Permutation limits

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- The space of permutations of size n will be denoted by S_n.

Viewing permutation as a measure (Hoppen et al., JCT(B) 2013)

- Given a permutation π on S_n, partition the unit square into n^2 squares of sizes $1/n$.

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- Thus we put mass on the $(i,j)^{th}$ box iff $\pi(i) = j$.

- With this definition the measure $L(\pi)$ has uniform marginals for any permutation π. Let $\mathcal{M}$ denote the set of all probability measure on $[0,1]^2$ with both marginals uniform.

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- Let $n = 3$, and let $\pi = (\pi(1), \pi(2), \pi(3)) = (1, 3, 2)$. Then the density of $L(\pi)$ is given below:

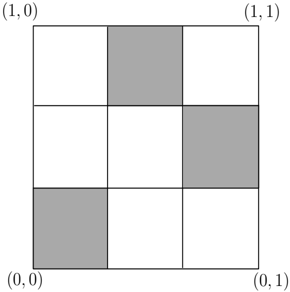

Figure: Permuton for $(1, 3, 2)$

- Here the white region has no density, and the shaded region has density 3.

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- Since $\mathcal{M}$ is closed with respect to weak topology, the limiting measure μ necessarily is in $\mathcal{M}$.
- Conversely given any measure $\mu \in \mathcal{M}$, there exists a sequence of permutations which converge to μ.
- The space $\mathcal{M}$ containing all permutations and their limits is a compact metric space.
- We will refer to $\mathcal{M}$ as the space of permutons.

Convergence of random permutations

- If $\{\pi_n\}_{n\geq 1}$ is a sequence of random permutations, then say that
$$\pi_n \xrightarrow{d} \mu,$$
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- The above definition is equivalent to:
$$[L(\pi_n)(f_1), \cdots, L(\pi_n)(f_k)] \xrightarrow{d} [\mu(f_1), \cdots, \mu(f_k)]$$
for every finite collection of continuous function $(f_1, \cdots, f_k)$ on the unit square.

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for every finite collection of continuous function $(f_1, \cdots, f_k)$ on the unit square.

- As a first example, if π_n is the identity permutation on S_n (i.e. $(\pi_n(i) = i)$ for $i \in [n]$), then π_n converges to the uniform distribution on the diagonal $x = y$.

Convergence of random permutations

- Similarly, if π_n is the reverse of the identity permutation on S_n (i.e. $(\pi_n(i) = n + 1 - i)$ for $i \in [n]$), then π_n converges to the uniform distribution on the other diagonal $x + y = 1$.

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- Similarly, if π_n is the reverse of the identity permutation on S_n (i.e. $(\pi_n(i) = n + 1 - i)$ for $i \in [n]$), then π_n converges to the uniform distribution on the other diagonal $x + y = 1$.

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- Next we give an example where the limiting measure μ is itself random.

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- In this case π_n converges in distribution to the random measure $F(U)$, where $U \sim U[0, 1]$ and $F : [0, 1] \mapsto \mathcal{M}$ is a deterministic mapping, defined below:

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- Given $s \in [0, 1]$, set $F(s)$ to be the joint law of the pair of random variables $\Big(V, s + V(\text{mod } 1)\Big)$, where $V \sim U[0, 1]$.

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- In fact, all the interesting examples of permutation limits come from sequence of permutations of growing size, as a convergent sequence of permutations $\{\pi_n\}_{n\geq 1}$ with bounded size is eventually constant (Hoppen et al., JCT(B) 2013).

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- In fact, all the interesting examples of permutation limits come from sequence of permutations of growing size, as a convergent sequence of permutations $\{\pi_n\}_{n\geq 1}$ with bounded size is eventually constant (Hoppen et al., JCT(B) 2013).
- Hence, throughout the rest of the talk we will assume that for any sequence $\{\pi_n\}_{n\geq 1}$ we have $\pi_n \in S_n$, and sometimes omit the subscript n.

Representing π as a discrete measure

- Define the discrete measure $\widetilde{L}(\pi)$ on the unit square by

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\left(\frac{i}{n},\frac{\pi(i)}{n}\right)}.$$

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- The two measures $L(\pi)$ and $\widetilde{L}(\pi)$ are close in the Kolmogorov-Smirnov distance, i.e.

$$\sup_{x,y\in[0,1]^2}\left|F_{L(\pi)}(x,y)-F_{\widetilde{L}(\pi)}(x,y)\right|\le\frac{2}{n}.$$

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$$\sup_{x,y\in[0,1]^2}\left|F_{L(\pi)}(x,y)-F_{\widetilde{L}(\pi)}(x,y)\right|\le\frac{2}{n}.$$

- Thus for most limiting operations, without loss of generality we can switch between $L(\pi)$ and $\widetilde{L}(\pi)$.

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  - Gibbs random permutation (Borga, Das, M., Winkler, IMRN 2023).

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- Focusing on the case when π_n is uniformly random on S_n, it will study detailed properties of the degree sequence of the permutation graph G_{π_n}.

- It will then explain how these results apply to the four examples mentioned above.

Part A: General results

A general estimate for “regular distributions”

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$$\frac{\mathbb{P}_n(\pi_n(i) = j)}{\mathbb{P}_n(\pi_n(i') = j')} = 1 + o(1) \text{ if } |i - i'| + |j - j'| = o(n).$$

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- Also assume that $\mathbb{P}_n$ is regular, in the sense that $\mathbb{P}_n(\pi_n(i) = j)$ is "jointly continuous" in $i, j \in n$.

- Under these two assumptions, we have (M., EJP 2016)

$$\mathbb{P}_n(\pi_n(i) = j) \approx \frac{1}{n}\rho(i/n, j/n).$$

Distribution of the permutation process

- In this case, for any $t \in (0,1]$ we have

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- In words, $Z(t)$ has the conditional law of Y given $X = t$, where $(X,Y) \sim \rho$.

- Also, for any finite collection of distinct numbers $t_1, t_2, \cdots, t_k \in (0,1]$, the components of the random vector

$$\frac{1}{n}(\pi_n(nt_1), \cdots, \pi_n(nt_k))$$

  are asymptotically mutually independent.

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- Conversely, convergence of permutation process implies convergence of permutation (no regularity required).
- Thus convergence of permutation process is a (slightly) stronger requirement than convergence of permutations.
- It is not hard to construct examples where permutation converges but process does not.
- However, since most common distributions are regular, these two are essentially equivalent.

Asymptotics of fixed points

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- Also the events $\{\pi_n(i) = i\}_{i\in[n]}$ are approximately independent.

- Thus $N_n(\pi_n)$ has an asymptotic Poisson distribution with mean

$$\int_0^1 \rho(x,x)dx.$$

(M., EJP 2016).

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- In this case $\{\pi_n\}_{n\geq 1}$ converges in probability to u, the uniform distribution on $[0, 1]^2$, for all θ.

- However, $N_n(\pi_n)$ converges in distribution to Poisson with mean e^{θ}, which depends on θ.

Comment about the regularity assumption

- In this case there is a natural bijection ϕ between the sets

$$A := \{\pi_n \in S_n : \pi_n(1) = 1, \pi_n(2) = 2\}$$
$$B := \{\pi_n \in S_n : \pi_n(1) = 2, \pi_n(2) = 1\}$$

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- Consequently $\mathbb{P}_{n,\theta}$ is not regular.

- Thus some regularity of the distribution is necessary for the number of fixed points to be determined by the limiting permuton.

Number of cycles of a given length

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- This generalizes the classical result for uniformly random permutations, which corresponds to the choice $\rho \equiv 1$.

Permutation Graphs

- Given a permutation $\pi \in S_n$, define a labeled graph G_π with vertex set $[n]$, and edges given by

$$\begin{aligned} G_\pi(i,j) =& 1 \text{ iff } (i-j)(\pi(i)-\pi(j)) < 0, \\ =& 0 \text{ otherwise.} \end{aligned}$$

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- Also the degree d_i of a vertex i in G_π is the number of inversions containing i.

Connection with dense graph limits

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- However, if π_n converges to a measure with density $\rho(x, y)$, then π_n^{-1} converges to a measure with density $\rho(y, x)$.

- Thus if a sequence π_n converges to a density $\rho(., .)$ which is not symmetric, the sequence which alternates between π_n and π_n^{-1} provides a counter example.

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- A natural question is behavior of the degree sequence, when π is chosen from some distribution.
- Surprisingly, this question has not been studied before in detail, even for the case when π is uniformly random.
- We will now study the degree sequence when π_n is from some regular distribution $\mathbb{P}_n$, under the assumption that $\{\pi_n\}_{n\geq 1}$ converges in probability to a non random measure μ with continuous density ρ.

Degree sequence of permutation graphs (Bhattacharya-M., AAP 2017)

- For any $t \in (0,1]$ we have

$$\frac{1}{n} d_{nt} \xrightarrow{d} D(t) := t + Z(t) - 2F_\rho(t, Z(t)).$$

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- Finally, the empirical degree distribution converges:

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\frac{d_i}{n}} \stackrel{d}{\to} U + V - 2F_\rho(U, V),$$

where $(U,V) \sim \rho$.

Part B: Degree sequence of G_π for π uniform

Uniform distribution on S_n

- In this case $\rho \equiv 1$, and so

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\frac{d_i}{n}} \xrightarrow{p} \mathcal{L}(U+V-2UV)$$

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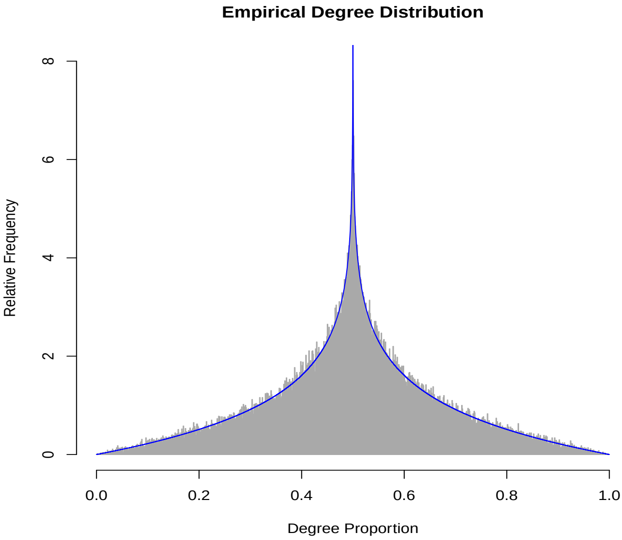
- This density vanishes at $x = 0, 1$, and blows up at $x = .5$.

Empirical degree distribution for $n = 10,000$



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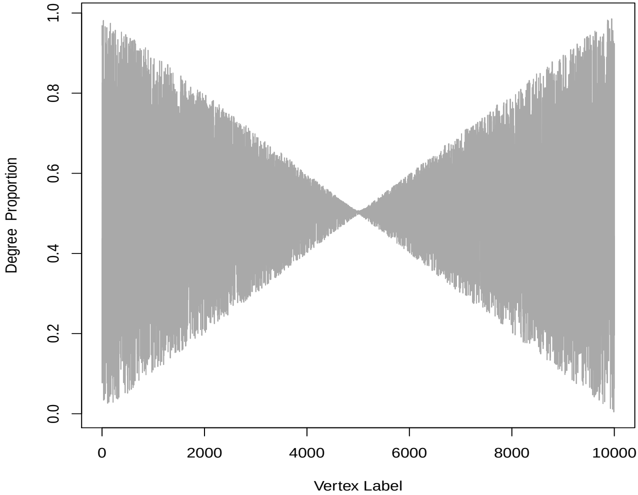
- Also setting $t = .5$ we have $D(.5) = .5$, and so

$$\frac{1}{n} d_{n/2} \stackrel{p}{\to} \frac{1}{2}.$$

Degree sequence for $n = 10,000$



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- We also show that $\frac{1}{\sqrt{n}} \min_{i \in [n]} d_i$ converges in distribution to a Rayleigh distribution.

Part C: Examples

Conditions to verify

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- The regularity condition on $\mathbb{P}_n$ has to be verified on a case by case basis, but is usually easy.
- The hard part is usually to verify that $\{\pi_n\}_{n \geq 1}$ indeed converges, and finding the limiting density ρ.

Handy tool: LDP for permutations

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- Then $L(\pi_n)$ (or $\widetilde{L}(\pi_n)$) satisfies a large deviation principle on the space $\mathcal{M}$ with speed n and the good rate function $I(\mu) = D(\mu||u)$, where $D(.||.)$ is the Kullback Leibler divergence (Trashorras, JTP 2008; M., AoS 2016; Kenyon-Kral-Radin-Winkler, RSA 2020).

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- By Varadhan's Lemma, if $T : \mathcal{M} \mapsto \mathbb{R}$ is a bounded continuous function, then we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{n\theta T(\widetilde{L}(\pi_n))} = \sup_{\mu \in \mathcal{M}} \{\theta T(\mu) - D(\mu||u)\}.$$

Exponential family on permutations

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- This framework can be used to deduce convergence of random permutations in exponential families.

Eg: Mallows model with Kendall's Tau

- Mallows model with Kendall's Tau is a one parameter exponential families on S_n with p.m.f.

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- This model was first introduced by C. Mallows in 1957, and has been widely studied in Statistics and Probability literature.

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- Also this model has nice independence properties built in, which makes it tractable for theoretical analysis.
- The law of large numbers for the LIS for this model has been established in Mueller-Starr, JTP 2013.
- However, the behavior of almost every other permutation statistics for this model was unknown.

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- This fits our generalized exponential family framework for the choice

$$T(\widetilde{L}(\pi_n)) := \frac{1}{n^2} \sum_{1 \le i < j \le n} 1\{(i-j)(\pi_n(i) - \pi_n(j)) < 0\}$$

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- Here the set A is

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=& \frac{1}{2n^2} \sum_{i,j=1}^{n} 1\{(i-j)(\pi_n(i) - \pi_n(j)) < 0\} \\
=& \frac{1}{2} (\widetilde{L}(\pi_n) \times \widetilde{L}(\pi_n))(A).
\end{aligned}
$$

- Here the set A is

$$
\{(x_1, y_1, x_2, y_2) \in [0,1]^4 : (x_1 - x_2)(y_1 - y_2) < 0\}.
$$

- Thus one only needs to check that the map $\mu \mapsto (\mu \times \mu)(A)$ is continuous with respect to weak topology on $\mathcal{M}$.

Eg: Mallows model with Kendall's Tau

- The boundary of A is

$$\partial A := \{(x_1, y_1, x_2, y_2) \in [0,1] : x_1 = x_2 \text{ or } y_1 = y_2\}.$$

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- This was shown in Starr, JMP 2009, where the author also finds an explicit density for the optimizing measure.

Eg: Exponential family with Linear Statistics

- Consider an exponential family on S_n with the p.m.f.

$$\mathbb{P}_{n,\theta}(\pi) = \frac{1}{Z_n(\theta)} e^{\theta \sum_{i=1}^n f(i/n, \pi(i)/n)}.$$

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- This class of models based on linear Statistics was introduced in M., AoS 2016 to study the above examples in a unified framework.

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- This fits our exponential family framework with

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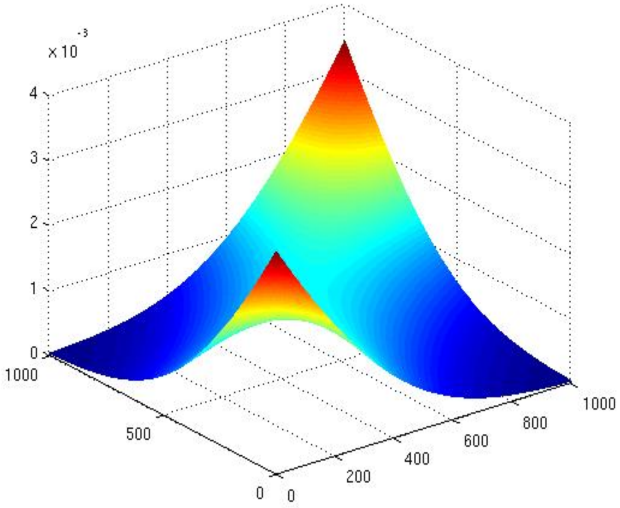
- There is an iterative Sinkhorn type algorithm to compute the density of the optimizing measure in M., AoS 2016, but no closed form formula is known for the density in this case.

Limiting density for $f(x, y) = -(x - y)^2, \theta = 10$



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- Given a measure $\mu \in \mathcal{M}$ with a continuous density ρ, let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be i.i.d. random vectors from ρ.

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- Given a measure $\mu \in \mathcal{M}$ with a continuous density ρ, let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be i.i.d. random vectors from ρ.
- Suppose there exists a pair (X_l, Y_l) such that $X_l = X_{(i)}$ and $Y_l = Y_{(j)}$, then set $\pi_n(i) = j$.

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- In this case, it was shown in Hoppen et al., JCT(B) 2013 that π_n converges in probability to μ, and so all our results apply.

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- We call π_n as the Gibbs random permutation, noting that its law depends on θ,μ (and the fact that we chose inversions in the Gibbs Hamiltonian).

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- Also (as indicated above), one can replace the Hamiltonian by general sub permutation counts.

LDP for μ random measures (Borga-Das-M.-Winkler, IMRN 2023)

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- In particular if $\mu = u$, then the minimizer is $\gamma = \nu$, thus giving $I_u(\nu) = D(\nu|u)$ (LDP rate function for uniform permutations).

Application (Borga-Das-M.-Winkler, IMRN 2023)

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- Consequently permutation convergence follows, and all our results apply.

- We give an example to show a sharp phase transition, i.e. uniqueness for θ small and non uniqueness for θ large.

- Thus, for θ large, there may or may not be a unique maximizer, and convergence of permutations is not guaranteed.

Conclusions

Future Scope

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  How to show uniqueness? Also, there are many other models that people care about.

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  The explicit limiting density for the Mallows model with Kendall's Tau was computed by Starr, JMP 2009.

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- A third direction is to bring more permutation statistics under this approach.

  LIS seems a very good candidate.

Thank you!