$, so Δ is a sum of p.s.d. matrices. □

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Corollary

The return probabilities $p_t(x, x)$ are monotone decreasing in t.

Proof sketch

The eigenvalues λ_i of Δ are nonnegative. If f_i are orthonormal eigenvectors of Δ, then

$$p_t(x, x) = \langle e^{-t\Delta} \mathbf{1}_x, \mathbf{1}_x \rangle = \sum_{i=1}^{|\mathrm{V}(G)|} e^{-t\lambda_i} |f_i(x)|^2 \,.$$

□

Both our problems concern random walks in random environments on Cayley graphs, where the law of the environment is invariant under group translations. In the first problem, the environment is primarily percolation.

The first problem is open for amenable Cayley graphs, whereas the second is open for nonamenable ones.

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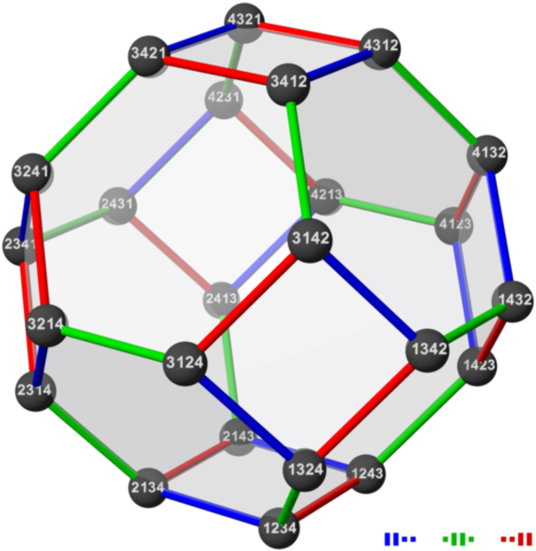
Note that for all $\gamma \in \Gamma$, we have $(\gamma x, \gamma xs)$ is an edge when (x, xs) is an edge, so Γ acts transitively on G by left multiplication.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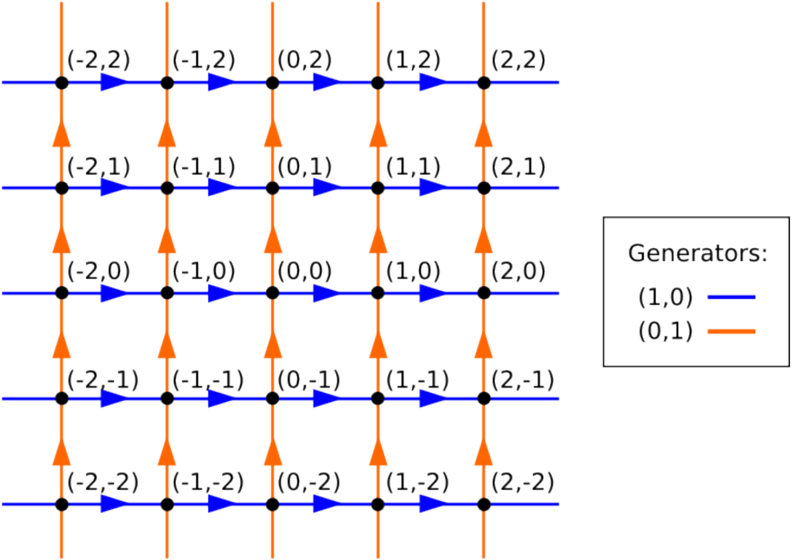
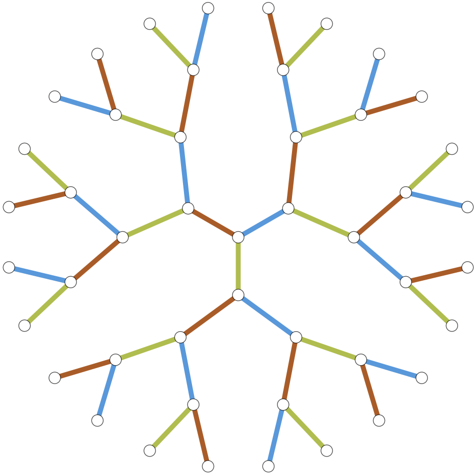
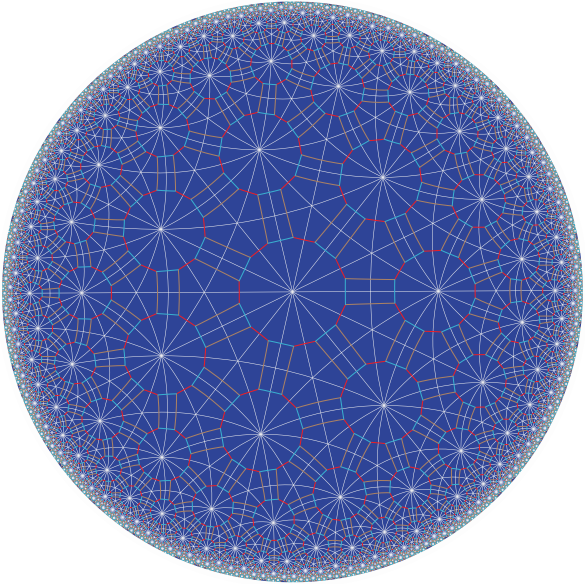


The standard Cayley diagram of $\mathbb{Z}^2$.

A Nonamenable Cayley Diagram, $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$



The presentation is $\langle a, b, c \mid a^2, b^2, c^2 \rangle$.



The presentation is $\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (bc)^3, (ca)^7 \rangle$.

PROBLEM ONE

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The motivating question:

Conjecture (Benjamini–L.–Schramm, 1999)

If G is an (infinite) Cayley graph on which (discrete-time) simple random walk escapes at zero speed, then a.s. simple random walk on each (infinite) cluster of Bernoulli percolation escapes at zero speed.

Speed here is the limit of graph distance divided by time as time $\to \infty$.

Bernoulli percolation is the random subgraph obtained by deleting each edge with the same probability and independently of all other edges.

Let $X = (X_t)_{t \geq 0}$ be (variable-speed) continuous-time random walk.

if G is a finite graph with two rate functions $r \leq r'$ and X_0 is a uniformly random vertex, then is

$$\mathbf{E}_r\big[\mathrm{dist}(X_0, X_1)\big] \leq \mathbf{E}_{r'}\big[\mathrm{dist}(X_0, X_1)\big]?$$

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Open Question (L.)

Is there some constant $c < \infty$ such that if G is a finite graph with two rate functions $r \leq r'$ and X_0 is a uniformly random vertex, then

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Conjecture (Benjamini–L.–Schramm, 1999)

if G is a finite graph with two rate functions $r \leq r'$, X_0 is a uniformly random vertex, and $\mathbf{H}$ is entropy, then

$$\mathbf{E}\Big[\mathbf{H}\big(\mathcal{L}_r(X_1 \mid X_0)\big)\Big] \leq \mathbf{E}\Big[\mathbf{H}\big(\mathcal{L}_{r'}(X_1 \mid X_0)\big)\Big].$$

Here, $\mathbf{H}\big((p_i)_i\big) := -\sum_i p_i \log p_i$.

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Theorem (essentially Kaimanovich, 1990, after Avez, 1974, ...)

Let G be a Cayley graph and ω be a random subgraph whose law is invariant under group translations. Let Z_t be simple random walk on ω starting at o. Then

$$\lim_{t \to \infty} \frac{\operatorname{dist}(Z_t, o)}{t} = 0$$

iff

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It follows that if G has subexponential growth, then these conditions hold.

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(Aside) A converse to our motivating conjecture:

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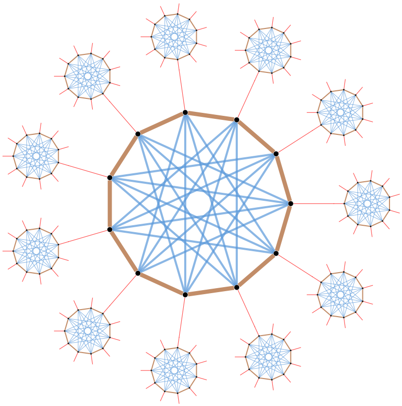
One might hope for monotonicity of the escape speed in general for continuous-time random walk; this would prove our motivating conjecture.

Counterexamples to Speed Monotonicity: $\mathbb{Z}/(n\mathbb{Z}) * \mathbb{Z}/(2\mathbb{Z})$ (Schramm, L.–White)



Increasing both rates within the cyclic group can decrease escape speed.

Converting rates to percolations: use rational rates to approximate the real rates and then parallel edges to convert to percolations.

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Note that changing time is the same as multiplying all rates by the same factor.

Conjecture (Benjamini–L.–Schramm, 1999)

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This conjecture remains open. Suppose we had a positive answer to

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When r'/r is constant, the inequality holds with $c = 1$, even if X_0 is fixed. In this sense, increasing time works better for entropy than for distance.

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PROBLEM
TWO

The motivating question:

Open Question (Fontes and Mathieu)

If G is an infinite Cayley graph, is $\mathbf{E}\big[p_1(o, o; r_G)\big]$ monotone decreasing in the rates, r_G, among random rate functions with invariant law? I.e., if $r_G(e) \leq r'_G(e)$ a.s. for all edges e and the laws of r_G and r'_G are invariant under left multiplication, then is $\mathbf{E}\big[p_1(o, o; r_G)\big] \geq \mathbf{E}\big[p_1(o, o; r'_G)\big]$?

This is open even on regular trees.

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The finite question:

Open Question (L., 2017)

If $G \succcurlyeq H$, then does continuous-time simple random walk satisfy

$$\frac{1}{|\mathsf{V}(G)|} \sum_{x \in \mathsf{V}(G)} p_1(x, x; G) \leq \frac{1}{|\mathsf{V}(H)|} \sum_{x \in \mathsf{V}(H)} p_1(x, x; H)?$$

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On every finite graph, G, the average return probability at each time is monotone decreasing in the rates, r_G.

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Proof sketch

If $r(e) \leq r'(e)$ for all edges e, then $\Delta_{(G,r)} \leq \Delta_{(G,r')}$, whence the ith eigenvalue of the former is at most the ith eigenvalue of the latter.

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On every finite graph, G, the average return probability at each time is monotone decreasing in the rates, r_G.

Proof sketch

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If $r(e) \leq r'(e)$ for all edges e, then $\Delta_{(G,r)} \leq \Delta_{(G,r')}$, whence the ith eigenvalue of the former is at most the ith eigenvalue of the latter. The sum of the return probabilities is the trace of $e^{-t\Delta}$, which equals $\sum_{i=1}^{|\mathsf{V}(G)|} e^{-t\lambda_i}$. □

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A rate function r on a Cayley graph is ***transitive*** if $r(x, xs)$ depends on the generator s but not on the vertex x.

Theorem (Brown, 2000)

If G is a Cayley graph, then $p_1(o, o; r_G)$ is monotone decreasing in the rates, r_G, among transitive rate functions. That is, if $r_G(x, xs) \leq r'_G(x, xs)$ for all vertices x and generators s, then $p_1(o, o; r_G) \geq p_1(o, o; r'_G)$.

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This is because for so-called equivariant operators A, $\operatorname{Tr} A := \langle A\mathbf{1}_o, \mathbf{1}_o \rangle$ defines a ***normalized trace***, meaning that $A \mapsto \operatorname{Tr} A$ is linear, $\operatorname{Tr} A \geq 0$ for $A \geq 0$, $\operatorname{Tr} I = 1$, and $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

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Theorem (Fontes and Mathieu, 2006)

If G is an amenable Cayley graph, then $\mathbf{E}\big[p_1(o, o; r_G)\big]$ is monotone decreasing in the rates, r_G, among random rate functions with invariant law.

Extension:

Theorem (Aldous and L., 2007)

If G is any Cayley graph, then $\mathbf{E}\big[p_1(o, o; r_G)\big]$ is monotone decreasing in the rates, r_G, among random rate functions provided the law of (r_G, r'_G) is invariant, where $r_G \leq r'_G$.

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The motivating question assumed only that the laws of r_G and r'_G are individually invariant, not the law of the pair. What is then required is to compare two different traces. We attempt to attack this problem via a similar question for finite graphs. Since we can always average on a finite graph to get a normalized trace, this appears impossible. But the essence is to compare two different traces, so we use two different graphs.

Recall that G ***dominates*** H, written $G \succcurlyeq H$, if there is a probability measure on pairs $(X, Y) \in \mathsf{V}(G) \times \mathsf{V}(H)$ such that (i) the marginal distributions of X and Y are each uniform and

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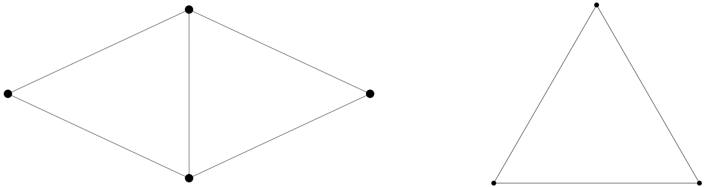
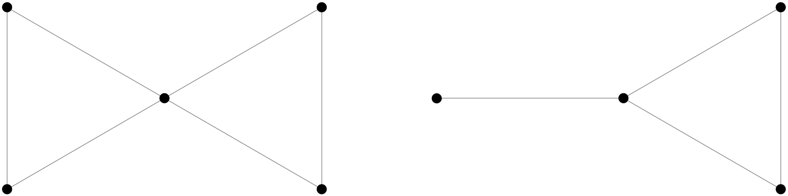
If there are rates on the edges, we require that the rooted isomorphism from (H, Y) to a subgraph of (G, X) is rate increasing.



The graph on the left dominates a triangle.

The graph on the left does *not* dominate the graph on the right:



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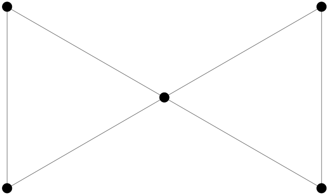
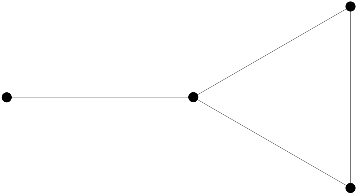


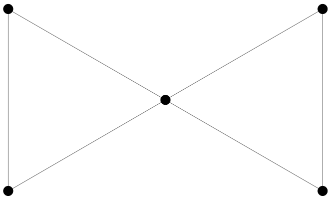


An edge fractionally tiles the graph on the left and tiles the graph on the right.

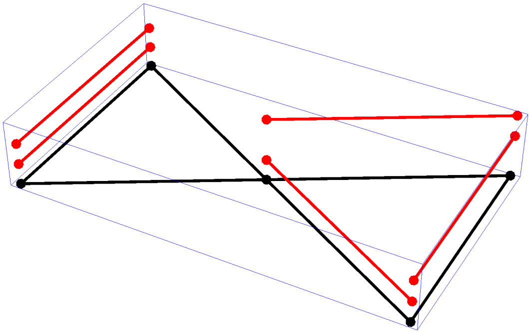
H ***fractionally tiles*** *G* if there is an integer number of copies of *H* in *G* such that each vertex of *G* is covered the same number of times by these copies of *H*.

If that latter number is 1, then *H* ***tiles*** *G*.

G:



If H fractionally tiles G, then $G \succcurlyeq H$:



Open Question (L., 2017)

If $G \succcurlyeq H$, then does continuous-time simple random walk satisfy

$$\frac{1}{|\mathsf{V}(G)|} \sum_{x \in \mathsf{V}(G)} p_t(x, x; G) \leq \frac{1}{|\mathsf{V}(H)|} \sum_{x \in \mathsf{V}(H)} p_t(x, x; H)?$$

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Domination of Finite Graphs: Model Question

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THANK YOU!

Supplementary Material

Why do we need X_0 to be uniform to get near monotonicity of entropy?

Let K_{2^n} and K_{n^2} be disjoint complete graphs. Choose $a \in \mathsf{V}(K_{2^n})$ and $b \in \mathsf{V}(K_{n^2})$. Join both a and b to a new vertex, o. Let r and r' be 1 on $\mathsf{E}(K_{2^n}) \cup \mathsf{E}(K_{n^2})$ and $r(o, a) := 1 =: r'(o, a)$. Let $r(o, b) := 0$ and $r'(o, b) := n$. At time 2, the r-random walk is mostly uniform on K_{2^n} with entropy, therefore, about $n \log 2$, whereas the r'-random walk is mostly uniform on K_{n^2} with entropy $2 \log n$.

(However, if we increase time on any graph with any fixed rates, we get decrease in the majorization order for any fixed X_0, and therefore increase in entropy.)

Theorem (Aldous and L., 2007)

If ν is a unimodular probability measure on rooted graphs with a pair of rate functions, r_1 and r_2, with $r_1 \le r_2$ a.s., then $\int p_1(o, o; r_1)\, d\nu \ge \int p_1(o, o; r_2)\, d\nu$.

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Proof.

We have $\Delta_1 \leq \Delta_2$, so $-\Delta_1 \geq -\Delta_2$. Therefore

$$\int p_1(o, o; r_1)\, d\nu = \mathrm{Tr}_\nu\, e^{-\Delta_1} \geq \mathrm{Tr}_\nu\, e^{-\Delta_2} = \int p_1(o, o; r_2)\, d\nu\,. \qquad \square$$

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If μ_1 and μ_2 are unimodular probability measures on rooted networks (G, o, r_i) such that there is a coupling (G, o, r_1, r_2) that is monotone, i.e., $r_1 \leq r_2$ a.s., then is $\int p_1(o, o)\, d\mu_1 \geq \int p_1(o, o)\, d\mu_2$?

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When there is a unimodular monotone coupling ν, we have $\mathrm{Tr}_{\mu_i}\, e^{-\Delta} = \mathrm{Tr}_\nu\, e^{-\Delta_i}$.

One can show that if f is any decreasing convex function and H fractionally tiles G, then

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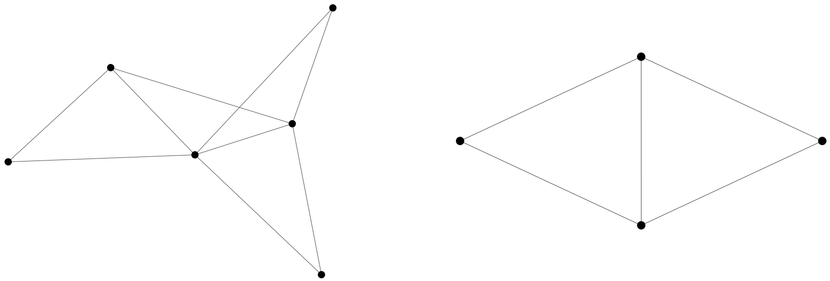
However, it is *not* true that this inequality holds whenever $G \succcurlyeq H$; a counter-example is provided by taking $f(s) := (4 - s)^+$ for these graphs:



The graph G on the left dominates the graph H on the right.

If H is transitive, then $G \succcurlyeq H$ iff every vertex of G belongs to a copy of H. If G is transitive, then $G \succcurlyeq H$ iff G contains a copy of H. In both cases, the independent coupling of roots works.

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If H fractionally tiles G, then $G \succcurlyeq H$. Conversely, if G is transitive and dominates H, then H fractionally tiles G.

Theorem (L., 2017)

Let G be a finite graph with positive rates r on its edges. Suppose that H_i is a subgraph of G with positive rates r_i on its edges for $i = 1, \ldots, k$ with the following two properties:

Then for all $t > 0$, we have

$$\frac{1}{|\mathsf{V}(G)|} \sum_{x \in \mathsf{V}(G)} p_t(x; G) \leq \frac{1}{\sum_{j=1}^{k} |\mathsf{V}(H_j)|} \sum_{i=1}^{k} \sum_{x \in \mathsf{V}(H_i)} p_t(x; H_i) \,.$$

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- for every $e \in \mathsf{E}(G)$,

$$w(e) \geq \frac{1}{m} \sum_{i\,;\ e \in \mathsf{E}(H_i)} w_i(e)\,.$$

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Note that $\log \tau(G)^{1/\mathsf{V}(G)} = \mathsf{V}(G)^{-1} \operatorname{tr}\log\Delta_o$.

An infinitary version of the conjecture holds. Define the ***tree entropy*** of μ as

$$\mathbf{h}(\mu) := \operatorname{Tr}_\mu \log\Delta = \int (\log\Delta)(o,o)\, d\mu(G,o)\,.$$

Theorem (L., 2005, 2010)

If $\mu_1 \neq \mu_2$ are unimodular probability measures on rooted weighted connected infinite graphs that both satisfy

$$\int \log w_G(o)\, d\mu_i(G, o) \in [-\infty, \infty)$$

and μ_1 stochastically dominates μ_2, then $\mathbf{h}(\mu_1) > \mathbf{h}(\mu_2)$.

This depends on another representation for tree entropy:

Theorem (L., 2010)

If μ is a unimodular probability measure on rooted weighted infinite graphs that satisfies

$$\int \log w_G(o)\, d\mu(G,o) \in [-\infty, \infty)\,,$$

then

$$\mathbf{h}(\mu) = \int_0^\infty \left(\frac{s}{1+s^2} - \int R(G,o,s)\, d\mu(G,o) \right) ds\,.$$

Here, given a network G, one of its vertices x, and a positive number s, let $R(G,x,s)$ be the effective resistance between x and ∞ in the network G^s formed from G by adding an edge of conductance s between every vertex and ∞, where ∞ is also a vertex of G^s.

This allows us to use Rayleigh's monotonicity principle pointwise.