Analytic closure for M1 neutrino transport

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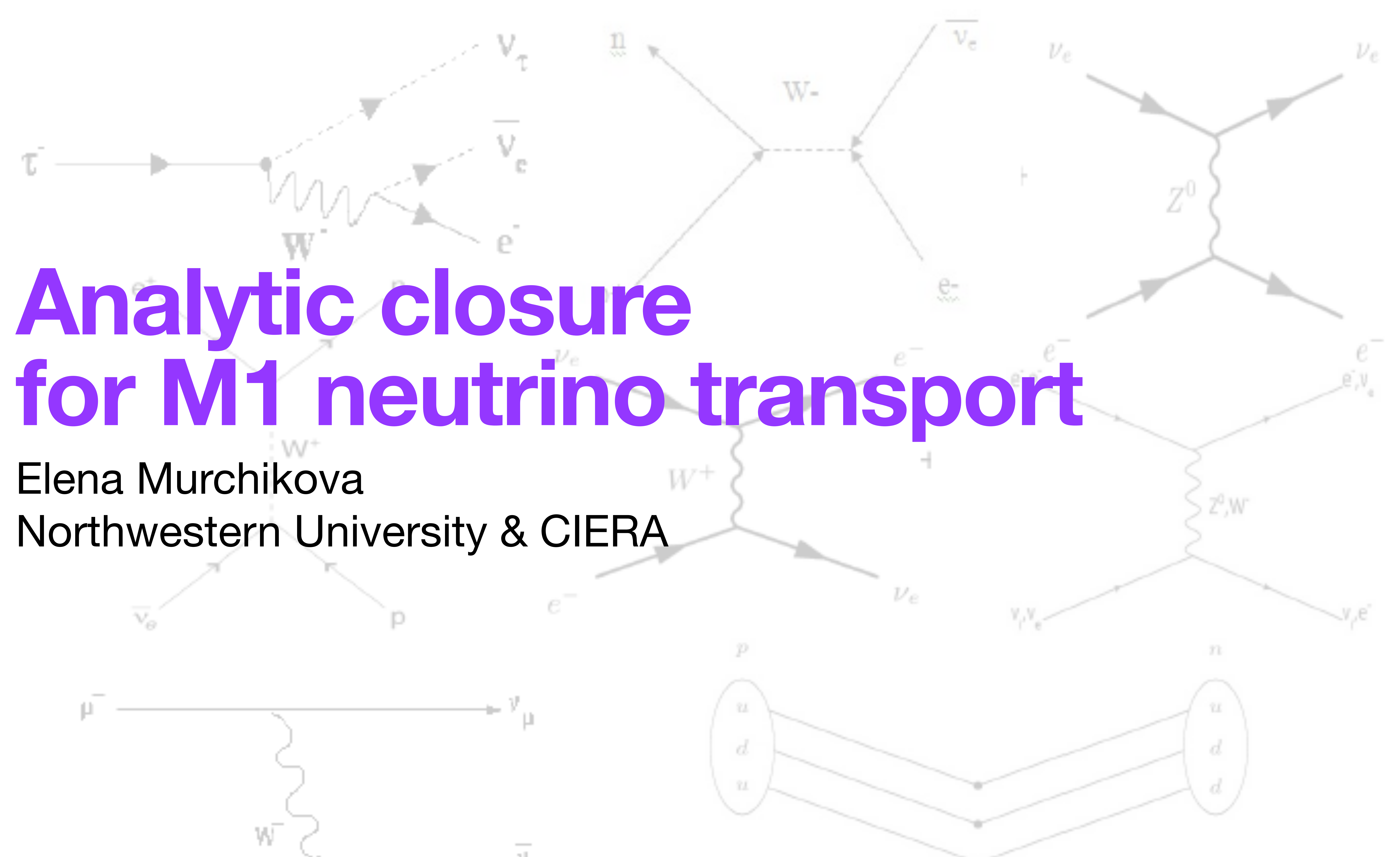

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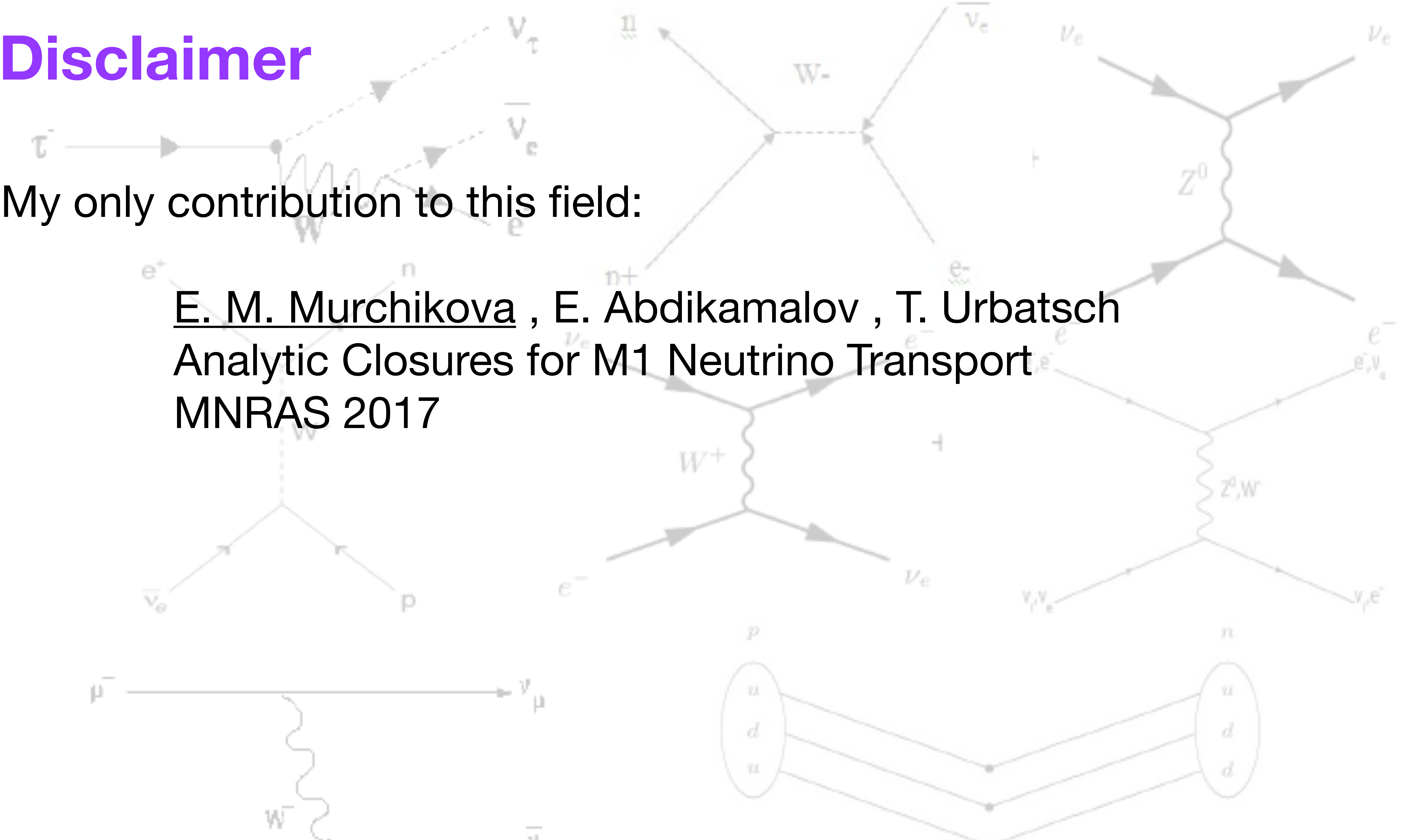
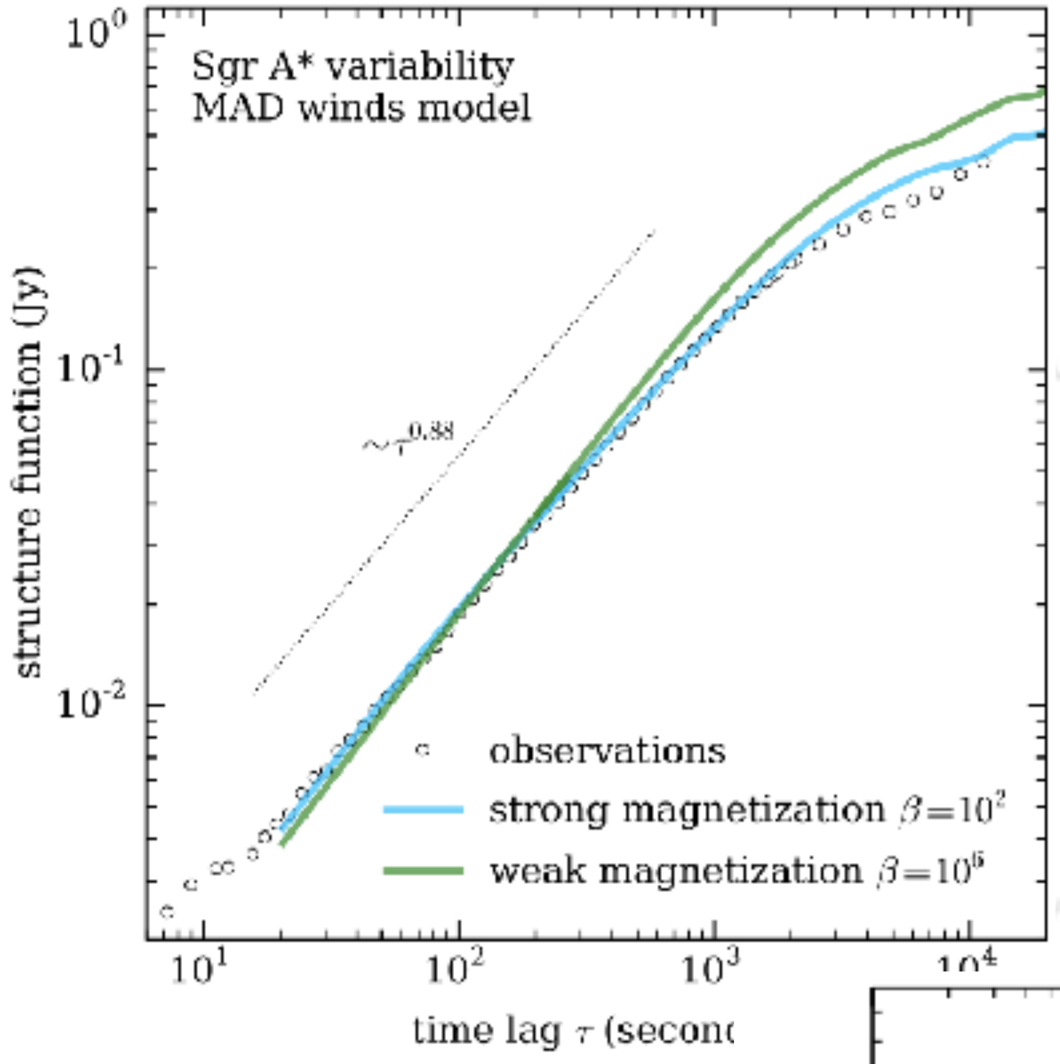


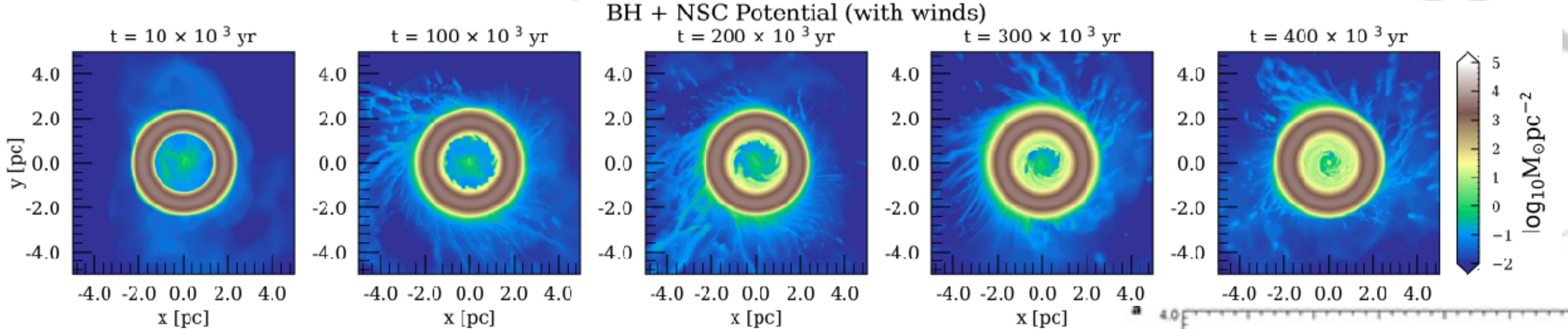
My only contribution to this field:

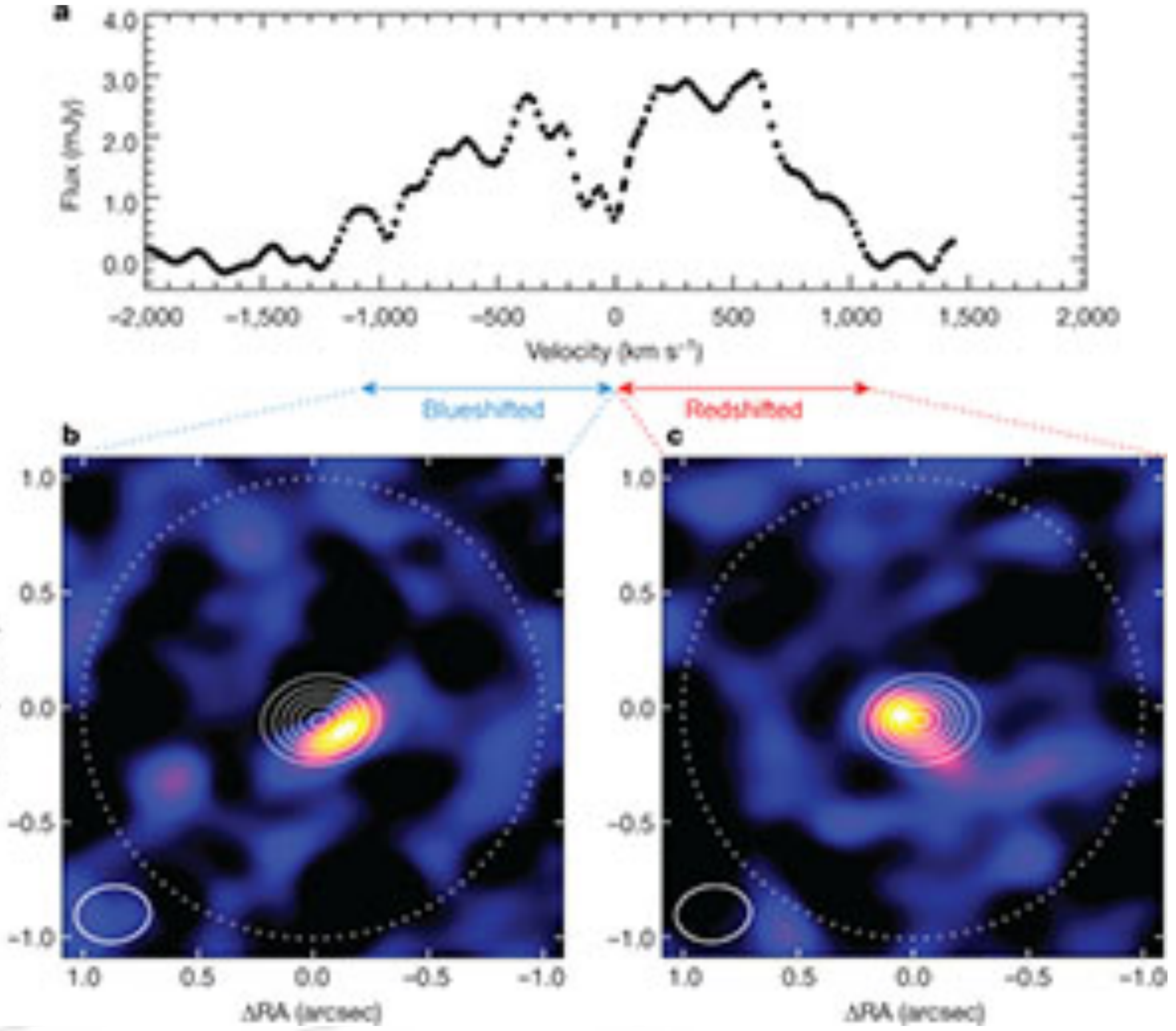
E. M. Murchikova , E. Abdikamalov , T. Urbatsch
Analytic Closures for M1 Neutrino Transport
MNRAS 2017

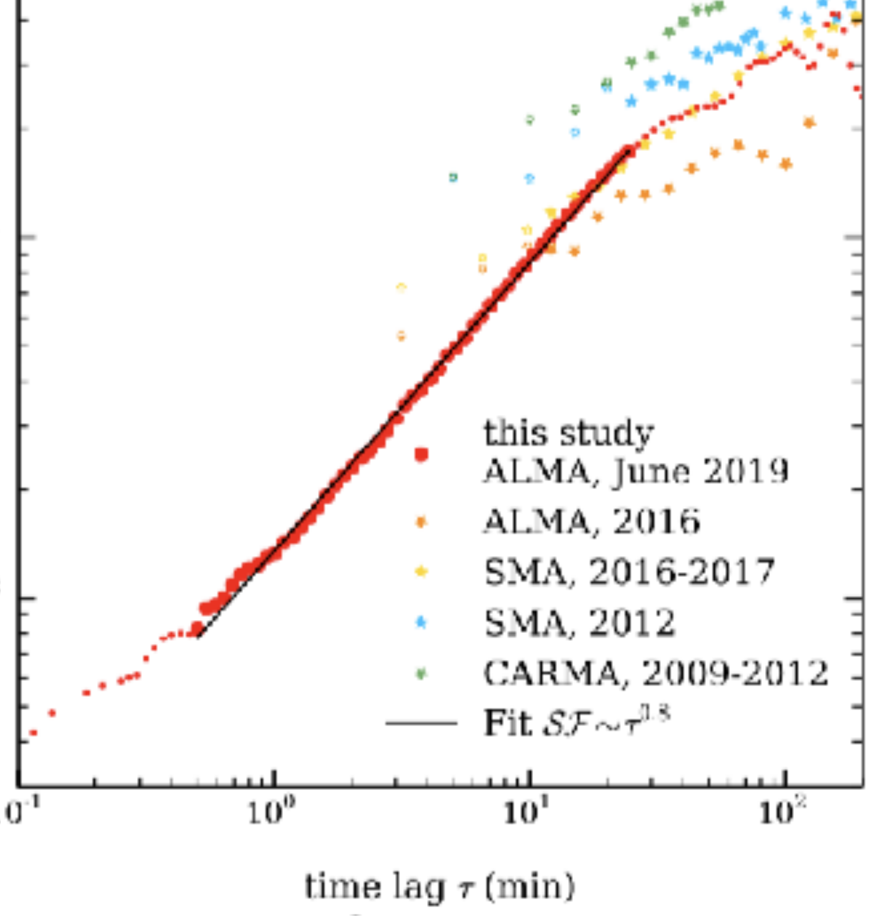
Disclaimer

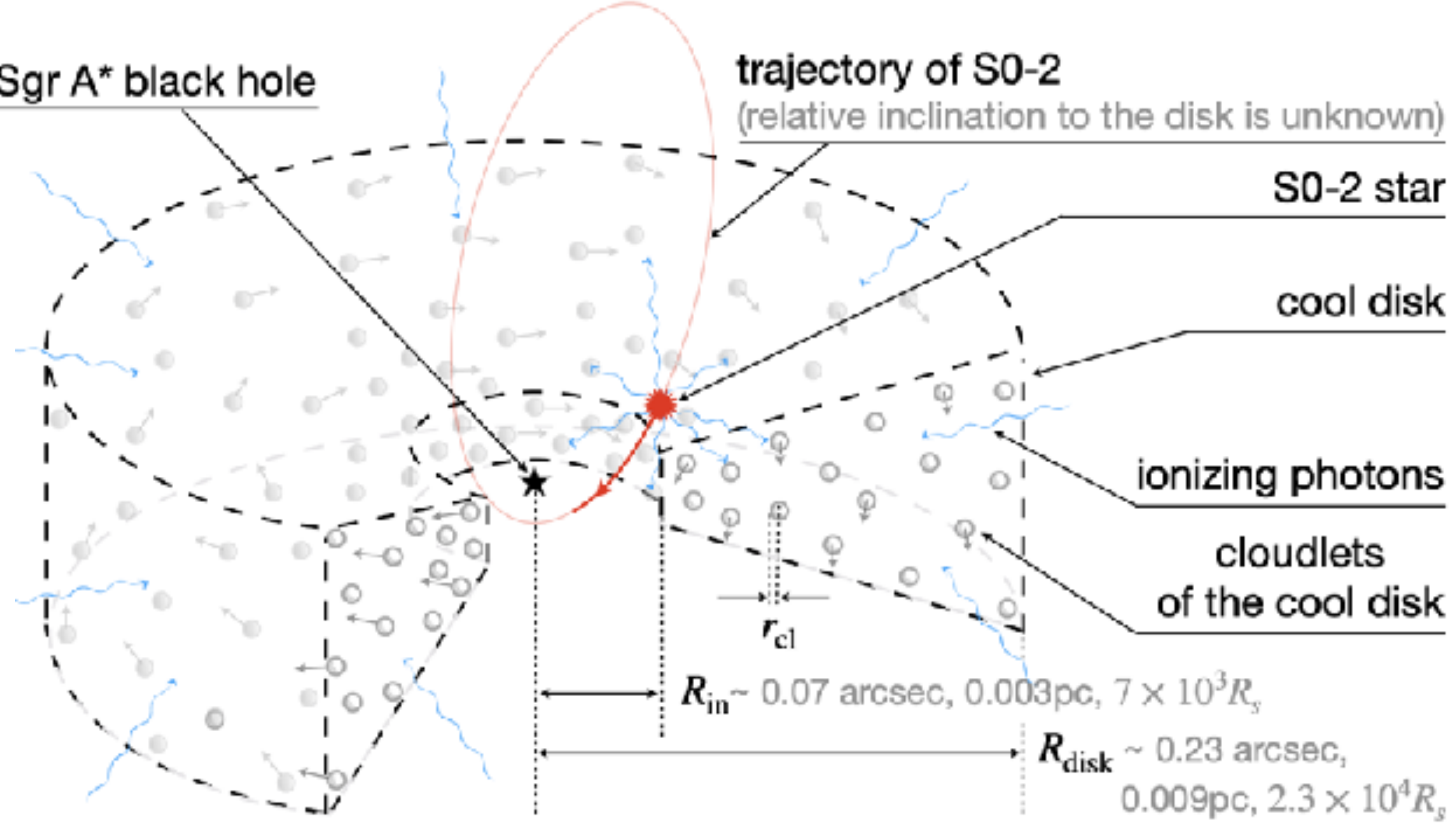
Actually I work on black holes, accretion, and variability doing theory, and ALMA observations…











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MNRAS 2017

with Ernazar Abdikamalov
(Nazarbayev University, Kazakhstan)

Boltzmann equation

$$\frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha} + \frac{dp^\alpha}{d\tau}\frac{\partial f}{\partial p^\alpha} = \varepsilon\, S(x^\mu, p^\mu, f)$$

where

$f(x^\mu, p^\mu)$	is neutrino distribution function,
$S(x^\mu, p^\mu, f)$	is collision term,
ε	is neutrino energy,
p^μ	is momentum,
τ	is affine parameter,

$\alpha, \mu = 0,1,2,4$

Boltzmann equation

$$\int \times \quad \Big| \Big| \frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha} + \frac{dp^\alpha}{d\tau}\frac{\partial f}{\partial p^\alpha} = \varepsilon\, S(x^\mu, p^\mu, f) \Big| \Big| \quad \times p^\alpha \ldots \delta(h\nu - \varepsilon) d^3p$$

can be rewritten in terms of moments

$$M[0] = \int \varepsilon f(x^\mu, p^\mu)\delta(h\nu - \varepsilon)d^3p = E_\nu$$ energy density

$$M[1] = \int p^\alpha f(x^\mu, p^\mu)\delta(h\nu - \varepsilon)d^3p = F_\nu{}^\alpha$$ flux

$$M[2] = \int p^\alpha p^\beta f(x^\mu, p^\mu)\, \delta(h\nu - \varepsilon)\frac{d^3p}{\varepsilon} = P_\nu{}^{\alpha\beta}$$ pressure tensor

…

$$M[N] = \int \varepsilon^2 \frac{p^{\alpha_1}}{\varepsilon} \ldots \frac{p^{\alpha_N}}{\varepsilon} f(x^\mu, p^\mu)\, \delta(h\nu - \varepsilon)\frac{d^3p}{\varepsilon}$$

…

Moment equations

Instead of Boltzmann equation

$$\int \times \left|\left| \frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha} + \frac{dp^\alpha}{d\tau}\frac{\partial f}{\partial p^\alpha} = \varepsilon\, S(x^\mu, p^\mu, f) \right|\right| \times p^\alpha \ldots \delta(h\nu - \varepsilon) d^3p$$

we get an infinite tower of moment equations

Diff Eq $(M[0], M[1])$

Diff Eq $(M[0], \ldots, M[2])$

Diff Eq $(M[0], \ldots, M[3])$

…

Diff Eq $(M[0], \ldots, M[N+1])$

…

Infinite set

More explicitly and in 1D

These equations look like

$$\frac{\partial E_\nu}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_\nu{}^r) = S[0]$$

$$\frac{\partial F_\nu{}^r}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_\nu{}^{rr}) = S[1]$$

Diff Eq $(M[0], \ldots, M[3])$

…

Diff Eq $(M[0], \ldots, M[N+1])$

…

Truncating the system for M1 scheme

We only keep two equations

$$\frac{\partial E_\nu}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 {F_\nu}^r) = S[0]$$

$$\frac{\partial {F_\nu}^r}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 {P_\nu}^{rr}) = S[1]$$

~~Diff Eq $(M[0], \ldots, M[3])$~~

~~…~~

~~Diff Eq $(M[0], \ldots, M[N+1])$~~

~~…~~

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Two equations
Three unknowns

Closing the system

$$\frac{\partial E_\nu}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_\nu^{\ r}\right) = S[0]$$

$$\frac{\partial F_\nu^{\ r}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_\nu^{\ rr}\right) = S[1]$$

Need one more equation to close the system. A function of $E_\nu, F_\nu^{\ r}, P_\nu^{\ rr}$.

We choose it to be

$P_\nu^{\ rr} = \mathrm{Function}(E_\nu, F_\nu^{\ r})$ ← **Closure**

Set of equations for M1 scheme in 1D

$$\frac{\partial E_\nu}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_\nu{}^r) = S[0]$$

$$\frac{\partial F_\nu{}^r}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_\nu{}^{rr}) = S[1]$$

$$P_\nu{}^{rr} = \text{Function}(E_\nu, F_\nu{}^r)$$

Three equations
Three unknowns

← **Closure**

Set of equations for M1 scheme in 1D

(The highest moment we are truly solving for is M1)

$$\frac{\partial E_\nu}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_\nu{}^r) = S[0]$$

$$\frac{\partial F_\nu{}^r}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_\nu{}^{rr}) = S[1]$$

$$P_\nu{}^{rr} = \text{Function}(E_\nu, F_\nu{}^r)$$

Three equations
Three unknowns

←

Closure

It is obvious that
the quality of the solution depends on the quality of the closure,
i.e. the quality of the extra equation we supplement the system with.

Common in literature expressions for closure

Kershaw

$$p = \frac{1}{3} + \frac{2}{3} f^2$$

Wilson

$$p = \frac{1}{3} - \frac{1}{3} f + f^2$$

Levermore

$$p = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f^2}}$$

Maximum Entropy (ME)

$$p = \frac{1}{3} + \frac{2f^2}{15}(3 - f + 3f^2)$$

Maximum Entropy Fermi-Dirac (MEFD)

$$p = \frac{1}{3} + \frac{2}{3}(1 - e)(1 - 2e)\frac{x^2(3 - x + 3x^2)}{5}, \quad x = \frac{f}{1 - e}$$

Janka 1

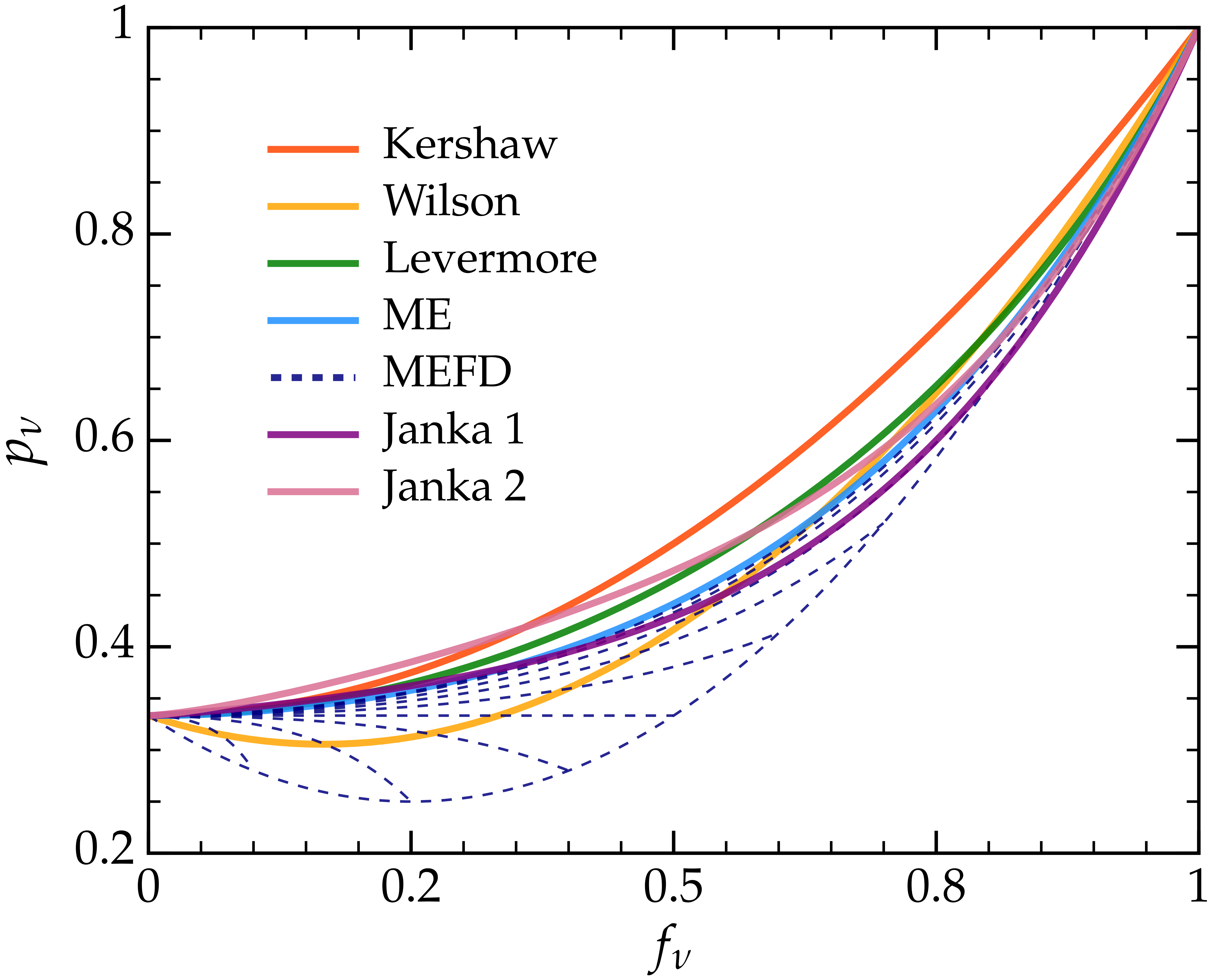
$$p = \frac{1}{3}(1 + \frac{1}{2} f^{1.3064} + \frac{3}{2} f^{4.1342})$$

Janka 2

$$p = \frac{1}{3}(1 + f^{1.3450} + f^{5.1717})$$

Here $e = \frac{E_\nu}{\nu^3}, f = \frac{F_\nu{}^r}{E_\nu}, p = \frac{P_\nu{}^{rr}}{E_\nu}$

Common in literature expressions for closure



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Common properties of the closures

In 1D:

$$F_\nu = 0 \quad \Rightarrow \quad P_\nu = \frac{1}{3} E_\nu$$

$$F_\nu = 1 \quad \Rightarrow \quad P_\nu = E_\nu$$

In higher dimensions:

$$P_\nu^{ij} = \frac{3p-1}{2} P^{ij}_{thin} + \frac{3(1-p)}{2} P^{ij}_{thick},$$

where $P^{ij}_{thick} = \frac{1}{3} E_\nu \delta^{ij}$, and

in thin limit the radiation exerts pressure only along the direction of the beam (n), i.e. $F^n_\nu = E_\nu$, $F^{i \neq n}_\nu = 0$, and

$P^{nn}_{thin} = E_\nu \frac{F^n_\nu F^n_\nu}{|F^2_\nu|}$, $P^{ij}_{thin} = 0$ if i or $j \neq n$.

Properties of the closures

In 1D:

$$F_\nu = 0 \quad \Rightarrow \quad P_\nu = \frac{1}{3} E_\nu$$

$$F_\nu = 1 \quad \Rightarrow \quad P_\nu = E_\nu$$

In higher dimensions:

whe

in th ssure only along the direction of the

beam $_\nu = E_\nu$, $F_\nu^{i\neq n} = 0$, and

$P^{nn}_{thin} = E_\nu \dfrac{F^n_\nu F^n_\nu}{|F^2_\nu|}$, $P^{ij}_{thin} = 0$ if i or $j \neq n$.

Inability of M1 to describe colliding beams,
or exerting pressure along the beam only
are not the issues of M1 scheme.

They are issues of our choices of closures.

Closures obtained as fits to simulations

Kershaw

$$p = \frac{1}{3} + \frac{2}{3} f^2$$

Maximum Entropy Fermi-Dirac (MEFD)

$$p = \frac{1}{3} + \frac{2}{3}(1-e)(1-2e)\frac{x^2(3-x+3x^2)}{5}, \quad x = \frac{f}{1-e}$$

Wilson

$$p = \frac{1}{3} - \frac{1}{3} f + f^2$$

simulations

Janka 1

$$p = \frac{1}{3}\left(1 + \frac{1}{2} f^{1.3064} + \frac{3}{2} f^{4.1342}\right)$$

Levermore

$$p = \frac{3+4f^2}{5+2\sqrt{4-3f^2}}$$

Janka 2

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Maximum Entropy (ME)

$$p = \frac{1}{3} + \frac{2f^2}{15}(3 - f + 3f^2)$$

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wild guess

simulations

Maximum Entropy Fermi-Dirac (MEFD)

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Closures obtained as fits to simulations

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wild guess

simulations

physical reasons

Maximum Entropy Fermi-Dirac (MEFD)

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Janka 2

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Here $e = \dfrac{E_\nu}{\nu^3}, f = \dfrac{F_\nu^{\,r}}{E_\nu}, p = \dfrac{P_\nu^{\,rr}}{E_\nu}$

Levermore closure

Levermore

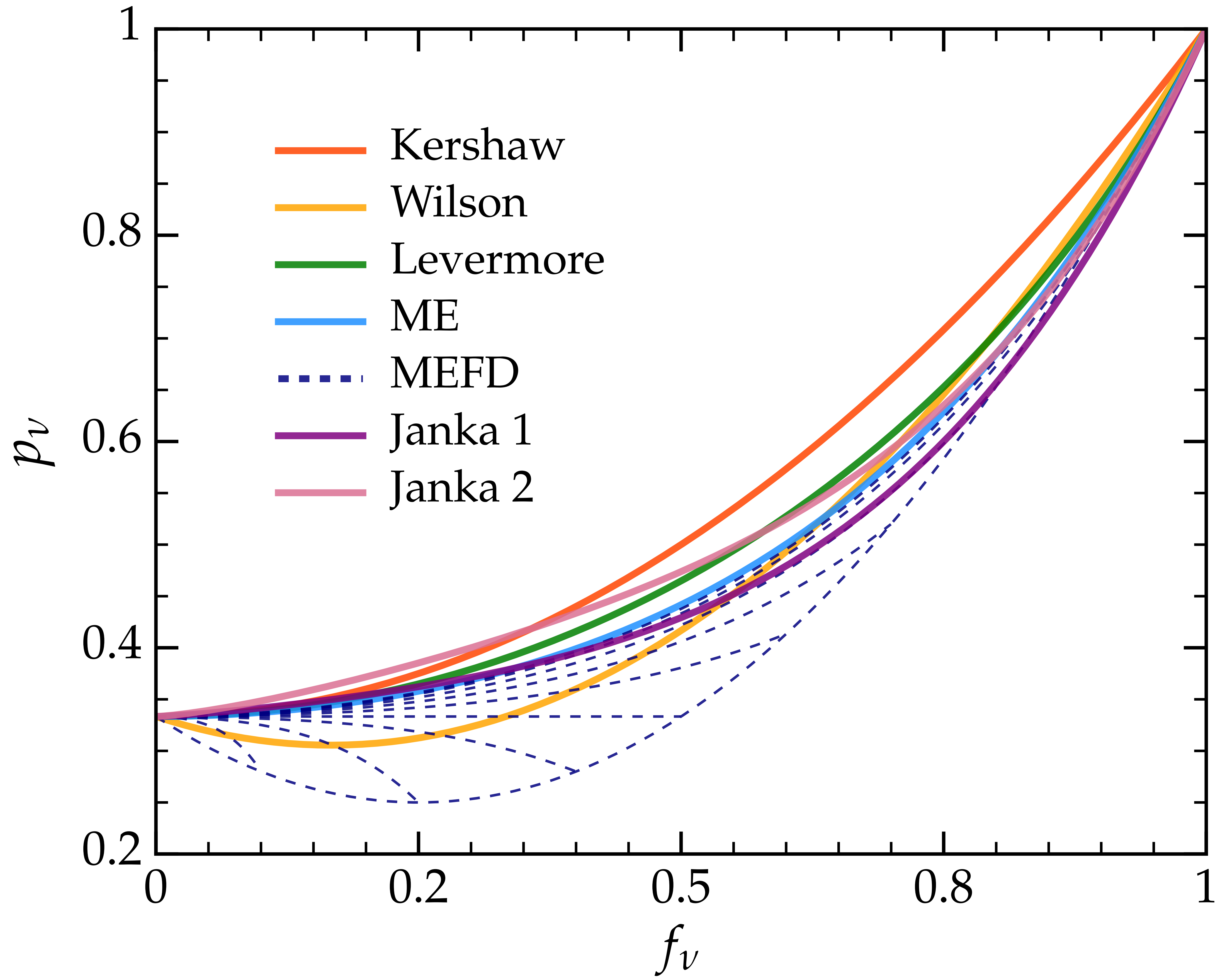
$$p = \frac{3+4f^2}{5+2\sqrt{4-3f^2}}$$

By Levermore 1984

Assumption:

In the rest frame of the fluid, i.e. the brake boosted to where $F_\nu = 0$ we have

$$P_\nu = \frac{1}{3}$$



Maximum Entropy Fermi-Dirac closure

MEFD

$$p = \frac{1}{3} + \frac{2}{3}(1-e)(1-2e)\frac{x^2(3-x+3x^2)}{5}, \quad x = \frac{f}{1-e}$$

By Chernohorsky & Bludman 1994

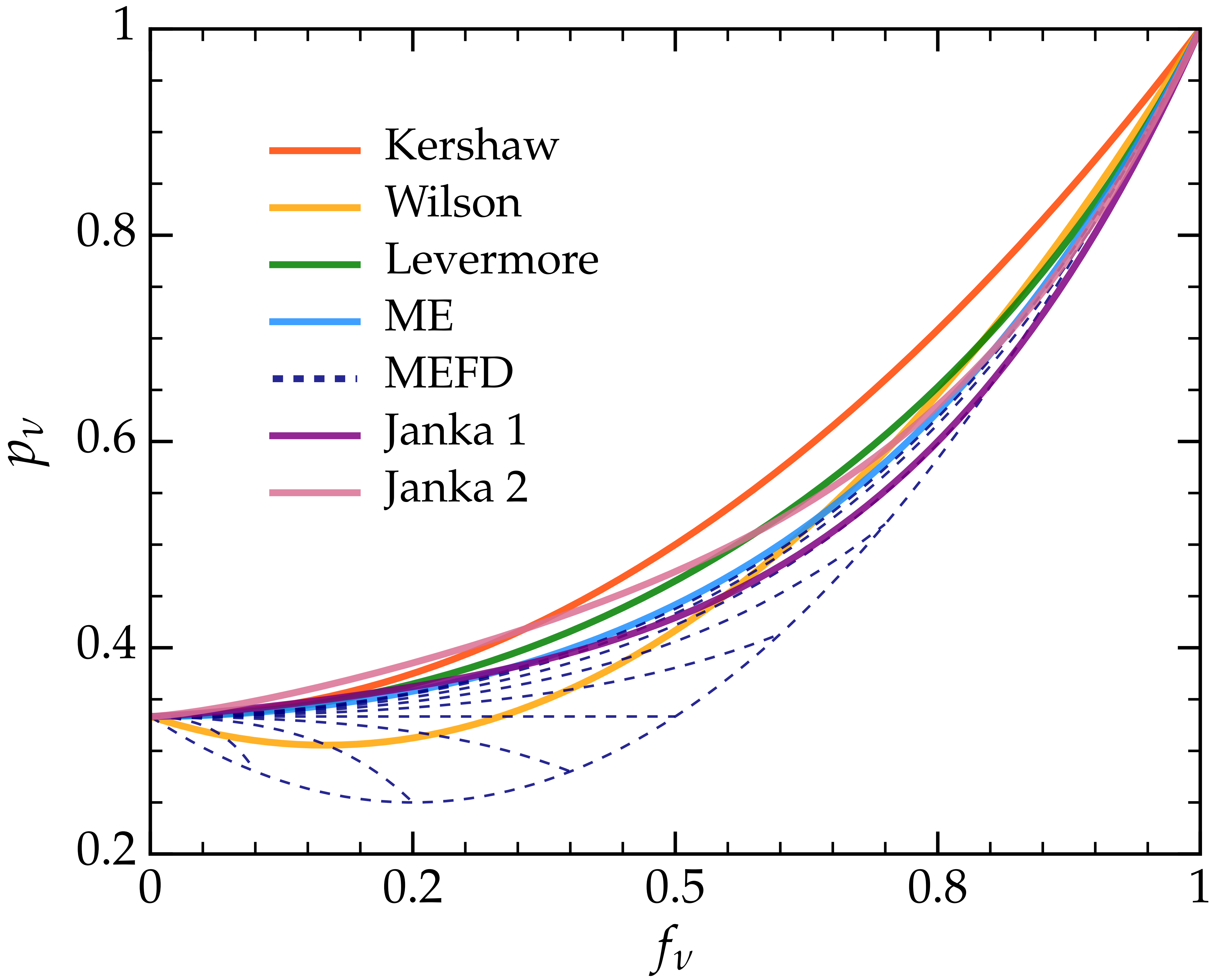
Assumption:

Entropy is maximized.
Statistics is Fermi-Dirac.

Bottom limiting curve is maximum packing limit.

Top limiting curve is where Fermi-Dirac statistics is not important.

It detents on two parameters.



Maximum Entropy closure

Maximum Entropy (ME)

$$p = \frac{1}{3} + \frac{2f^2}{15}(3 - f + 3f^2)$$

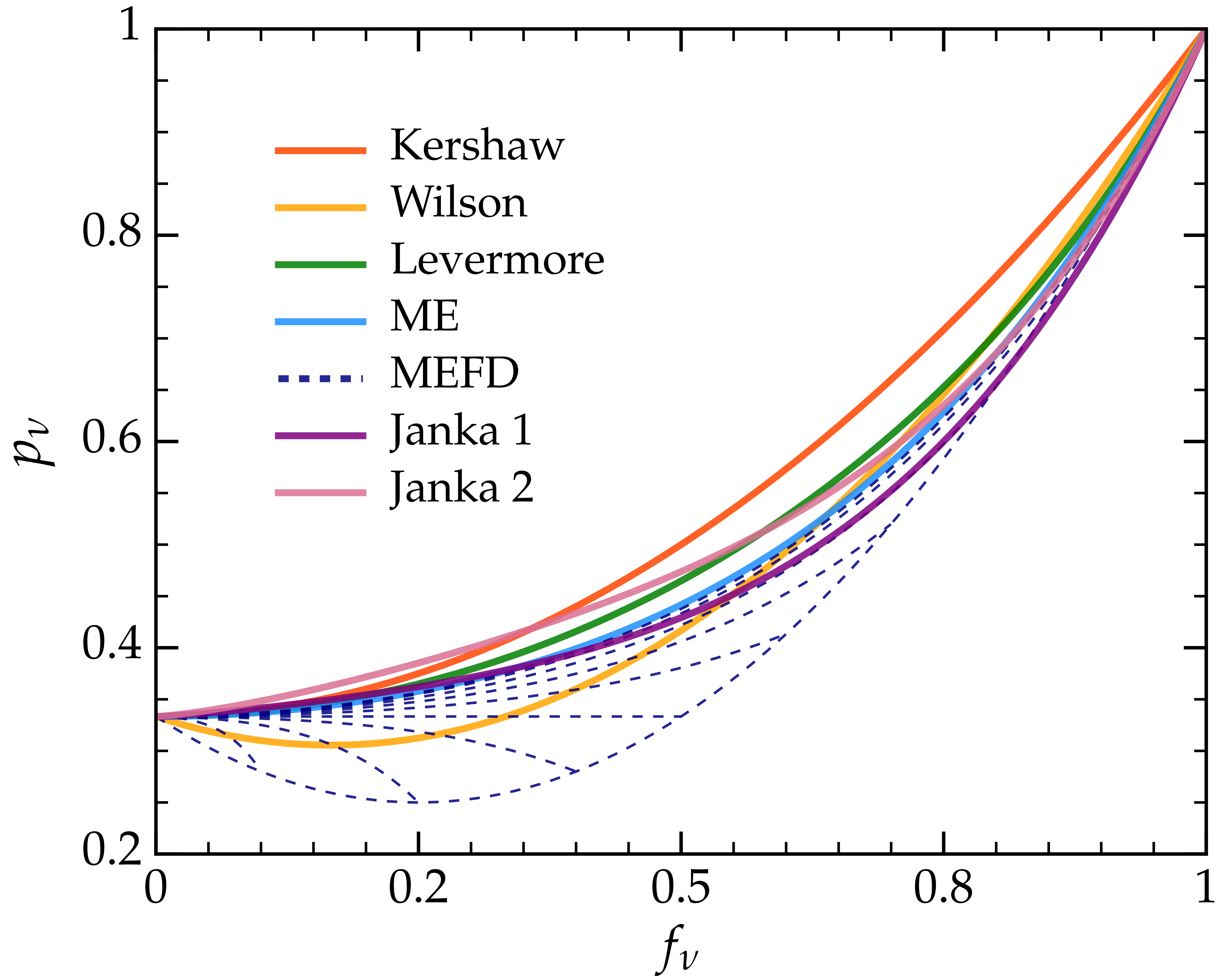
By Minerbo 1978

Assumption:

Entropy is maximized.

Boltzmann statistics.

Top limiting curve of MEFD.



Kershaw closure

Kershaw

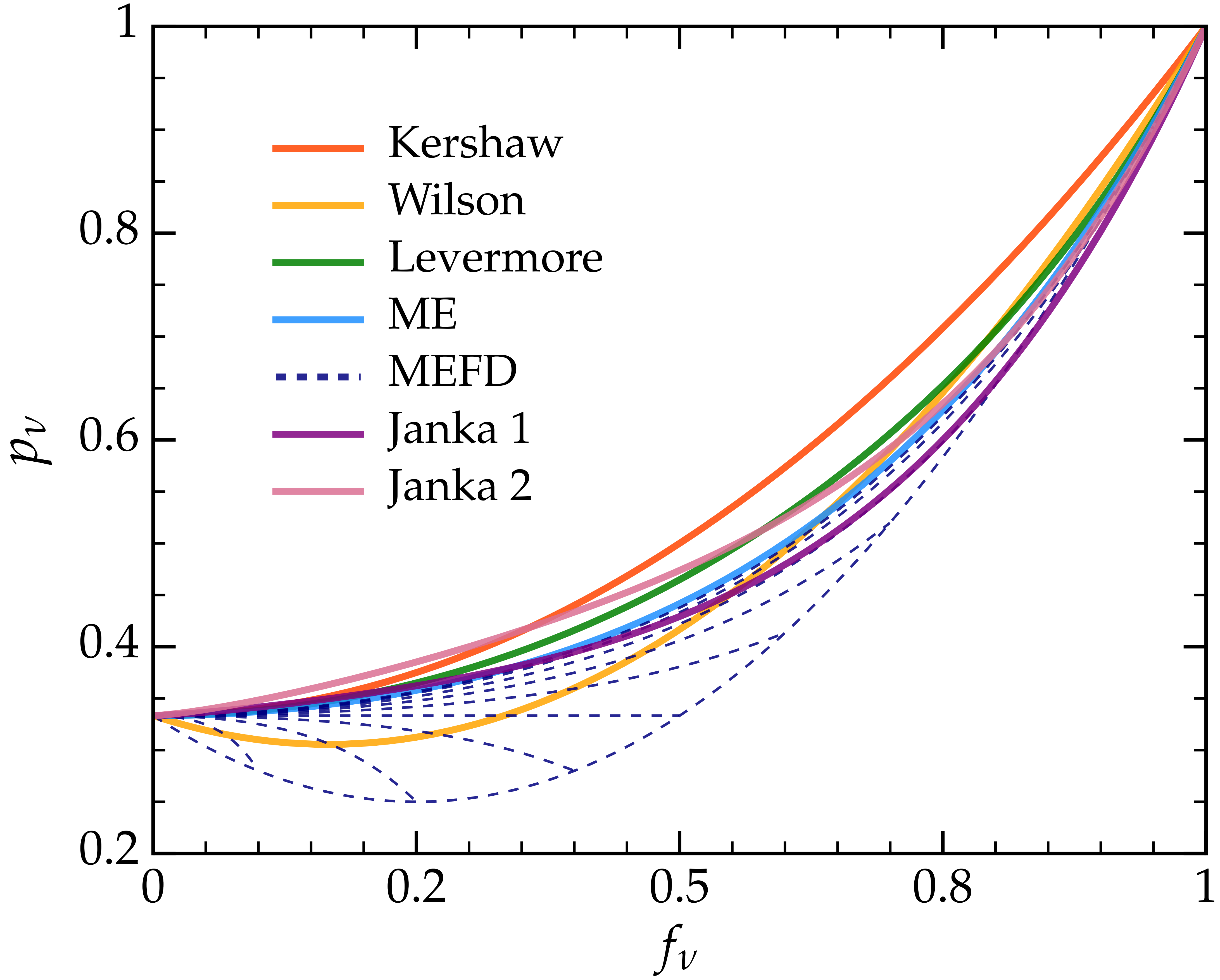
$$p = \frac{1}{3} + \frac{2}{3} f^2$$

By Kershaw 1976

Assumption:

Smooth interpolation between $\frac{1}{3}$ and 1.

Nothing else.



The purpose of the study was to test various closures and determine the best performer.

Setup:

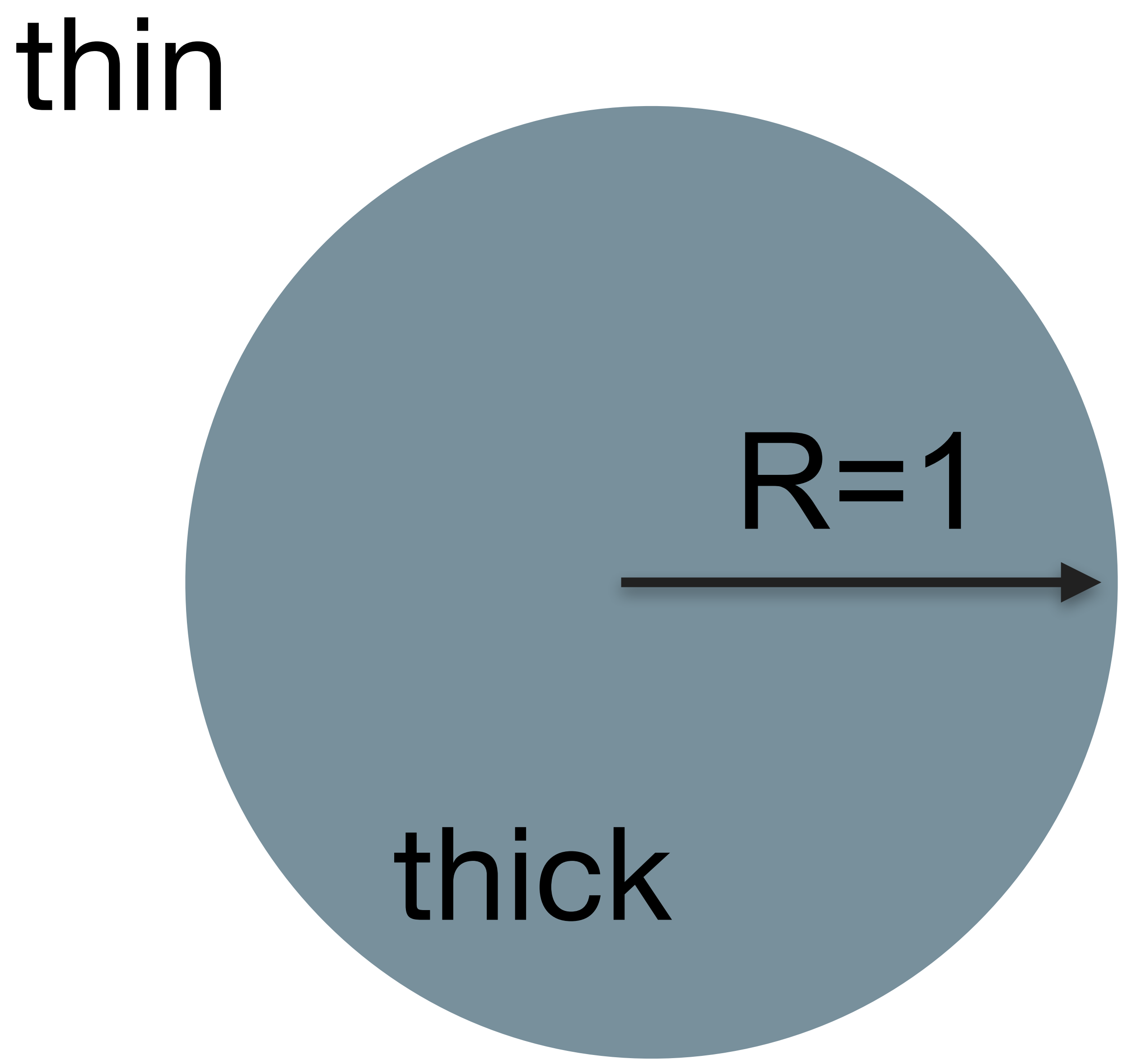
Spherically symmetry (i.e. 1D problem)
Three PNS post bounce configurations, and a uniform sphere.

GR1D with closure (O'Connor et al 2015)
vs
MC neutrino transport code (Abdikamalov et al 2012) as truth

Uniform sphere

Simplest test

Has analytic solution

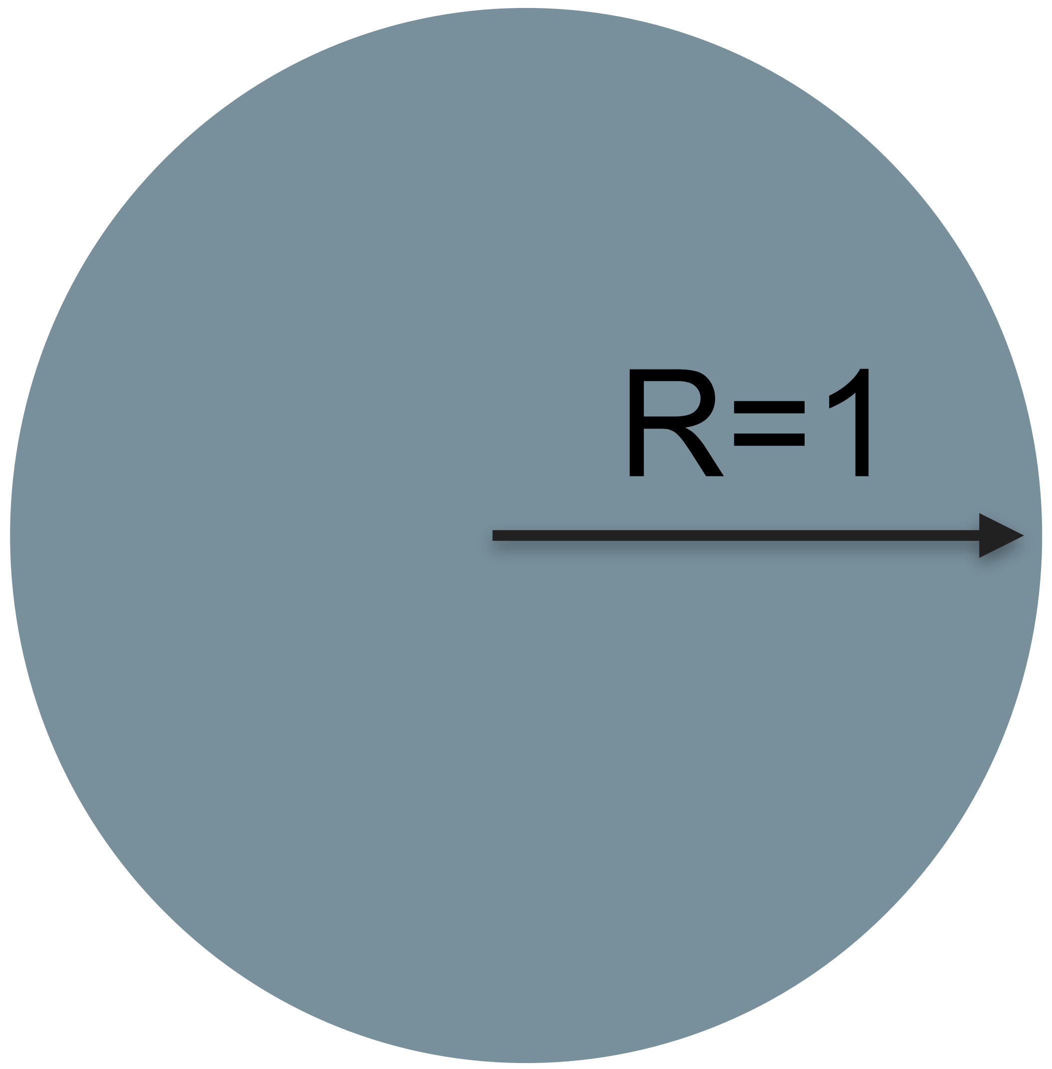


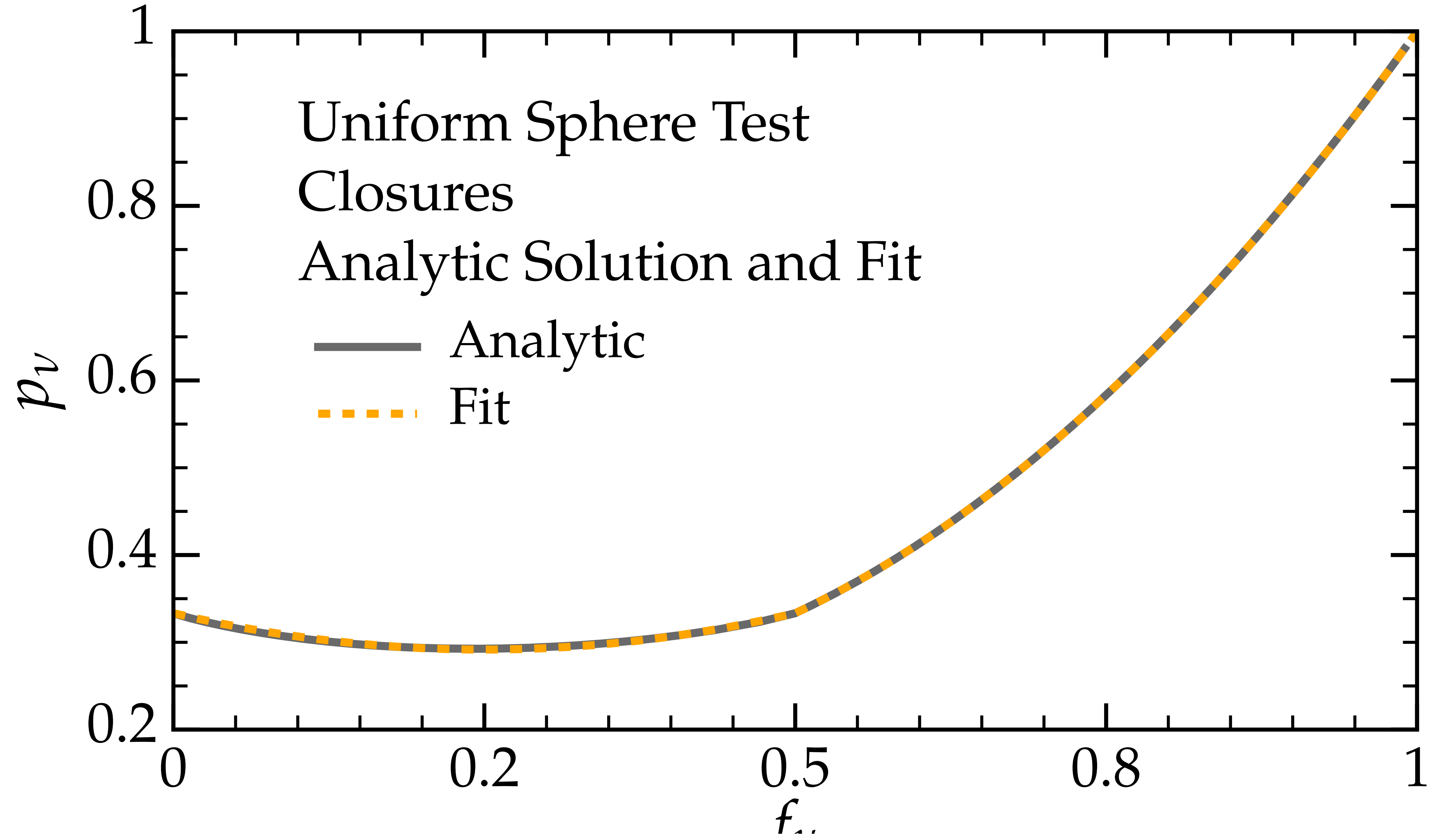
Neutrinos streaming from the center of the sphere

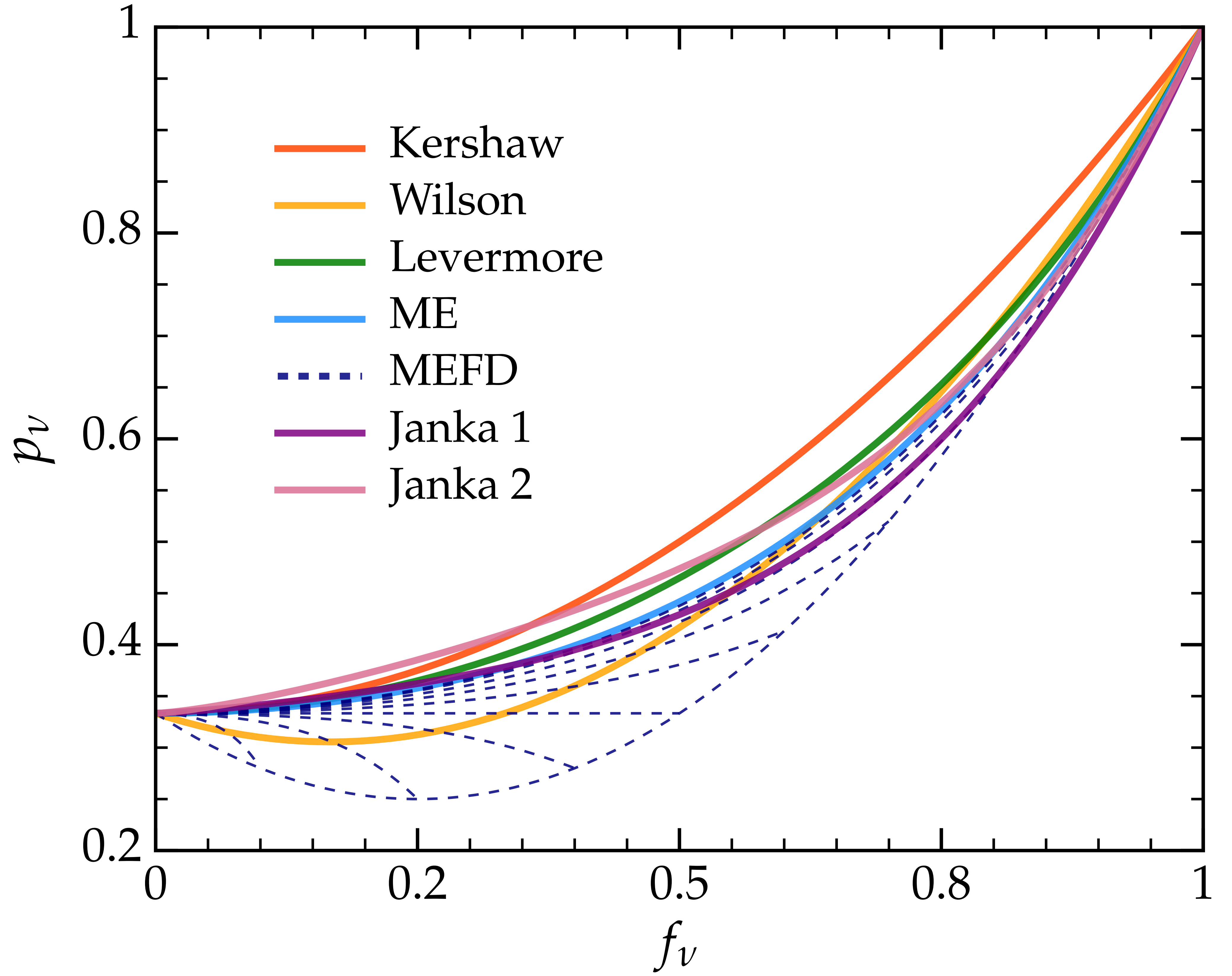
Uniform sphere

Fit: $$p = \frac{1}{3} - \frac{1}{3}f + \frac{2}{3}f^2 \text{ for } f \leq \frac{1}{2}$$

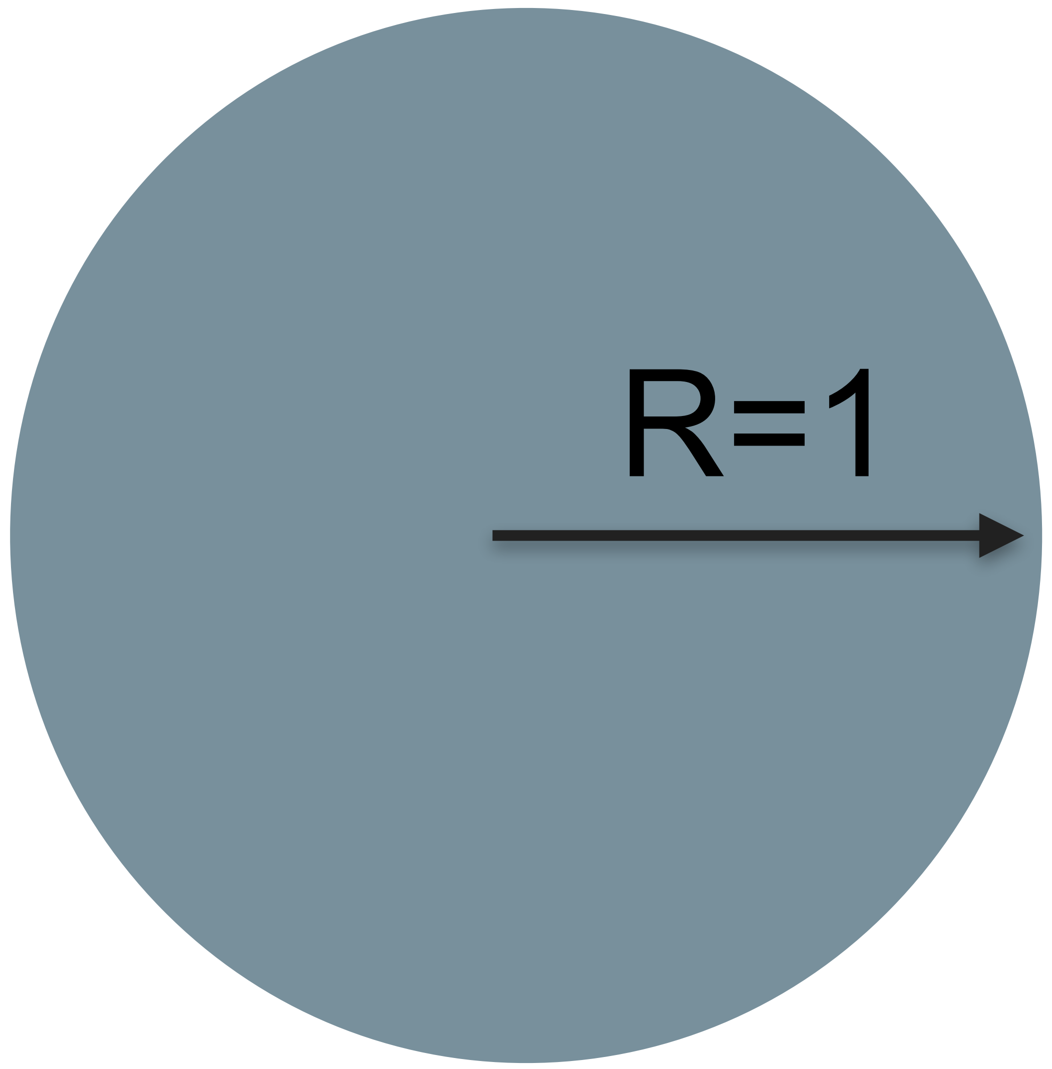
$$p = \frac{1}{3} - \frac{2}{3}f + \frac{4}{3}f^2 \text{ for } f > \frac{1}{2}$$

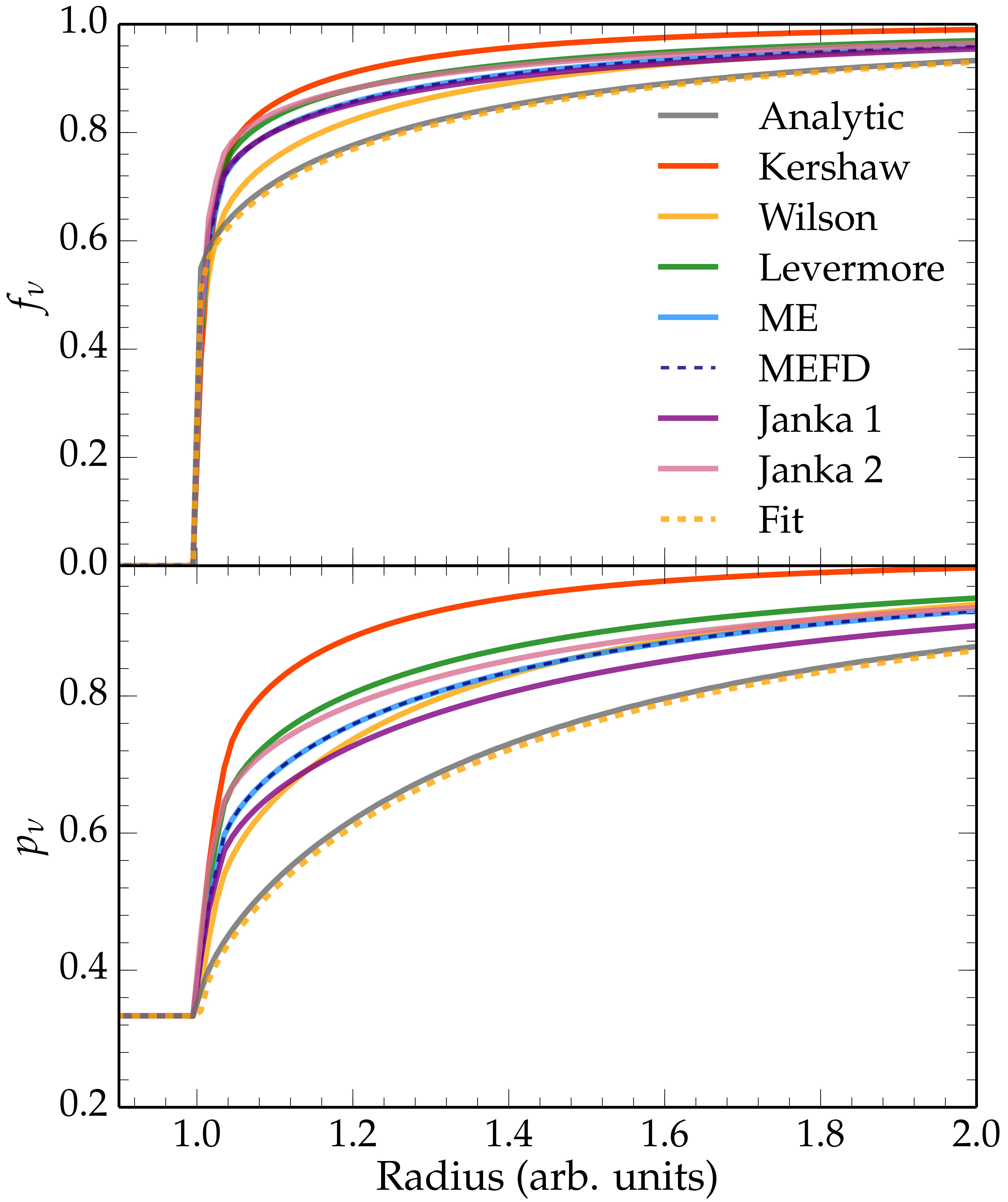






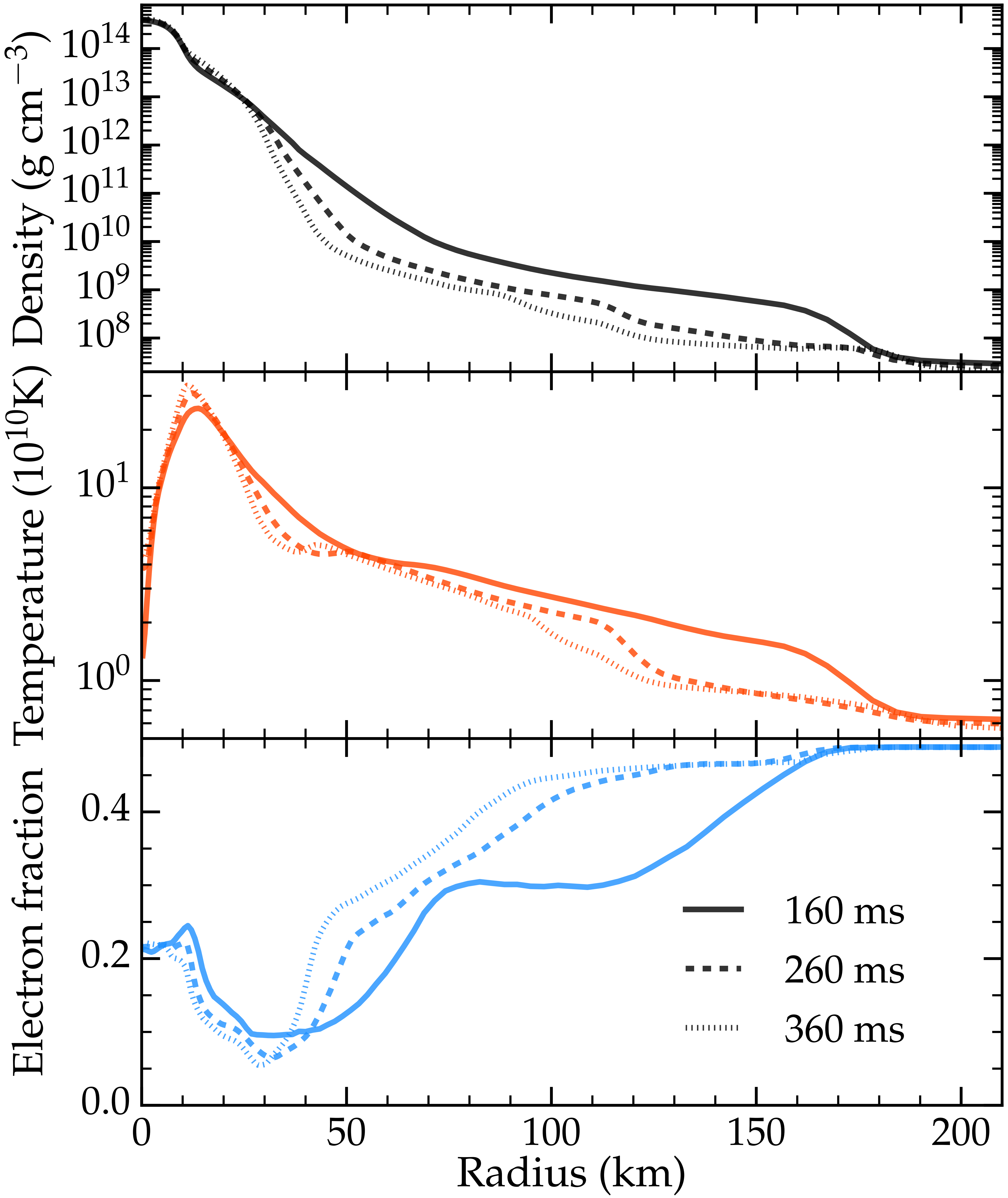
Uniform sphere





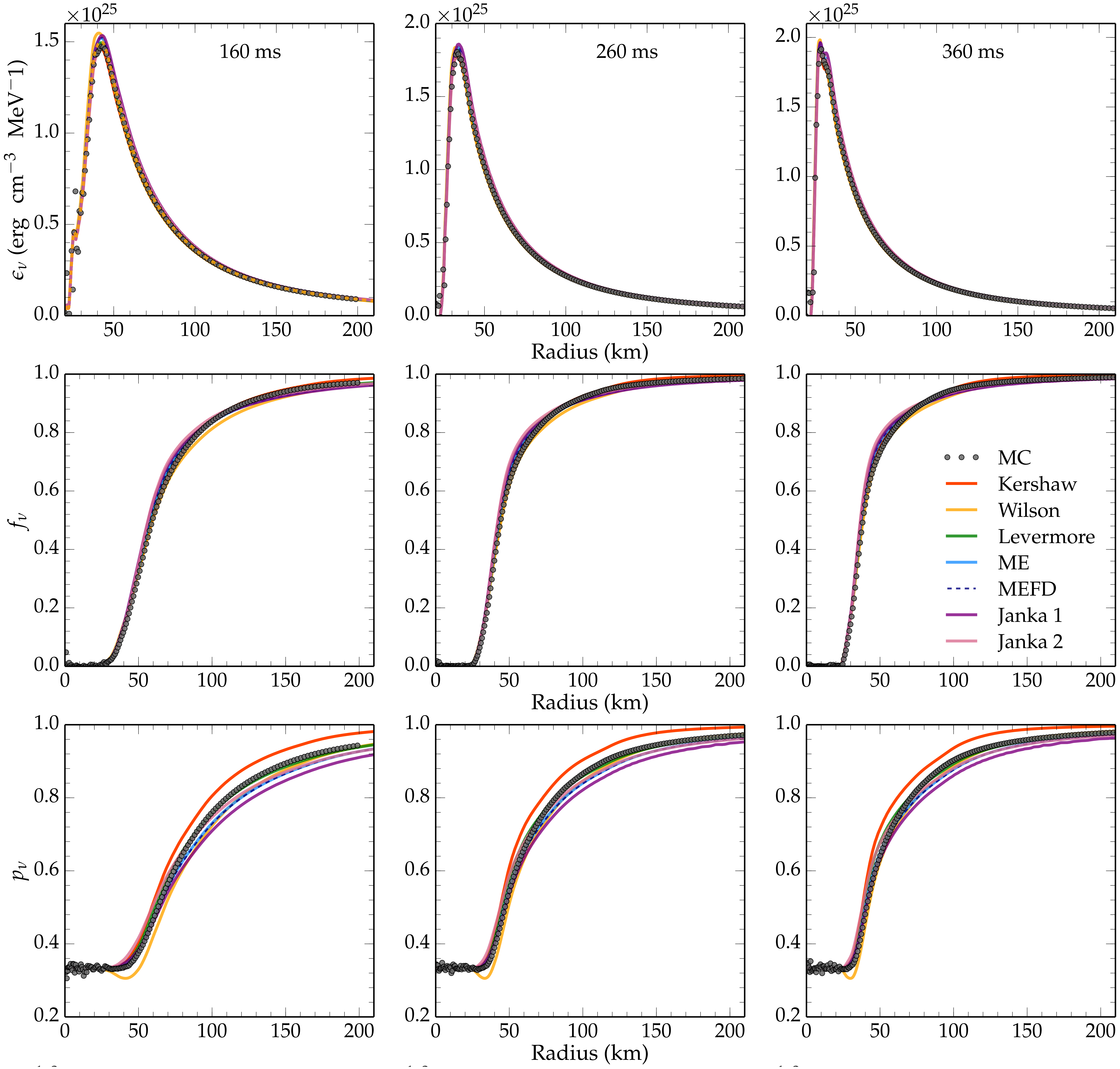
Protoneutron star

Three models



Protoneutron star

Typical output



To quantify the quality of fits

Mean square deviation

$$\delta Y = \sqrt{\frac{1}{N_X}\sum_{X_{\min}}^{X_{\max}}\left[1-\frac{Y_{GR1D}(X_i)}{Y_{MC}(X_i)}\right]^2}$$

Spectrum weighted deviation

$$\bar{\delta Y} = \frac{\sum w_{E_i}\,\delta Y_{E_i}}{\sum w_{E_i}}, \qquad w_{E_i} = S_{E_i}/S_{E_{\mathrm{max}}}$$

Protoneutron star

Closure performances (integrated)

Spectrum weighted deviations

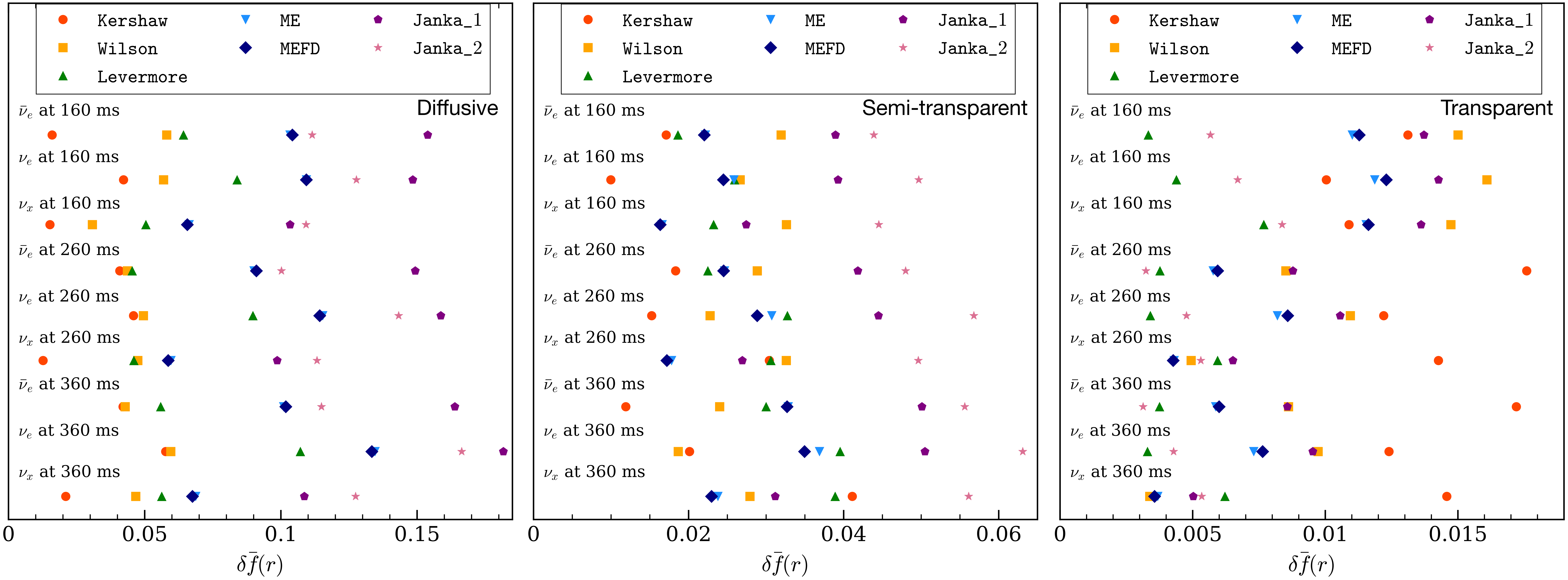
Protoneutron star

Closure performances (by flux value)

energy

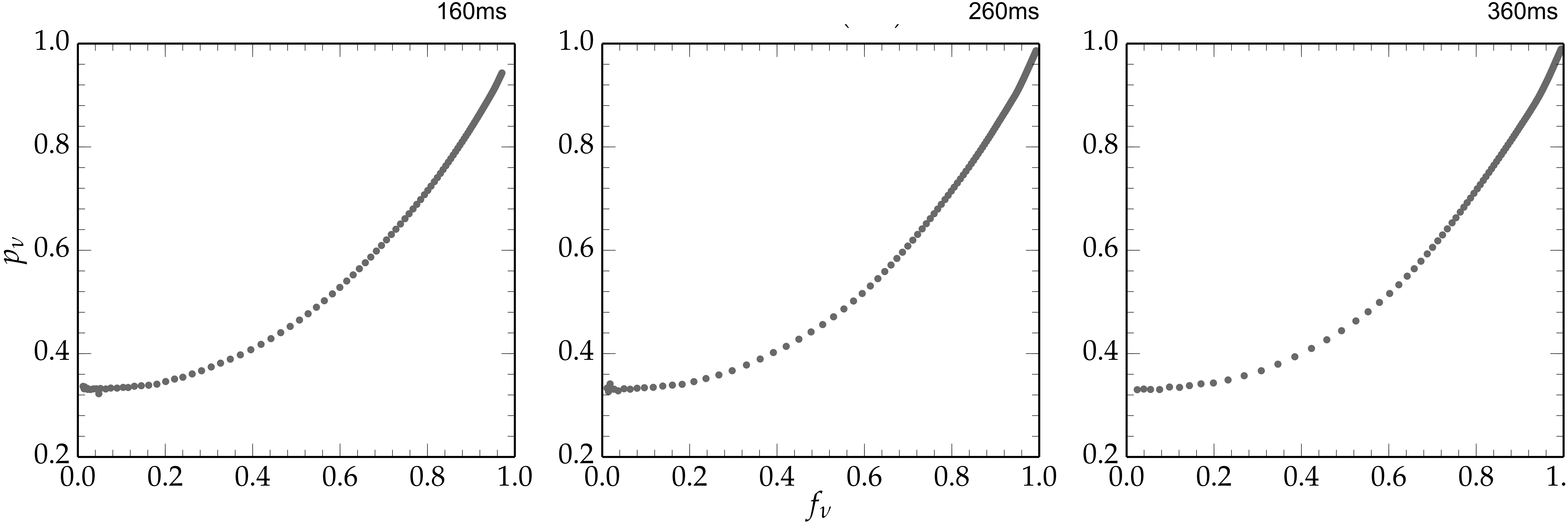
Protoneutron star

Closure performances (by flux value)



flux

Protoneutron star



Protoneutron star

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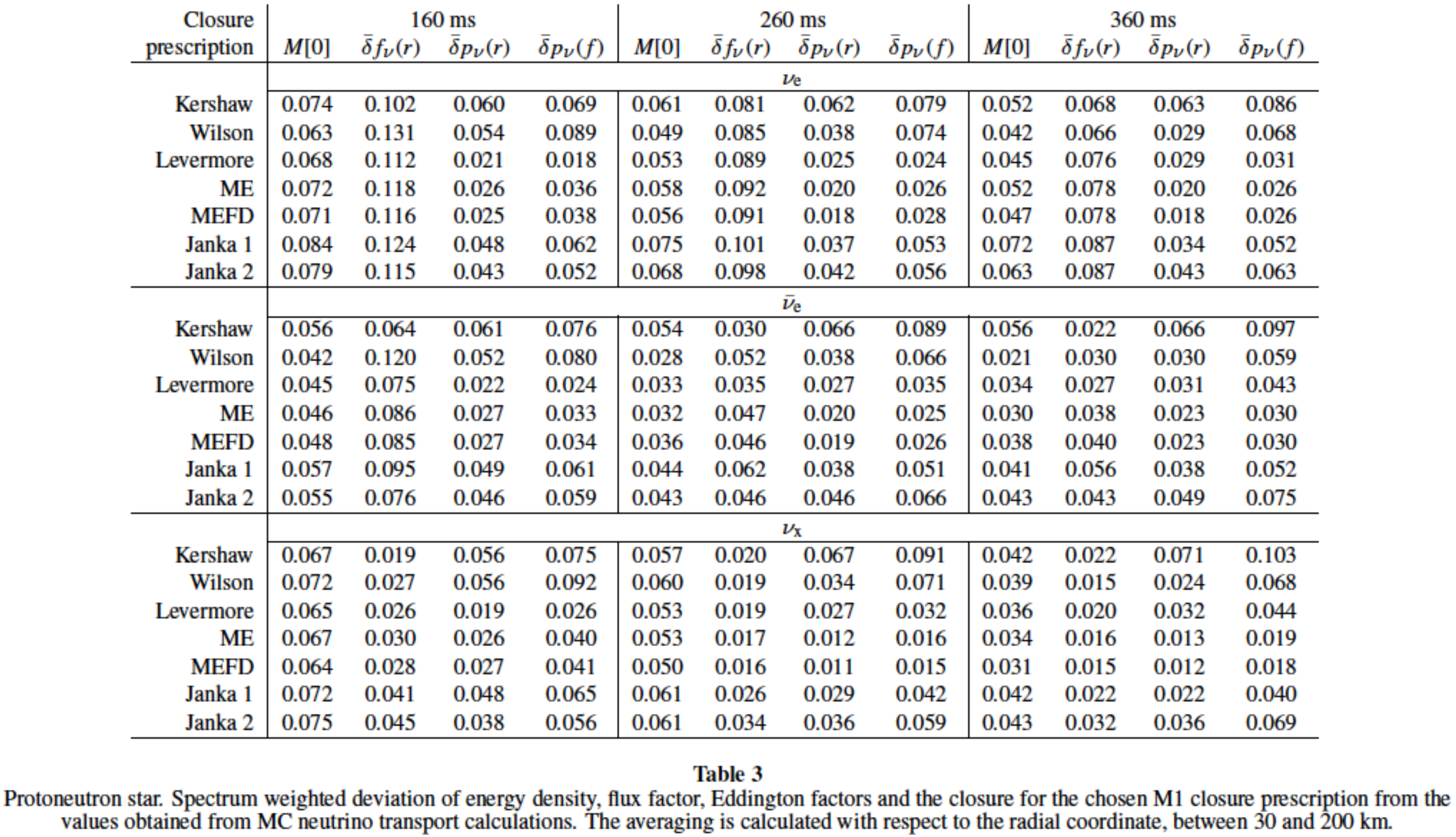
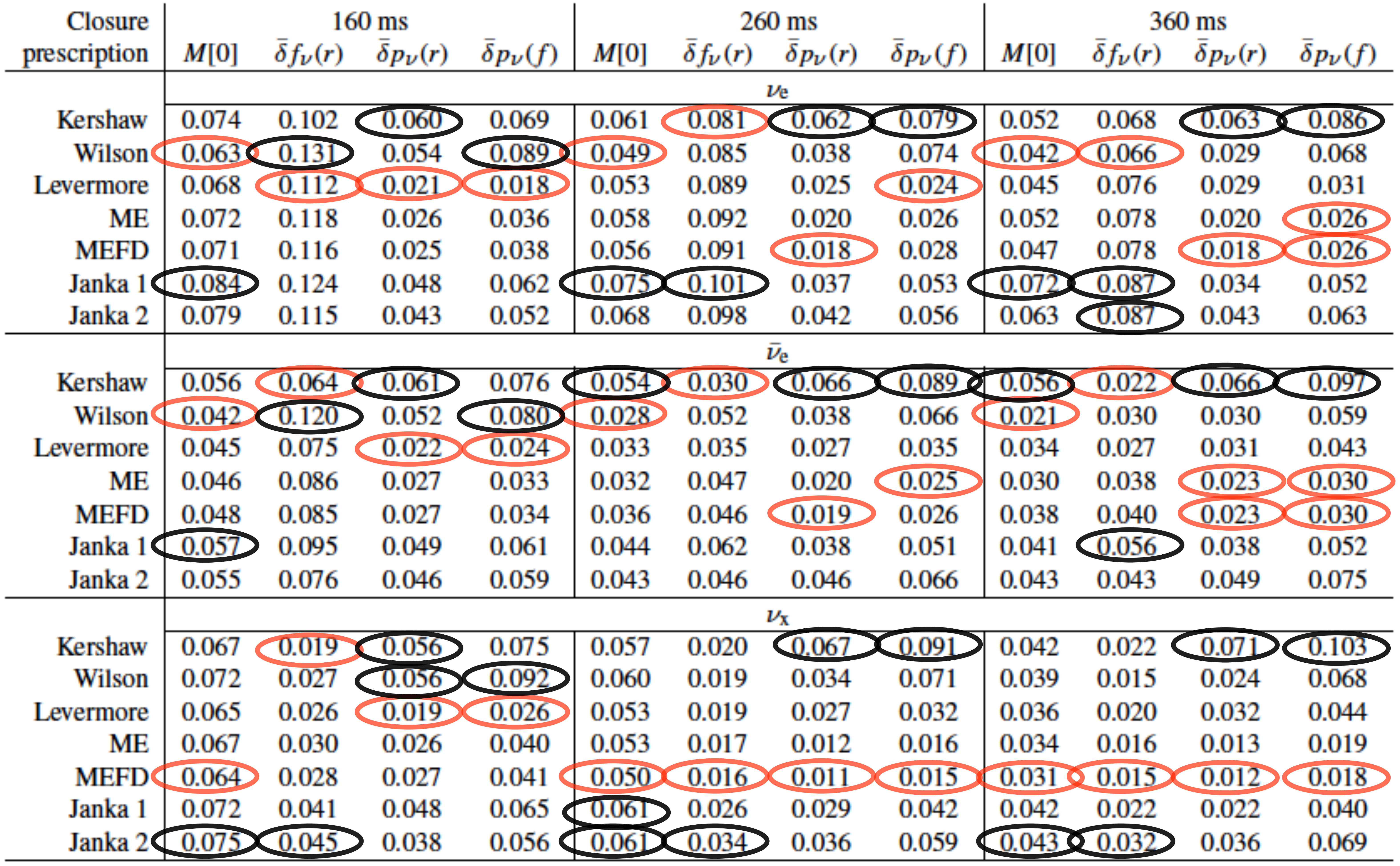


Closure prescription	160 ms $M[0]$	160 ms $\bar{\delta} f_\nu(r)$	160 ms $\bar{\delta} p_\nu(r)$	160 ms $\bar{\delta} p_\nu(f)$	260 ms $M[0]$	260 ms $\bar{\delta} f_\nu(r)$	260 ms $\bar{\delta} p_\nu(r)$	260 ms $\bar{\delta} p_\nu(f)$	360 ms $M[0]$	360 ms $\bar{\delta} f_\nu(r)$	360 ms $\bar{\delta} p_\nu(r)$	360 ms $\bar{\delta} p_\nu(f)$
						ν_e						
Kershaw	0.074	0.102	0.060	0.069	0.061	0.081	0.062	0.079	0.052	0.068	0.063	0.086
Wilson	0.063	0.131	0.054	0.089	0.049	0.085	0.038	0.074	0.042	0.066	0.029	0.068
Levermore	0.068	0.112	0.021	0.018	0.053	0.089	0.025	0.024	0.045	0.076	0.029	0.031
ME	0.072	0.118	0.026	0.036	0.058	0.092	0.020	0.026	0.052	0.078	0.020	0.026
MEFD	0.071	0.116	0.025	0.038	0.056	0.091	0.018	0.028	0.047	0.078	0.018	0.026
Janka 1	0.084	0.124	0.048	0.062	0.075	0.101	0.037	0.053	0.072	0.087	0.034	0.052
Janka 2	0.079	0.115	0.043	0.052	0.068	0.098	0.042	0.056	0.063	0.087	0.043	0.063
						$\bar{\nu}_e$						
Kershaw	0.056	0.064	0.061	0.076	0.054	0.030	0.066	0.089	0.056	0.022	0.066	0.097
Wilson	0.042	0.120	0.052	0.080	0.028	0.052	0.038	0.066	0.021	0.030	0.030	0.059
Levermore	0.045	0.075	0.022	0.024	0.033	0.035	0.027	0.035	0.034	0.027	0.031	0.043
ME	0.046	0.086	0.027	0.033	0.032	0.047	0.020	0.025	0.030	0.038	0.023	0.030
MEFD	0.048	0.085	0.027	0.034	0.036	0.046	0.019	0.026	0.038	0.040	0.023	0.030
Janka 1	0.057	0.095	0.049	0.061	0.044	0.062	0.038	0.051	0.041	0.056	0.038	0.052
Janka 2	0.055	0.076	0.046	0.059	0.043	0.046	0.046	0.066	0.043	0.043	0.049	0.075
						ν_x						
Kershaw	0.067	0.019	0.056	0.075	0.057	0.020	0.067	0.091	0.042	0.022	0.071	0.103
Wilson	0.072	0.027	0.056	0.092	0.060	0.019	0.034	0.071	0.039	0.015	0.024	0.068
Levermore	0.065	0.026	0.019	0.026	0.053	0.019	0.027	0.032	0.036	0.020	0.032	0.044
ME	0.067	0.030	0.026	0.040	0.053	0.017	0.012	0.016	0.034	0.016	0.013	0.019
MEFD	0.064	0.028	0.027	0.041	0.050	0.016	0.011	0.015	0.031	0.015	0.012	0.018
Janka 1	0.072	0.041	0.048	0.065	0.061	0.026	0.029	0.042	0.042	0.022	0.022	0.040
Janka 2	0.075	0.045	0.038	0.056	0.061	0.034	0.036	0.059	0.043	0.032	0.036	0.069

Table 3

Protoneutron star. Spectrum weighted deviation of energy density, flux factor, Eddington factors and the closure for the chosen M1 closure prescription from the values obtained from MC neutrino transport calculations. The averaging is calculated with respect to the radial coordinate, between 30 and 200 km.

Protoneutron star



Closure prescription	160 ms				260 ms				360 ms			
	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$
	ν_e											
Kershaw	0.074	0.102	0.060	0.069	0.061	0.081	0.062	0.079	0.052	0.068	0.063	0.086
Wilson	0.063	0.131	0.054	0.089	0.049	0.085	0.038	0.074	0.042	0.066	0.029	0.068
Levermore	0.068	0.112	0.021	0.018	0.053	0.089	0.025	0.024	0.045	0.076	0.029	0.031
ME	0.072	0.118	0.026	0.036	0.058	0.092	0.020	0.026	0.052	0.078	0.020	0.026
MEFD	0.071	0.116	0.025	0.038	0.056	0.091	0.018	0.028	0.047	0.078	0.018	0.026
Janka 1	0.084	0.124	0.048	0.062	0.075	0.101	0.037	0.053	0.072	0.087	0.034	0.052
Janka 2	0.079	0.115	0.043	0.052	0.068	0.098	0.042	0.056	0.063	0.087	0.043	0.063
	$\bar{\nu}_e$											
Kershaw	0.056	0.064	0.061	0.076	0.054	0.030	0.066	0.089	0.056	0.022	0.066	0.097
Wilson	0.042	0.120	0.052	0.080	0.028	0.052	0.038	0.066	0.021	0.030	0.030	0.059
Levermore	0.045	0.075	0.022	0.024	0.033	0.035	0.027	0.035	0.034	0.027	0.031	0.043
ME	0.046	0.086	0.027	0.033	0.032	0.047	0.020	0.025	0.030	0.038	0.023	0.030
MEFD	0.048	0.085	0.027	0.034	0.036	0.046	0.019	0.026	0.038	0.040	0.023	0.030
Janka 1	0.057	0.095	0.049	0.061	0.044	0.062	0.038	0.051	0.041	0.056	0.038	0.052
Janka 2	0.055	0.076	0.046	0.059	0.043	0.046	0.046	0.066	0.043	0.043	0.049	0.075
	ν_x											
Kershaw	0.067	0.019	0.056	0.075	0.057	0.020	0.067	0.091	0.042	0.022	0.071	0.103
Wilson	0.072	0.027	0.056	0.092	0.060	0.019	0.034	0.071	0.039	0.015	0.024	0.068
Levermore	0.065	0.026	0.019	0.026	0.053	0.019	0.027	0.032	0.036	0.020	0.032	0.044
ME	0.067	0.030	0.026	0.040	0.053	0.017	0.012	0.016	0.034	0.016	0.013	0.019
MEFD	0.064	0.028	0.027	0.041	0.050	0.016	0.011	0.015	0.031	0.015	0.012	0.018
Janka 1	0.072	0.041	0.048	0.065	0.061	0.026	0.029	0.042	0.042	0.022	0.022	0.040
Janka 2	0.075	0.045	0.038	0.056	0.061	0.034	0.036	0.059	0.043	0.032	0.036	0.069

Table 3

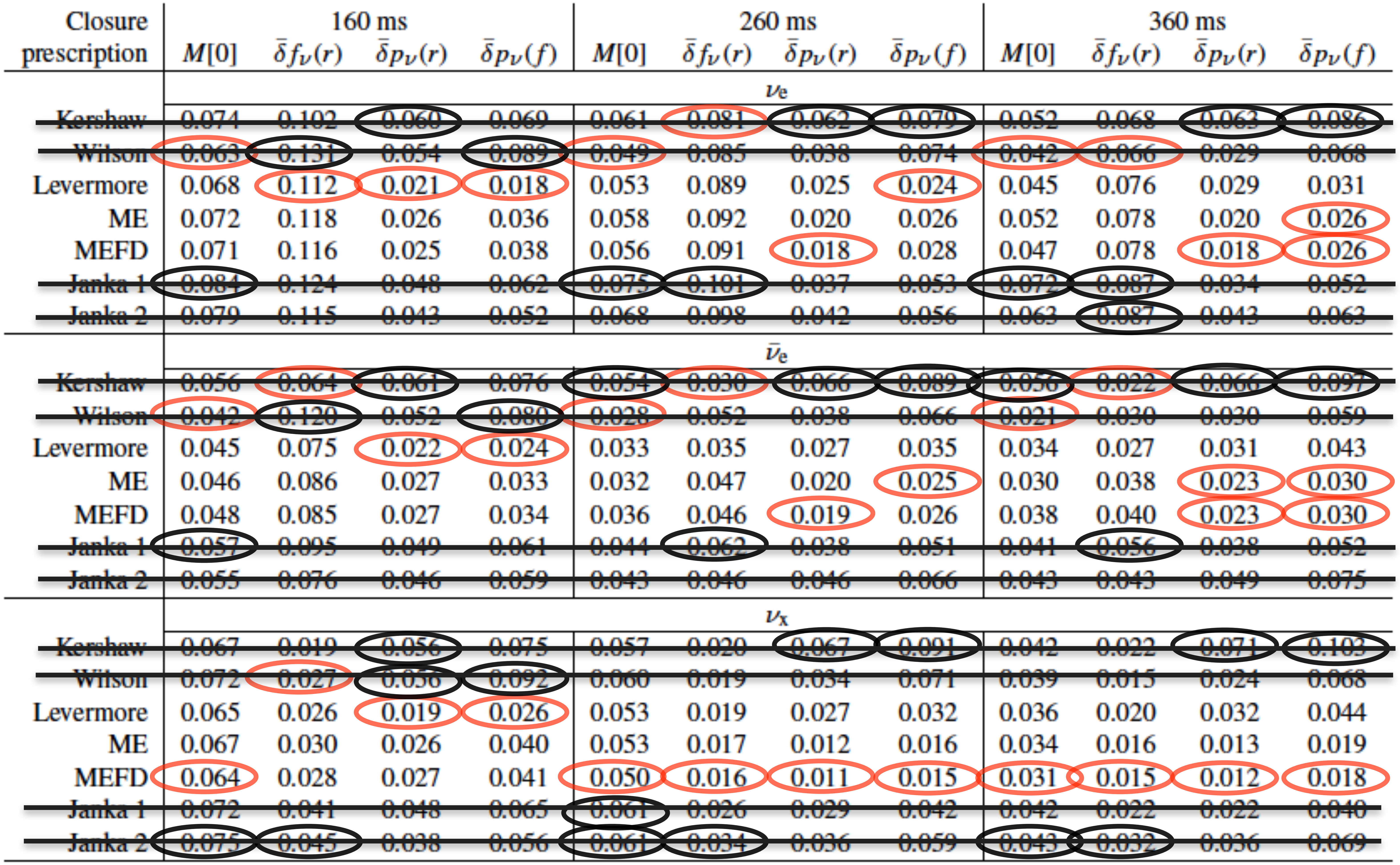
Protoneutron star. Spectrum weighted deviation of energy density, flux factor, Eddington factors and the closure for the chosen M1 closure prescription from the values obtained from MC neutrino transport calculations. The averaging is calculated with respect to the radial coordinate, between 30 and 200 km.

○ best fit

○ worst fit

Protoneutron star

Crossing out closures with worst fits



Closure prescription	160 ms				260 ms				360 ms			
	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$
						ν_e						
~~Kershaw~~	~~0.074~~	~~0.102~~	~~0.060~~	~~0.069~~	~~0.061~~	~~0.081~~	~~0.062~~	~~0.079~~	~~0.052~~	~~0.068~~	~~0.063~~	~~0.086~~
~~Wilson~~	~~0.063~~	~~0.131~~	~~0.054~~	~~0.089~~	~~0.049~~	~~0.085~~	~~0.038~~	~~0.074~~	~~0.042~~	~~0.066~~	~~0.029~~	~~0.068~~
Levermore	0.068	0.112	0.021	0.018	0.053	0.089	0.025	0.024	0.045	0.076	0.029	0.031
ME	0.072	0.118	0.026	0.036	0.058	0.092	0.020	0.026	0.052	0.078	0.020	0.026
MEFD	0.071	0.116	0.025	0.038	0.056	0.091	0.018	0.028	0.047	0.078	0.018	0.026
~~Janka 1~~	~~0.084~~	~~0.124~~	~~0.048~~	~~0.062~~	~~0.075~~	~~0.101~~	~~0.037~~	~~0.053~~	~~0.072~~	~~0.087~~	~~0.034~~	~~0.052~~
~~Janka 2~~	~~0.079~~	~~0.115~~	~~0.043~~	~~0.052~~	~~0.068~~	~~0.098~~	~~0.042~~	~~0.056~~	~~0.063~~	~~0.087~~	~~0.043~~	~~0.063~~
						$\bar{\nu}_e$						
~~Kershaw~~	~~0.056~~	~~0.064~~	~~0.061~~	~~0.076~~	~~0.054~~	~~0.030~~	~~0.066~~	~~0.089~~	~~0.056~~	~~0.022~~	~~0.066~~	~~0.097~~
~~Wilson~~	~~0.042~~	~~0.120~~	~~0.052~~	~~0.080~~	~~0.028~~	~~0.052~~	~~0.038~~	~~0.066~~	~~0.021~~	~~0.030~~	~~0.030~~	~~0.059~~
Levermore	0.045	0.075	0.022	0.024	0.033	0.035	0.027	0.035	0.034	0.027	0.031	0.043
ME	0.046	0.086	0.027	0.033	0.032	0.047	0.020	0.025	0.030	0.038	0.023	0.030
MEFD	0.048	0.085	0.027	0.034	0.036	0.046	0.019	0.026	0.038	0.040	0.023	0.030
~~Janka 1~~	~~0.057~~	~~0.095~~	~~0.049~~	~~0.061~~	~~0.044~~	~~0.062~~	~~0.038~~	~~0.051~~	~~0.041~~	~~0.056~~	~~0.038~~	~~0.052~~
~~Janka 2~~	~~0.055~~	~~0.076~~	~~0.046~~	~~0.059~~	~~0.043~~	~~0.046~~	~~0.046~~	~~0.066~~	~~0.043~~	~~0.043~~	~~0.049~~	~~0.075~~
						ν_x						
~~Kershaw~~	~~0.067~~	~~0.019~~	~~0.056~~	~~0.075~~	~~0.057~~	~~0.020~~	~~0.067~~	~~0.091~~	~~0.042~~	~~0.022~~	~~0.071~~	~~0.103~~
~~Wilson~~	~~0.072~~	~~0.027~~	~~0.056~~	~~0.092~~	~~0.060~~	~~0.019~~	~~0.034~~	~~0.071~~	~~0.039~~	~~0.015~~	~~0.024~~	~~0.068~~
Levermore	0.065	0.026	0.019	0.026	0.053	0.019	0.027	0.032	0.036	0.020	0.032	0.044
ME	0.067	0.030	0.026	0.040	0.053	0.017	0.012	0.016	0.034	0.016	0.013	0.019
MEFD	0.064	0.028	0.027	0.041	0.050	0.016	0.011	0.015	0.031	0.015	0.012	0.018
~~Janka 1~~	~~0.072~~	~~0.041~~	~~0.048~~	~~0.065~~	~~0.061~~	~~0.026~~	~~0.029~~	~~0.042~~	~~0.042~~	~~0.022~~	~~0.022~~	~~0.040~~
~~Janka 2~~	~~0.075~~	~~0.045~~	~~0.038~~	~~0.056~~	~~0.061~~	~~0.034~~	~~0.036~~	~~0.059~~	~~0.043~~	~~0.032~~	~~0.036~~	~~0.069~~

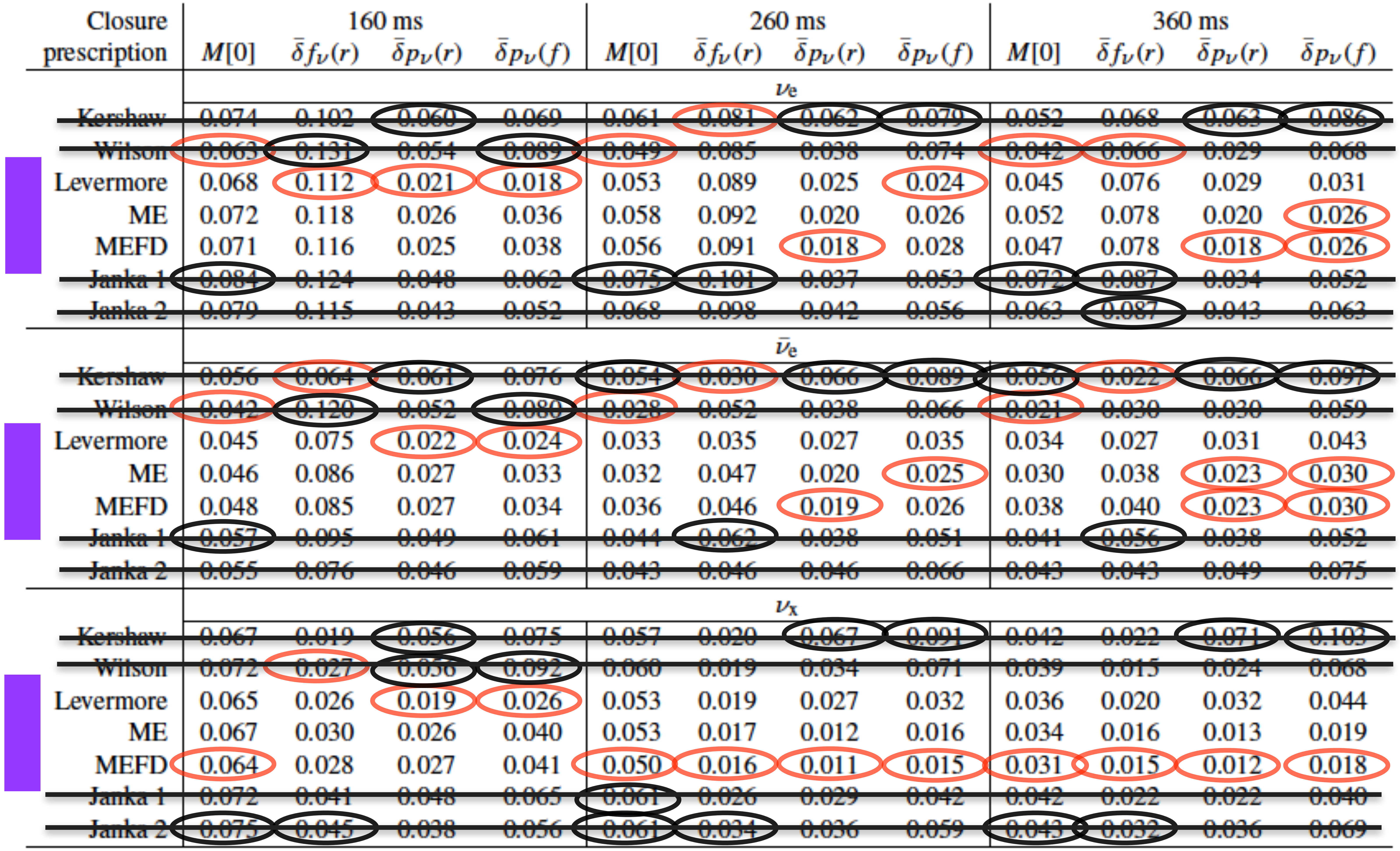
Table 3
Protoneutron star. Spectrum weighted deviation of energy density, flux factor, Eddington factors and the closure for the chosen M1 closure prescription from the values obtained from MC neutrino transport calculations. The averaging is calculated with respect to the radial coordinate, between 30 and 200 km.

○ best fit

○ worst fit

Protoneutron star

Only the three physical closures survive



Closure prescription	160 ms M[0]	160 ms $\bar{\delta} f_\nu(r)$	160 ms $\bar{\delta} p_\nu(r)$	160 ms $\bar{\delta} p_\nu(f)$	260 ms M[0]	260 ms $\bar{\delta} f_\nu(r)$	260 ms $\bar{\delta} p_\nu(r)$	260 ms $\bar{\delta} p_\nu(f)$	360 ms M[0]	360 ms $\bar{\delta} f_\nu(r)$	360 ms $\bar{\delta} p_\nu(r)$	360 ms $\bar{\delta} p_\nu(f)$
						ν_e						
~~Kershaw~~	~~0.074~~	~~0.102~~	~~0.060~~	~~0.069~~	~~0.061~~	~~0.081~~	~~0.062~~	~~0.079~~	~~0.052~~	~~0.068~~	~~0.063~~	~~0.086~~
~~Wilson~~	~~0.063~~	~~0.131~~	~~0.054~~	~~0.089~~	~~0.049~~	~~0.085~~	~~0.038~~	~~0.074~~	~~0.042~~	~~0.066~~	~~0.029~~	~~0.068~~
Levermore	0.068	0.112	0.021	0.018	0.053	0.089	0.025	0.024	0.045	0.076	0.029	0.031
ME	0.072	0.118	0.026	0.036	0.058	0.092	0.020	0.026	0.052	0.078	0.020	0.026
MEFD	0.071	0.116	0.025	0.038	0.056	0.091	0.018	0.028	0.047	0.078	0.018	0.026
~~Janka 1~~	~~0.084~~	~~0.124~~	~~0.048~~	~~0.062~~	~~0.075~~	~~0.101~~	~~0.037~~	~~0.053~~	~~0.072~~	~~0.087~~	~~0.034~~	~~0.052~~
~~Janka 2~~	~~0.079~~	~~0.115~~	~~0.043~~	~~0.052~~	~~0.068~~	~~0.098~~	~~0.042~~	~~0.056~~	~~0.063~~	~~0.087~~	~~0.043~~	~~0.063~~
						$\bar{\nu}_e$						
~~Kershaw~~	~~0.056~~	~~0.064~~	~~0.061~~	~~0.076~~	~~0.054~~	~~0.030~~	~~0.066~~	~~0.089~~	~~0.056~~	~~0.022~~	~~0.066~~	~~0.097~~
~~Wilson~~	~~0.042~~	~~0.120~~	~~0.052~~	~~0.080~~	~~0.028~~	~~0.052~~	~~0.038~~	~~0.066~~	~~0.021~~	~~0.030~~	~~0.030~~	~~0.059~~
Levermore	0.045	0.075	0.022	0.024	0.033	0.035	0.027	0.035	0.034	0.027	0.031	0.043
ME	0.046	0.086	0.027	0.033	0.032	0.047	0.020	0.025	0.030	0.038	0.023	0.030
MEFD	0.048	0.085	0.027	0.034	0.036	0.046	0.019	0.026	0.038	0.040	0.023	0.030
~~Janka 1~~	~~0.057~~	~~0.095~~	~~0.049~~	~~0.061~~	~~0.044~~	~~0.062~~	~~0.038~~	~~0.051~~	~~0.041~~	~~0.056~~	~~0.038~~	~~0.052~~
~~Janka 2~~	~~0.055~~	~~0.076~~	~~0.046~~	~~0.059~~	~~0.043~~	~~0.046~~	~~0.046~~	~~0.066~~	~~0.043~~	~~0.043~~	~~0.049~~	~~0.075~~
						ν_x						
~~Kershaw~~	~~0.067~~	~~0.019~~	~~0.056~~	~~0.075~~	~~0.057~~	~~0.020~~	~~0.067~~	~~0.091~~	~~0.042~~	~~0.022~~	~~0.071~~	~~0.103~~
~~Wilson~~	~~0.072~~	~~0.027~~	~~0.056~~	~~0.092~~	~~0.060~~	~~0.019~~	~~0.034~~	~~0.071~~	~~0.039~~	~~0.015~~	~~0.024~~	~~0.068~~
Levermore	0.065	0.026	0.019	0.026	0.053	0.019	0.027	0.032	0.036	0.020	0.032	0.044
ME	0.067	0.030	0.026	0.040	0.053	0.017	0.012	0.016	0.034	0.016	0.013	0.019
MEFD	0.064	0.028	0.027	0.041	0.050	0.016	0.011	0.015	0.031	0.015	0.012	0.018
~~Janka 1~~	~~0.072~~	~~0.041~~	~~0.048~~	~~0.065~~	~~0.061~~	~~0.026~~	~~0.029~~	~~0.042~~	~~0.042~~	~~0.022~~	~~0.022~~	~~0.040~~
~~Janka 2~~	~~0.075~~	~~0.045~~	~~0.038~~	~~0.056~~	~~0.061~~	~~0.034~~	~~0.036~~	~~0.059~~	~~0.043~~	~~0.032~~	~~0.036~~	~~0.069~~

Table 3

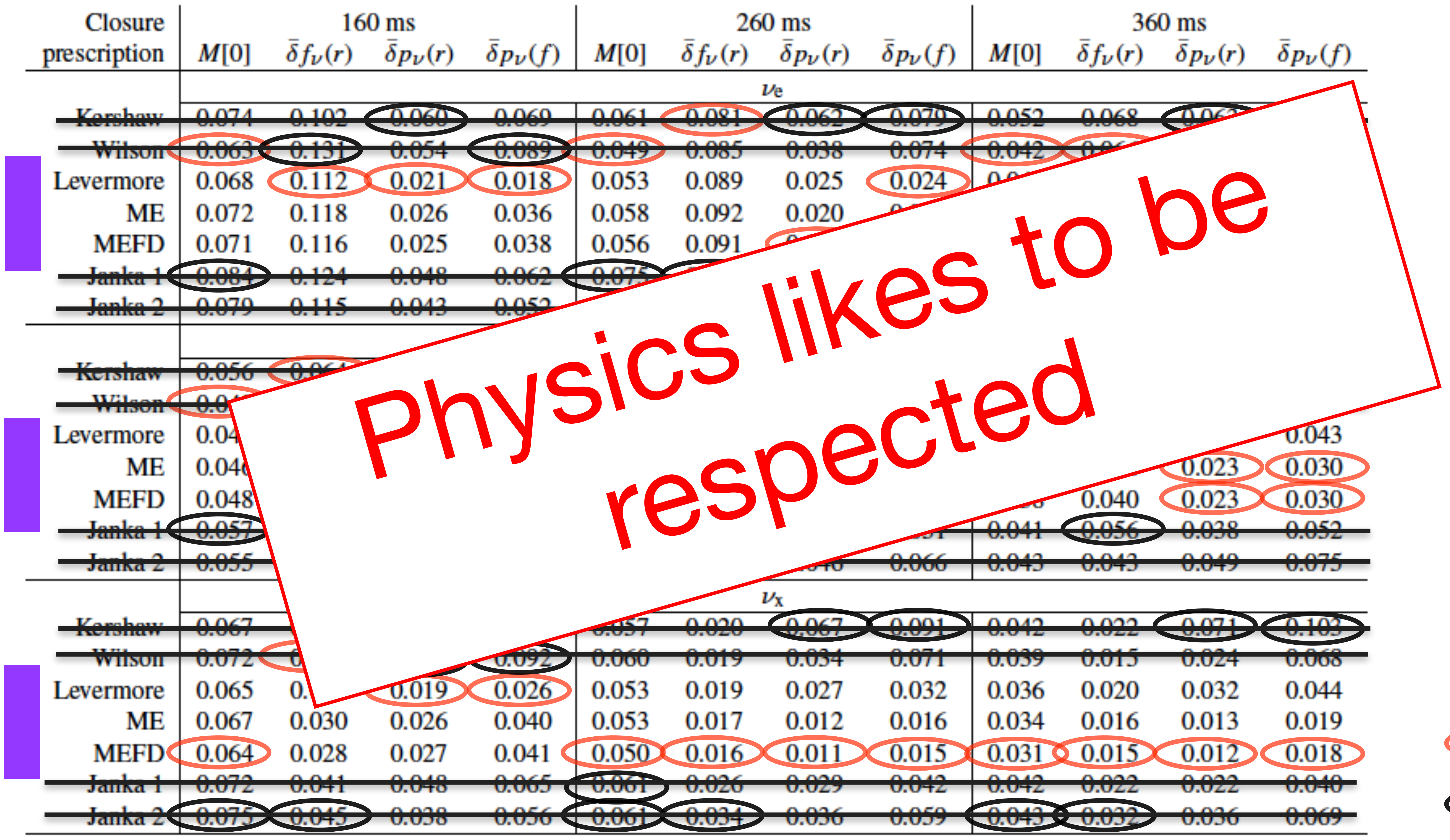
Protoneutron star. Spectrum weighted deviation of energy density, flux factor, Eddington factors and the closure for the chosen M1 closure prescription from the values obtained from MC neutrino transport calculations. The averaging is calculated with respect to the radial coordinate, between 30 and 200 km.

○ best fit

○ worst fit

Protoneutron star

Only the three physical closures survive



Closure	160 ms				260 ms				360 ms			
prescription	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$	$M[0]$	$\bar{\delta} f_\nu(r)$	$\bar{\delta} p_\nu(r)$	$\bar{\delta} p_\nu(f)$
					ν_e							
~~Kershaw~~	~~0.074~~	~~0.102~~	~~0.060~~	~~0.069~~	~~0.061~~	~~0.081~~	~~0.062~~	~~0.079~~	~~0.052~~	~~0.068~~	[illegible]	[illegible]
~~Wilson~~	~~0.063~~	~~0.131~~	~~0.054~~	~~0.089~~	~~0.049~~	~~0.085~~	~~0.038~~	~~0.074~~	~~0.042~~	[illegible]	[illegible]	[illegible]
Levermore	0.068	0.112	0.021	0.018	0.053	0.089	0.025	0.024	[illegible]	[illegible]	[illegible]	[illegible]
ME	0.072	0.118	0.026	0.036	0.058	0.092	0.020	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
MEFD	0.071	0.116	0.025	0.038	0.056	0.091	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
~~Janka 1~~	~~0.084~~	~~0.124~~	~~0.048~~	~~0.062~~	~~0.075~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
~~Janka 2~~	~~0.079~~	~~0.115~~	~~0.043~~	~~0.052~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
					[illegible]							
~~Kershaw~~	~~0.056~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
~~Wilson~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Levermore	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.043
ME	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.023	0.030
MEFD	0.048	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.040	0.023	0.030
~~Janka 1~~	~~0.057~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	~~0.041~~	~~0.056~~	~~0.038~~	~~0.052~~
~~Janka 2~~	~~0.055~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	~~0.066~~	~~0.043~~	~~0.043~~	~~0.049~~	~~0.075~~
					ν_x							
~~Kershaw~~	~~0.067~~	[illegible]	[illegible]	[illegible]	~~0.057~~	~~0.020~~	~~0.067~~	~~0.091~~	~~0.042~~	~~0.022~~	~~0.071~~	~~0.103~~
~~Wilson~~	~~0.072~~	[illegible]	[illegible]	~~0.092~~	~~0.060~~	~~0.019~~	~~0.034~~	~~0.071~~	~~0.039~~	~~0.015~~	~~0.024~~	~~0.068~~
Levermore	0.065	[illegible]	0.019	0.026	0.053	0.019	0.027	0.032	0.036	0.020	0.032	0.044
ME	0.067	0.030	0.026	0.040	0.053	0.017	0.012	0.016	0.034	0.016	0.013	0.019
MEFD	0.064	0.028	0.027	0.041	0.050	0.016	0.011	0.015	0.031	0.015	0.012	0.018
~~Janka 1~~	~~0.072~~	~~0.041~~	~~0.048~~	~~0.065~~	~~0.061~~	~~0.026~~	~~0.029~~	~~0.042~~	~~0.042~~	~~0.022~~	~~0.022~~	~~0.040~~
~~Janka 2~~	~~0.075~~	~~0.045~~	~~0.038~~	~~0.056~~	~~0.061~~	~~0.034~~	~~0.036~~	~~0.059~~	~~0.043~~	~~0.032~~	~~0.036~~	~~0.069~~

Physics likes to be respected

○ best fit
○ worst fit

Table 3
Protoneutron star. Spectrum weighted deviation of energy density, flux factor, Eddington factors and the closure for the chosen M1 closure prescription from the values obtained from MC neutrino transport calculations. The averaging is calculated with respect to the radial coordinate, between 30 and 200 km.

Conclusions

There is no single best closure.

Best closure is a function of neutrino type, neutrino energy and neutrino specie.

Deviations of energy, flux, pressure, and closure are non-linear. Sometimes the worst fitting closure produces the best fitting flux.

One may want to choose the closure depending on what quantity they want to estimate with the highest accuracy.

It useful to obtain closure from MC calculations and then feeding the result to M1 code. However this may not be worth for small corrections especially in spectral case. It is only clearly when deviation of closure from the previously estimated is getting large.

Thank you!

Well… Actually

Instead of Boltzmann equation

$$\int \times \quad \Bigg| \Bigg| \frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha} + \xcancel{\frac{dp^\alpha}{d\tau}\frac{\partial f}{\partial p^\alpha}} = \varepsilon\, S(x^\mu, p^\mu, f) \Bigg| \Bigg| \quad \times p^\alpha \ldots \delta(h\nu - \varepsilon) d^3p$$

we get an infinite tower of moment equations

Diff Eq $(M[0], M[1])$

Diff Eq $(M[0], \ldots, M[2])$

Diff Eq $(M[0], \ldots, M[3])$

…

Diff Eq $(M[0], \ldots, M[N+1])$

…

Well… Actually

Instead of Boltzmann equation

$$\int \times \;\Big|\Big| \frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha} + \boxed{\frac{dp^\alpha}{d\tau}\frac{\partial f}{\partial p^\alpha}} = \varepsilon\, S(x^\mu, p^\mu, f) \;\Big|\Big| \quad \times p^\alpha \ldots \delta(h\nu - \varepsilon) d^3p$$

we get an infinite tower of moment equations

Diff Eq $(M[0], M[1], \boxed{M[2]})$

Diff Eq $(M[0], \ldots, M[2], \boxed{M[3]})$

Diff Eq $(M[0], \ldots, M[3], \boxed{M[4]})$

…

Diff Eq $(M[0], \ldots, M[N+1], \boxed{M[N+2]})$

…

We'd need two closures

(See e.g. Richers, at al Phys Rev D (2020))

But we will ignore this here.