Recent Developments in the Eigenvalue Distribution of Sparse-Spectral Limiting Operators

Azita Mayeli

City University of New York

Recent Progress on Optimal Point Distributions and Related Fields

ICERM Jun 3 - 7, 2024

- OVERVIEW OF THE TOPIC:
  - This presentation is about spatio-spectral limiting operators (SSLO), a concept at the intersection of harmonic analysis and signal processing.
  - These operators were previously studied extensively in one dimension, with a particular emphasis on the asymptotic distribution of their eigenvalues.

- OVERVIEW OF THE TOPIC:
  - This presentation is about spatio-spectral limiting operators (SSLO), a concept at the intersection of harmonic analysis and signal processing.
  - These operators were previously studied extensively in one dimension, with a particular emphasis on the asymptotic distribution of their eigenvalues.
- IMPORTANCE IN MATHEMATICS:
  - SSLOs are used to analyze and manipulate signals, particularly in the context of their spatial and frequency components.
  - Their primary role of these operators is to limit or 'filter' a signal in both spatial and spectral domains.

- RELEVANCE IN APPLICATIONS:
  - The study of these operators and their eigenvalue distributions is significant in mathematical spectral analysis, particularly in understanding the concentration of functions in space and frequency domains.
  - It holds significance in practical applications such as signal processing and scientific imaging, including MRI (2D imaging), cryoelectron microscopy (3D imaging), and geodesy.

- RELEVANCE IN APPLICATIONS:
  - The study of these operators and their eigenvalue distributions is significant in mathematical spectral analysis, particularly in understanding the concentration of functions in space and frequency domains.
  - It holds significance in practical applications such as signal processing and scientific imaging, including MRI (2D imaging), cryoelectron microscopy (3D imaging), and geodesy.
- OBJECTIVES OF THIS PRESENTATION:
  - The main objectives include:

    - Introduction to SSLO operators

- RELEVANCE IN APPLICATIONS:
  - The study of these operators and their eigenvalue distributions is significant in mathematical spectral analysis, particularly in understanding the concentration of functions in space and frequency domains.
  - It holds significance in practical applications such as signal processing and scientific imaging, including MRI (2D imaging), cryoelectron microscopy (3D imaging), and geodesy.
- OBJECTIVES OF THIS PRESENTATION:
  - The main objectives include:

    - Introduction to SSLO operators

    - Spectral properties of the operators

- RELEVANCE IN APPLICATIONS:
  - The study of these operators and their eigenvalue distributions is significant in mathematical spectral analysis, particularly in understanding the concentration of functions in space and frequency domains.
  - It holds significance in practical applications such as signal processing and scientific imaging, including MRI (2D imaging), cryoelectron microscopy (3D imaging), and geodesy.
- OBJECTIVES OF THIS PRESENTATION:
  - The main objectives include:

    - Introduction to SSLO operators

    - Spectral properties of the operators

    - Results on non-asymptotic bounds on the distribution of eigenvalues of the operators

SSLO: Spatio-Spectral Limiting Operator

Let $Q \subset \mathbb{R}^d$ with positive and finite Lebesgue measures

Let $S \subset \mathbb{R}^d$ with positive and finite Lebesgue measures.

SSLO: Spatio-Spectral Limiting Operator

Let $Q \subset \mathbb{R}^d$ with positive and finite Lebesgue measures

Let $S \subset \mathbb{R}^d$ with positive and finite Lebesgue measures.

- Defn. "space cut-off" operator $P_Q : L^2(\mathbb{R}^d) \to L^2(Q)$

$$P_Q(\psi) = \chi_Q \cdot \psi$$

SSLO: Spatio-Spectral Limiting Operator

Let $Q \subset \mathbb{R}^d$ with positive and finite Lebesgue measures

Let $S \subset \mathbb{R}^d$ with positive and finite Lebesgue measures.

- Defn. "space cut-off" operator $P_Q : L^2(\mathbb{R}^d) \to L^2(Q)$

$$P_Q(\psi) = \chi_Q \cdot \psi$$

- Defn. "frequency cut-off" operator $B_S : L^2(\mathbb{R}^d) \to B(S)$

$$B_S(\psi) = \psi * \widehat{\chi_S}$$

Notation: $B(S)$ or PW_S is the space of functions with Fourier support in S.

SSLO: Spatio-Spectral Limiting Operator

Let $Q \subset \mathbb{R}^d$ with positive and finite Lebesgue measures

Let $S \subset \mathbb{R}^d$ with positive and finite Lebesgue measures.

- Defn. "space cut-off" operator $P_Q : L^2(\mathbb{R}^d) \to L^2(Q)$

$$P_Q(\psi) = \chi_Q \cdot \psi$$

- Defn. "frequency cut-off" operator $B_S : L^2(\mathbb{R}^d) \to B(S)$

$$B_S(\psi) = \psi * \widehat{\chi_S}$$

Notation: $B(S)$ or PW_S is the space of functions with Fourier support in S.

- Defn. *SSLO*

$$\mathcal{T}_{Q,S} := B_S P_Q B_S \ : B(S) \to B(S)$$

SSLO: Spatio-Spectral Limiting Operator

Let $Q \subset \mathbb{R}^d$ with positive and finite Lebesgue measures

Let $S \subset \mathbb{R}^d$ with positive and finite Lebesgue measures.

- Defn. "space cut-off" operator $P_Q : L^2(\mathbb{R}^d) \to L^2(Q)$

$$P_Q(\psi) = \chi_Q \cdot \psi$$

- Defn. "frequency cut-off" operator $B_S : L^2(\mathbb{R}^d) \to B(S)$

$$B_S(\psi) = \psi * \widehat{\chi_S}$$

  Notation: $B(S)$ or PW_S is the space of functions with Fourier support in S.

- Defn. *SSLO*

$$\mathcal{T}_{Q,S} := B_S P_Q B_S \ : B(S) \to B(S)$$

- By the spectral theory, the eigenvalues are monotonic in $[0,1]$, and decrease to zero:

$$1 > \lambda_1(S,Q) \geqslant \lambda_2(S,Q) \geqslant \cdots \geqslant \lambda_k(S,Q) \geqslant \cdots > 0$$

SSLO: Spatio-Spectral Limiting Operator

- The eigenfunctions $\{\psi_k\}$ of SSLOs have frequency support in S.
- $\{\psi_k\}$ is an orthonormal basis for $B(S)$.

SSLO: Spatio-Spectral Limiting Operator

- The eigenfunctions $\{\psi_k\}$ of SSLOs have frequency support in S.
- $\{\psi_k\}$ is an orthonormal basis for $B(S)$.

Example. Let $Q = [0, 1]$ and $S = [-W, W]$.

SSLO: Spatio-Spectral Limiting Operator

- The eigenfunctions $\{\psi_k\}$ of SSLOs have frequency support in S.
- $\{\psi_k\}$ is an orthonormal basis for $B(S)$.

Example. Let $Q = [0, 1]$ and $S = [-W, W]$. The operator $\mathcal{T}$ is known as *time-frequency limiting operators*.

SSLO: Spatio-Spectral Limiting Operator

- The eigenfunctions $\{\psi_k\}$ of SSLOs have frequency support in S.
- $\{\psi_k\}$ is an orthonormal basis for $B(S)$.

Example. Let $Q = [0, 1]$ and $S = [-W, W]$. The operator $\mathcal{T}$ is known as *time-frequency limiting operators*.

- The eigenfunctions $\{\psi_k\}$ are prolate spheroidal wave functions (PSWFs).

SSLO: Spatio-Spectral Limiting Operator

- The eigenfunctions $\{\psi_k\}$ of SSLOs have frequency support in S.
- $\{\psi_k\}$ is an orthonormal basis for $B(S)$.

Example. Let $Q = [0, 1]$ and $S = [-W, W]$. The operator $\mathcal{T}$ is known as *time-frequency limiting operators*.

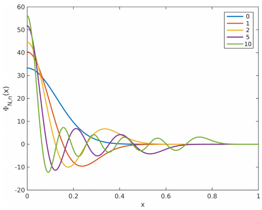
- The eigenfunctions $\{\psi_k\}$ are prolate spheroidal wave functions (PSWFs).
- These eigenfunctions are defined recursively through Zernike's polynomials.

SSLO: Spatio-Spectral Limiting Operator

- The eigenfunctions $\{\psi_k\}$ of SSLOs have frequency support in S.
- $\{\psi_k\}$ is an orthonormal basis for $B(S)$.

Example. Let $Q = [0,1]$ and $S = [-W, W]$. The operator $\mathcal{T}$ is known as *time-frequency limiting operators.*

- The eigenfunctions $\{\psi_k\}$ are prolate spheroidal wave functions (PSWFs).
- These eigenfunctions are defined recursively through Zernike's polynomials.



Eigenfunctions Prolates
Roy R. Lederman, 2017

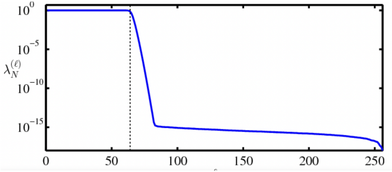


FIGURE: Eigenvalues λ_k
Z. Zhu, S. Karnik, M. A. Davenport, J. Romberg and M. B. Wakin, IEEE, 2018

Slepian 1964 - generalized prolate spheroidal functions (GPSFs) - Both the space and frequency domains are balls

P. Greengard, V. Rokhlin, K. Serkh 2018 - computation of approximation of GPSFs, interpolation and numerical computations ...

Background

Where do these operators originate, and what makes them significant?

Where do these operators originate, and what makes them significant?

These SSLO operators $B_S P_Q B_S$ were explored in dimension $d = 1$ in a series of Bell Lab papers by H. Landau- H. Pollak -D. Slepian, H. Widom and I. Daubechies between 1960-1980.

Where do these operators originate, and what makes them significant?

These SSLO operators $B_S P_Q B_S$ were explored in dimension $d = 1$ in a series of Bell Lab papers by H. Landau- H. Pollak -D. Slepian, H. Widom and I. Daubechies between 1960-1980.

The authors studied this operators to find a friendly solution to a practical paradox:

"An entire function cannot vanish on any interval without vanishing identically. Thus a function cannot be simultaneously band-limited and time-limited.

Where do these operators originate, and what makes them significant?

These SSLO operators $B_S P_Q B_S$ were explored in dimension $d = 1$ in a series of Bell Lab papers by H. Landau- H. Pollak -D. Slepian, H. Widom and I. Daubechies between 1960-1980.

The authors studied this operators to find a friendly solution to a practical paradox:

"An entire function cannot vanish on any interval without vanishing identically. Thus a function cannot be simultaneously band-limited and time-limited. And yet, in the practical design of communications systems, that assumption, and some of its theoretical consequences such as sampling, are exploited in an essential way without generating contradictions.

Where do these operators originate, and what makes them significant?

These SSLO operators $B_S P_Q B_S$ were explored in dimension $d = 1$ in a series of Bell Lab papers by H. Landau- H. Pollak -D. Slepian, H. Widom and I. Daubechies between 1960-1980.

The authors studied this operators to find a friendly solution to a practical paradox:

"An entire function cannot vanish on any interval without vanishing identically. Thus a function cannot be simultaneously band-limited and time-limited. And yet, in the practical design of communications systems, that assumption, and some of its theoretical consequences such as sampling, are exploited in an essential way without generating contradictions. Thus the problem of reconciling the time-limited property and frequency-limited property presents an interesting challenge" *H. Landau*

Where do these operators originate, and what makes them significant?

These SSLO operators $B_S P_Q B_S$ were explored in dimension $d = 1$ in a series of Bell Lab papers by H. Landau- H. Pollak -D. Slepian, H. Widom and I. Daubechies between 1960-1980.

The authors studied this operators to find a friendly solution to a practical paradox:

"An entire function cannot vanish on any interval without vanishing identically. Thus a function cannot be simultaneously band-limited and time-limited. And yet, in the practical design of communications systems, that assumption, and some of its theoretical consequences such as sampling, are exploited in an essential way without generating contradictions. Thus the problem of reconciling the time-limited property and frequency-limited property presents an interesting challenge" *H. Landau*

Addressing this question is equivalent to addressing the following problem:

Problem: How can we construct bandlimited functions with maximum 'energy' within a specified compact region?

$$\text{Construct } f: \ \text{supp}\hat{f} \subset S \text{ with max } \|f\|_{L^2(Q)}$$

Problem: How can we construct bandlimited functions with maximum 'energy' within a specified compact region?

$$\text{Construct } f: \ \text{supp}\hat{f} \subset S \text{ with max } \|f\|_{L^2(Q)}$$

Optimization problem

$$\begin{aligned} \text{Maximize:} \quad & \|P_Q f\|^2 \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

Problem: How can we construct bandlimited functions with maximum 'energy' within a specified compact region?

$$\text{Construct } f: \ \text{supp}\hat{f} \subset S \text{ with max } \|f\|_{L^2(Q)}$$

Optimization problem

$$\begin{aligned} \text{Maximize:} \quad & \|P_Q f\|^2 \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

We can express the objective function as follows:

$$\|P_Q f\|^2_{L^2(\mathbb{R}^d)} = \langle P_Q f, f \rangle = \langle P_Q B_S f, B_S f \rangle = \langle B_S P_Q B_S f, f \rangle$$

Problem: How can we construct bandlimited functions with maximum 'energy' within a specified compact region?

Construct f : $\text{supp}\hat{f} \subset S$ with max $\|f\|_{L^2(Q)}$

Optimization problem

$$\begin{aligned} \text{Maximize:} \quad & \|P_Q f\|^2 \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

We can express the objective function as follows:

$$\|P_Q f\|^2_{L^2(\mathbb{R}^d)} = \langle P_Q f, f \rangle = \langle P_Q B_S f, B_S f \rangle = \langle B_S P_Q B_S f, f \rangle$$

New look

$$\begin{aligned} \text{Maximize:} \quad & \langle B_S P_Q B_S f, f \rangle \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

Problem: How can we construct bandlimited functions with maximum 'energy' within a specified compact region?

Construct f : $\text{supp}\hat{f} \subset S$ with max $\|f\|_{L^2(Q)}$

Optimization problem

$$\begin{aligned} \text{Maximize:} \quad & \|P_Q f\|^2 \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

We can express the objective function as follows:

$$\|P_Q f\|^2_{L^2(\mathbb{R}^d)} = \langle P_Q f, f \rangle = \langle P_Q B_S f, B_S f \rangle = \langle B_S P_Q B_S f, f \rangle$$

New look

$$\begin{aligned} \text{Maximize:} \quad & \langle B_S P_Q B_S f, f \rangle \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

By the variational characterization of eigenvalues, the maximizers of the optimization problem correspond to the eigenfunctions of the SSLO with maximum eigenvalues.

Problem: How can we construct bandlimited functions with maximum 'energy' within a specified compact region?

$$\text{Construct } f: \ \text{supp}\hat{f} \subset S \text{ with max } \|f\|_{L^2(Q)}$$

Optimization problem

$$\begin{aligned} \text{Maximize:} \quad & \|P_Q f\|^2 \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

We can express the objective function as follows:

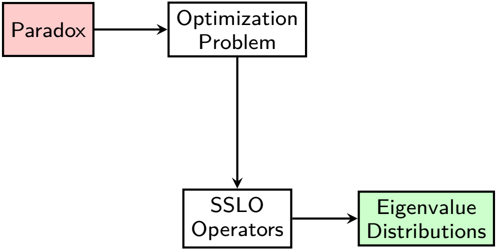
$$\|P_Q f\|^2_{L^2(\mathbb{R}^d)} = \langle P_Q f, f\rangle = \langle P_Q B_S f, B_S f\rangle = \langle B_S P_Q B_S f, f\rangle$$

New look

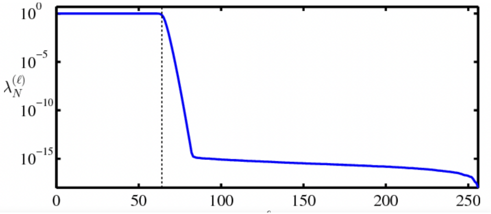
$$\begin{aligned} \text{Maximize:} \quad & \langle B_S P_Q B_S f, f\rangle \\ \text{Subject to:} \quad & B_S(f) = f \\ & \|f\| = 1 \end{aligned}$$

By the variational characterization of eigenvalues, the maximizers of the optimization problem correspond to the eigenfunctions of the SSLO with maximum eigenvalues.

Summary: To solve the paradox, one approach is to explore the SSLO operators, and a method to explore these operators is to understand the distribution of their eigenvalues.


Paradox
Optimization Problem
SSLO Operators
Eigenvalue Distributions

Recall:

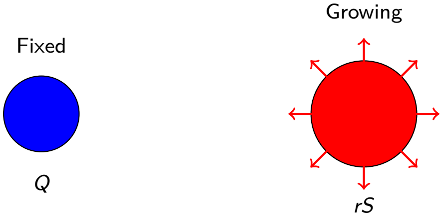


Z. Zhu, S. Karnik, M. A. Davenport, J. Romberg and M. B. Wakin, IEEE Signal Processing Letters, vol. 25, no. 1, pp. 95-99, Jan. 2018.

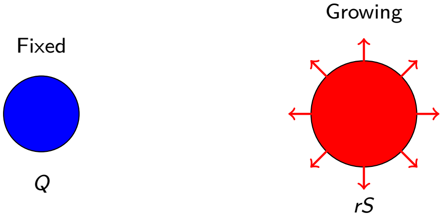
1. Counting of eigenvalues near "1"
2. Counting of eigenvalues in "plunge region" (descends abruptly)
3. Estimating the rate of decay at tail

(Landau '75) Let $Q, S \subset \mathbb{R}^d$ be measurable sets.

(Landau '75) Let $Q, S \subset \mathbb{R}^d$ be measurable sets. Let rS be an isotropic dilation of S, $r > 0$.

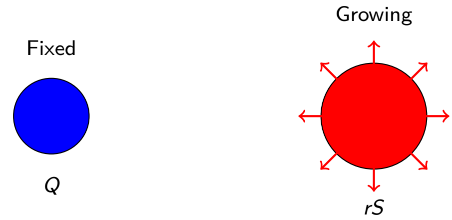


(Landau '75) Let $Q, S \subset \mathbb{R}^d$ be measurable sets. Let rS be an isotropic dilation of S, $r > 0$.



Landau [1] considers $B_{rS} P_Q B_{rS}$ and describes its distributions as follows:

(Landau '75) Let $Q, S \subset \mathbb{R}^d$ be measurable sets. Let rS be an isotropic dilation of S, $r > 0$.



Landau [1] considers $B_{rS} P_Q B_{rS}$ and describes its distributions as follows:

Theorem

Let $\{\lambda_k(Q, rS)\}_{k \geqslant 0}$ denote the sequence of eigenvalues of the operator $B_{rS} P_Q B_{rS}$.

(Landau '75) Let $Q, S \subset \mathbb{R}^d$ be measurable sets. Let rS be an isotropic dilation of S, $r > 0$.

Landau [1] considers $B_{rS}P_QB_{rS}$ and describes its distributions as follows:

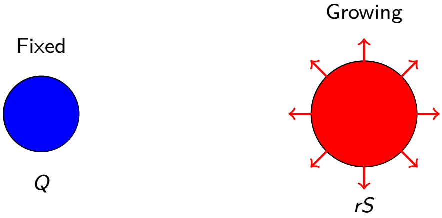
Theorem

Let $\{\lambda_k(Q, rS)\}_{k\geqslant 0}$ *denote the sequence of eigenvalues of the operator* $B_{rS}P_QB_{rS}$.

With $\epsilon \in (0, 1)$, *let*

$$N_\epsilon(r) = \{k : \ \lambda_k(Q, rS) \geqslant \epsilon\}$$

(Landau '75) Let $Q, S \subset \mathbb{R}^d$ be measurable sets. Let rS be an isotropic dilation of S, $r > 0$.



Landau [1] considers $B_{rS} P_Q B_{rS}$ and describes its distributions as follows:

Theorem

Let $\{\lambda_k(Q, rS)\}_{k \geqslant 0}$ denote the sequence of eigenvalues of the operator $B_{rS} P_Q B_{rS}$.

With $\epsilon \in (0,1)$, let

$$N_\epsilon(r) = \{k : \ \lambda_k(Q, rS) \geqslant \epsilon\}$$

Then

$$\lim_{r \to \infty} \frac{N_\epsilon(r)}{r^d} = (2\pi)^{-d} |Q| \cdot |S| \tag{1}$$

A key observation is that Landau's result

$$\lim_{r\to\infty} r^{-d} N_\epsilon(r) = (2\pi)^{-d}|Q|\cdot|S|$$

lacks a quantitative rate of convergence as $r \to \infty$.

[2] P. Greengard, K. Serkh, *On generalized prolate spheroidal functions*, https://arxiv.org/pdf/1811.02733.pdf, 2018.

A key observation is that Landau's result

$$\lim_{r\to\infty} r^{-d} N_\epsilon(r) = (2\pi)^{-d}|Q|\cdot|S|$$

lacks a quantitative rate of convergence as $r \to \infty$.

For fast computing tasks in numerical analysis and related fields, it is crucial to determine both the quantitative rate of convergence and non-asymptotic bounds on the number of SSLO eigenvalues, particularly those not close to 0 or 1.

Tasks such as: interpolations, integration, differentiation and sampling of bandlimited functions. [2]

[2] P. Greengard, K. Serkh, *On generalized prolate spheroidal functions*, https://arxiv.org/pdf/1811.02733.pdf, 2018.

Background

Here, we pose the following problem:

Quantaitive estimation: Estimate the distribution of the eigenvalues of the operator $T_{Q,S}$ for given measurable subsets Q and S in $\mathbb{R}^d$.

More specifically, for $\epsilon \in (0, 1/2)$, one is interested

1. to find bounds for
$$N_\epsilon := \sharp\{k : \lambda_k(Q,S) \geqslant \epsilon\}$$
2. to find bounds for
$$M_\epsilon := \sharp\{k : \lambda_k(Q,S) \in (\epsilon, 1-\epsilon)\}$$
3. Rate of decay $\lambda_k(Q,S) \to 0$?

Background

Here, we pose the following problem:

Quantaitive estimation: Estimate the distribution of the eigenvalues of the operator $T_{Q,S}$ for given measurable subsets Q and S in $\mathbb{R}^d$.

More specifically, for $\epsilon \in (0, 1/2)$, one is interested

1. to find bounds for

$$N_\epsilon := \sharp\{k : \lambda_k(Q,S) \geqslant \epsilon\}$$

2. to find bounds for

$$M_\epsilon := \sharp\{k : \lambda_k(Q,S) \in (\epsilon, 1-\epsilon)\}$$

3. Rate of decay $\lambda_k(Q,S) \to 0$?

Practical Importance: Notice that any function with $\mathrm{supp}(\hat{f}) \subset S$ with time concentration in Q can be represented by

$$f = \sum_\lambda c_\lambda \psi_\lambda$$

For numerical computations (in tasks such as interpolations, integration, differentiation and sampling of bandlimited functions) and efficiency of algorithms, understanding the clustering behaviour of eigenvalues is crucial.

The space and frequency domains have a strong influence on the investigation of the eigenvalue distribution of the SSLO operators, as observed by Landau.

The space and frequency domains have a strong influence on the investigation of the eigenvalue distribution of the SSLO operators, as observed by Landau.

Therefore, it is crucial to address this issue by considering specific cases of domains.

The space and frequency domains have a strong influence on the investigation of the eigenvalue distribution of the SSLO operators, as observed by Landau.

Therefore, it is crucial to address this issue by considering specific cases of domains.

For this purpose, we looked at the following scenarios:

- cube-cube: $Q = [0, 1]^d$ and $S = [-W, W]^d$.

The space and frequency domains have a strong influence on the investigation of the eigenvalue distribution of the SSLO operators, as observed by Landau.

Therefore, it is crucial to address this issue by considering specific cases of domains.

For this purpose, we looked at the following scenarios:

- cube-cube: $Q = [0,1]^d$ and $S = [-W, W]^d$.
- cube-convex: $Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.

The space and frequency domains have a strong influence on the investigation of the eigenvalue distribution of the SSLO operators, as observed by Landau.

Therefore, it is crucial to address this issue by considering specific cases of domains.

For this purpose, we looked at the following scenarios:

- cube-cube: $Q = [0,1]^d$ and $S = [-W, W]^d$.
- cube-convex: $Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.
- Q and S with maximal Ahlfors regular boundary

The space and frequency domains have a strong influence on the investigation of the eigenvalue distribution of the SSLO operators, as observed by Landau.

Therefore, it is crucial to address this issue by considering specific cases of domains.

For this purpose, we looked at the following scenarios:

- cube-cube: $Q = [0,1]^d$ and $S = [-W, W]^d$.
- cube-convex: $Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.
- Q and S with maximal Ahlfors regular boundary

To clarify, to the best of our knowledge, the quantitative version of Landau's result in higher dimensions had not been explored until we began our work in 2023.

Outline for the rest of the presentation:

PART I

- Definition of domains with maximally Ahlfors regular boundary
- Quantitative results for the set of eigenvalues of $B_S P_Q B_S$, when both domains Q and S are "regular".

Outline for the rest of the presentation:

PART I

- Definition of domains with maximally Ahlfors regular boundary
- Quantitative results for the set of eigenvalues of $B_S P_Q B_S$, when both domains Q and S are "regular".

PART II

- Discuss the techniques employed
  - Quantitative results for when Q and S are cubes.

Outline for the rest of the presentation:

Part I

- Definition of domains with maximally Ahlfors regular boundary
- Quantitative results for the set of eigenvalues of $B_S P_Q B_S$, when both domains Q and S are "regular".

Part II

- Discuss the techniques employed
  - Quantitative results for when Q and S are cubes.

Part III

- Quantitative results for the set of eigenvalues for when Q is a cube and S is convex.
  - Whitney decomposition of domain
  - construction of wave packets
  - Using the wave packets for counting

PART I

- Definition of domains with maximally Ahlfors regular boundary
- Quantitative upper bound for the set of eigenvalues of $B_S P_Q B_S$ near 1, when both domains Q and S are maximally A. regular domains.

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ ($d \geqslant 2$) has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ ($d \geqslant 2$) has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Theorem (k.hughes, a.israel, a.m)

Let $F \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial F}$.

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ ($d \geqslant 2$) has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Theorem (k.hughes, a.israel, a.m)

Let $F \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial F}$.
Let $S \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial S}$.

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ $(d \geqslant 2)$ has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Theorem (k.hughes, a.israel, a.m)

Let $F \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial F}$.
Let $S \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial S}$.

Consider the SSLO $\quad B_S P_F B_S$.

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ ($d \geqslant 2$) has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Theorem (k.hughes, a.israel, a.m)

Let $F \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial F}$. Let $S \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial S}$.

Consider the SSLO $\quad B_S P_F B_S$.

Then for any $\epsilon \in (0, 1/2)$

$$\#\{k : \lambda_k(F,S) > \epsilon\} = (2\pi)^{-d}|F| \cdot |S| + \mathrm{Err}(F,S,\epsilon),$$

when $\mathcal{H}_{d-1}(\partial F) \cdot \mathcal{H}_{d-1}(\partial S)$ is large enough.

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ $(d \geqslant 2)$ has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Theorem (k.hughes, a.israel, a.m)

Let $F \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial F}$.
Let $S \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial S}$.

Consider the SSLO $\quad B_S P_F B_S$.

Then for any $\epsilon \in (0, 1/2)$

$$\#\{k : \lambda_k(F,S) > \epsilon\} = (2\pi)^{-d}|F| \cdot |S| + \mathrm{Err}(F,S,\epsilon),$$

when $\mathcal{H}_{d-1}(\partial F) \cdot \mathcal{H}_{d-1}(\partial S)$ *is large enough.*

Here,

$$|\mathrm{Err}(F,S,\epsilon)| \leqslant C_d \frac{\mathcal{H}_{d-1}(\partial F)}{\kappa_{\partial F}} \frac{\mathcal{H}_{d-1}(\partial S)}{\kappa_{\partial S}} \left\{ \textit{sharp up to logarithmic factors} \right\}$$

Defn: We say that a (bounded) set $\Omega \subset \mathbb{R}^d$ $(d \geqslant 2)$ has *maximally Ahlfors regular* boundary with *regularity constant* $\kappa_{\partial\Omega} > 0$ provided that $\forall\ x \in \partial\Omega$

$$\mathcal{H}_{d-1}(\partial\Omega \cap B(x,r)) \geqslant \kappa_{\partial\Omega} r^{d-1}, \qquad 0 < r \leqslant \mathcal{H}_{d-1}(\partial\Omega)^{1/(d-1)}.$$

Theorem (K.Hughes, A.Israel, A.M)

Let $F \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial F}$.
Let $S \subset \mathbb{R}^d$ has max. Ahlfors regular boundaries with regularity constants $\kappa_{\partial S}$.

Consider the SSLO $\quad B_S P_F B_S$.

Then for any $\epsilon \in (0, 1/2)$

$$\#\{k : \lambda_k(F,S) > \epsilon\} = (2\pi)^{-d}|F| \cdot |S| + \mathrm{Err}(F,S,\epsilon),$$

when $\mathcal{H}_{d-1}(\partial F) \cdot \mathcal{H}_{d-1}(\partial S)$ *is large enough.*

Here,

$$|\mathrm{Err}(F,S,\epsilon)| \leqslant C_d \frac{\mathcal{H}_{d-1}(\partial F)}{\kappa_{\partial F}} \frac{\mathcal{H}_{d-1}(\partial S)}{\kappa_{\partial S}} \left\{\textit{sharp up to logarithmic factors}\right\}$$

$\lambda_k(F,S)$ are eigenvalues of the $B_S P_F B_S$.

$$|\mathrm{Err}(F,S,\epsilon)| \lesssim_d \frac{\mathcal{H}_{d-1}(\partial F)}{\kappa_{\partial F}} \frac{\mathcal{H}_{d-1}(\partial S)}{\kappa_{\partial S}} \Big\{ \log\left(\mathcal{H}_{d-1}(\partial F)\mathcal{H}_{d-1}(\partial S)\right) \log(\min\{\epsilon, 1-\epsilon\}^{-1})^d + \log\left(\mathcal{H}_{d-1}(\partial F)\mathcal{H}_{d-1}(\partial S)\right)^3 \log(\min\{\epsilon, 1-\epsilon\}^{-1}) \Big\}$$

$$|\mathrm{Err}(F,S,\epsilon)| \lesssim_d \frac{\mathcal{H}_{d-1}(\partial F)}{\kappa_{\partial F}} \frac{\mathcal{H}_{d-1}(\partial S)}{\kappa_{\partial S}} \Big\{ \log\left(\mathcal{H}_{d-1}(\partial F)\mathcal{H}_{d-1}(\partial S)\right) \log(\min\{\epsilon, 1-\epsilon\}^{-1})^d + \log\left(\mathcal{H}_{d-1}(\partial F)\mathcal{H}_{d-1}(\partial S)\right)^3 \log(\min\{\epsilon, 1-\epsilon\}^{-1}) \Big\}$$

Our Result:

$$\boxed{\#\{k : \lambda_k(F,S) > \epsilon\} = (2\pi)^{-d}|F| \cdot |S| + \mathrm{Err}(F,S,\epsilon),}$$

$$|\mathrm{Err}(F,S,\epsilon)| \lesssim_d \frac{\mathcal{H}_{d-1}(\partial F)}{\kappa_{\partial F}} \frac{\mathcal{H}_{d-1}(\partial S)}{\kappa_{\partial S}} \Big\{ \log\left(\mathcal{H}_{d-1}(\partial F)\mathcal{H}_{d-1}(\partial S)\right) \log(\min\{\epsilon, 1-\epsilon\}^{-1})^d + \log\left(\mathcal{H}_{d-1}(\partial F)\mathcal{H}_{d-1}(\partial S)\right)^3 \log(\min\{\epsilon, 1-\epsilon\}^{-1}) \Big\}$$

Our Result:

$$\boxed{\#\{k : \lambda_k(F,S) > \epsilon\} = (2\pi)^{-d}|F| \cdot |S| + \mathrm{Err}(F,S,\epsilon),}$$

Landau's result:

$$\boxed{\lim_{r\to\infty} r^{-d}\{k : \ \lambda_k(Q, rS) \geqslant \epsilon\} = (2\pi)^{-d}|Q| \cdot |S|}$$

Part II Discuss the techniques employed

- Quantitative upper bound for cube-cube case

Let $Q = [0,1]^d$, and $S = [-W, W]^d$ for some $W \geqslant 2\pi$.

Let $Q = [0,1]^d$, and $S = [-W, W]^d$ for some $W \geqslant 2\pi$.

THEOREM (CUBE-CUBE CASE, A. ISRAEL, A.M)

Let $\mathcal{T}_{Q,S} = B_S P_Q B_S$ be THE SSLO.

Let $\{\lambda_k\}$ be the eigenvalues of $\mathcal{T}_{Q,S}$. For $\epsilon \in (0, 1/2)$, we have

Let $Q = [0,1]^d$, and $S = [-W, W]^d$ for some $W \geqslant 2\pi$.

Theorem (cube-cube case, A. Israel, A.M)

Let $\mathcal{T}_{Q,S} = B_S P_Q B_S$ be THE SSLO.

Let $\{\lambda_k\}$ be the eigenvalues of $\mathcal{T}_{Q,S}$. For $\epsilon \in (0, 1/2)$, we have

$$\begin{aligned} \left| \# \{k : \lambda_k > \epsilon\} - (W/\pi)^d \right| &\lesssim_d B_d(\epsilon, W), \\ \# \{k : \lambda_k \in (\epsilon, 1-\epsilon)\} &\lesssim_d B_d(\epsilon, W). \end{aligned} \tag{2}$$

where

$$B_d(\epsilon, W) \leqslant W^{d-1} \log(W) \log(1/\epsilon) + (\log(W) \log(1/\epsilon))^d.$$

Let $Q = [0,1]^d$, and $S = [-W, W]^d$ for some $W \geqslant 2\pi$.

Theorem (cube-cube case, A. Israel, A.M)

Let $\mathcal{T}_{Q,S} = B_S P_Q B_S$ be THE SSLO.

Let $\{\lambda_k\}$ be the eigenvalues of $\mathcal{T}_{Q,S}$. For $\epsilon \in (0, 1/2)$, we have

$$\begin{aligned} \left| \# \{k : \lambda_k > \epsilon\} - (W/\pi)^d \right| &\lesssim_d B_d(\epsilon, W), \\ \# \{k : \lambda_k \in (\epsilon, 1-\epsilon)\} &\lesssim_d B_d(\epsilon, W). \end{aligned} \tag{2}$$

where

$$B_d(\epsilon, W) \leqslant W^{d-1} \log(W) \log(1/\epsilon) + (\log(W) \log(1/\epsilon))^d.$$

Remark: Notice $\mu_d(S)\mu_d(Q)/(2\pi)^d = (W/\pi)^d$.

Proof outline: d-folding the results in 1-dimension.

Proof techniques for cube-cube case: Reduction to the 1-dimensional case

- Let $I = [0, 1]$ and and $J = [-W, W]$.
- Let $\mathcal{T}_{I,J}$ denote the SSLO operator associated to $I = [0, 1]$ and $J = [-W, W]$.

Proof techniques for cube-cube case: Reduction to the 1-dimensional case

- Let $I = [0,1]$ and and $J = [-W, W]$.
- Let $\mathcal{T}_{I,J}$ denote the SSLO operator associated to $I = [0,1]$ and $J = [-W, W]$.
- Then the SSLO operator associated to $Q = I^d$ and $S = J^d$ is
$$\mathcal{T}_{Q,S} = \otimes_{k=1}^{d} \mathcal{T}_{I,J}.$$

Proof techniques for cube-cube case: Reduction to the 1-dimensional case

- Let $I = [0,1]$ and and $J = [-W, W]$.
- Let $\mathcal{T}_{I,J}$ denote the SSLO operator associated to $I = [0,1]$ and $J = [-W, W]$.
- Then the SSLO operator associated to $Q = I^d$ and $S = J^d$ is
$$\mathcal{T}_{Q,S} = \otimes_{k=1}^{d} \mathcal{T}_{I,J}.$$
- The set of eigenvalues of $\mathcal{T}_{Q,S}$ is given by the product of the eigenvalues of each one-dimensional operators:
$$\{\lambda_{\otimes \mathcal{T}_{I,J}}\} = \left\{ \prod_{k=1}^{d} \lambda_{\mathcal{T}_{I,J}} \right\}.$$

Proof techniques for cube-cube case: Reduction to the 1-dimensional case

- Let $I = [0, 1]$ and and $J = [-W, W]$.
- Let $\mathcal{T}_{I,J}$ denote the SSLO operator associated to $I = [0, 1]$ and $J = [-W, W]$.
- Then the SSLO operator associated to $Q = I^d$ and $S = J^d$ is
$$\mathcal{T}_{Q,S} = \otimes_{k=1}^{d} \mathcal{T}_{I,J}.$$
- The set of eigenvalues of $\mathcal{T}_{Q,S}$ is given by the product of the eigenvalues of each one-dimensional operators:
$$\{\lambda_{\otimes \mathcal{T}_{I,J}}\} = \left\{ \prod_{k=1}^{d} \lambda_{\mathcal{T}_{I,J}} \right\}.$$
- In $d = 1$, we have
$$|\#\{k : \lambda_k > \epsilon\} - W/\pi| \lesssim \log(W) \log(1/(\epsilon(1-\epsilon))).$$

Proof techniques for cube-cube case: Reduction to the 1-dimensional case

- Let $I = [0, 1]$ and and $J = [-W, W]$.
- Let $\mathcal{T}_{I,J}$ denote the SSLO operator associated to $I = [0, 1]$ and $J = [-W, W]$.
- Then the SSLO operator associated to $Q = I^d$ and $S = J^d$ is
$$\mathcal{T}_{Q,S} = \otimes_{k=1}^{d} \mathcal{T}_{I,J}.$$
- The set of eigenvalues of $\mathcal{T}_{Q,S}$ is given by the product of the eigenvalues of each one-dimensional operators:
$$\{\lambda_{\otimes \mathcal{T}_{I,J}}\} = \left\{ \prod_{k=1}^{d} \lambda_{\mathcal{T}_{I,J}} \right\}.$$
- In $d = 1$, we have
$$|\#\{k : \lambda_k > \epsilon\} - W/\pi| \lesssim \log(W) \log(1/(\epsilon(1-\epsilon))).$$
- Apply the results in $d = 1$; we obtain the first inequality in our theorem:
$$\left| M_\epsilon(\otimes \mathcal{T}) - (W/\pi)^d \right| \lesssim_d W^{d-1} \log(W) \log(1/\epsilon) + (\log(W) \log(1/\epsilon))^d$$

QED.

Eigenvalues DONOT change under affine transformation (rescaling+translation):

If we change (Q, S) into $(\delta Q, \delta^{-1} S + \alpha)$, the eigenvalue set of SSLO does not change.

Eigenvalues DONOT change under affine transformation (rescaling+translation):

If we change (Q, S) into $(\delta Q, \delta^{-1} S + \alpha)$, the eigenvalue set of SSLO does not change.

Corollary

Let Q_1 and Q_2 be cubes in $\mathbb{R}^d$ with sidelengths δ_i that are sufficiently large. Consider the operator

$$T_{Q_1,Q_2} = B_{Q_2} P_{Q_1} B_{Q_2}.$$

Then for every $\epsilon \in (0, 1/2)$,

$$M_\epsilon(T_{Q_1,Q_2}) \lesssim \log(\delta_1\delta_2)^d \log(\epsilon^{-1})^d + (\delta_1\delta_2)^{d-1} \log(\delta_1\delta_2) \log(\epsilon^{-1}).$$

Technique of proof: By affine transformation, we reduce the case into $S = [-W, W]^d$ and $Q = [0, 1]^d$ for some large W.

Corollary (exponential decay property)

Let Q and S be compact sets and let $\Delta = diam_\infty(Q) \cdot diam_\infty(S)$. Then

$$\lambda_k(Q,S) \lesssim \exp\left(-c(\Delta)k^{1/d}\right), \quad \text{for } k \geqslant 1.$$

Sketch of proof:

- We dilate and translation $(Q,S) \mapsto (\alpha^{-1}Q + x, \alpha S + \xi)$, so that $Q \subset [0,1]^d$ and $S \subset [-\Delta, \Delta]^d$.

Corollary (exponential decay property)

Let Q and S be compact sets and let $\Delta = diam_\infty(Q) \cdot diam_\infty(S)$. Then

$$\lambda_k(Q,S) \lesssim \exp\left(-c(\Delta)k^{1/d}\right), \quad \text{for } k \geqslant 1.$$

Sketch of proof:

- We dilate and translation $(Q,S) \mapsto (\alpha^{-1}Q + x, \alpha S + \xi)$, so that $Q \subset [0,1]^d$ and $S \subset [-\Delta, \Delta]^d$.
- We use the following observations due to Landau '67: For all $k \geqslant 0$,

$$\lambda_k(Q,S) = \lambda_k(S,Q), \tag{3}$$

$$\lambda_k(Q,S_1) \leqslant \lambda_k(Q,S_2), \quad \text{if } S_1 \subset S_2, \tag{4}$$

$$\lambda_k(Q_1,S) \leqslant \lambda_k(Q_2,S), \quad \text{if } Q_1 \subset Q_2. \tag{5}$$

Corollary (exponential decay property)

Let Q and S be compact sets and let $\Delta = diam_\infty(Q) \cdot diam_\infty(S)$. Then

$$\lambda_k(Q,S) \lesssim \exp\left(-c(\Delta)k^{1/d}\right), \quad \textit{for } k \geqslant 1.$$

Sketch of proof:

- We dilate and translation $(Q,S) \mapsto (\alpha^{-1}Q + x, \alpha S + \xi)$, so that $Q \subset [0,1]^d$ and $S \subset [-\Delta,\Delta]^d$.
- We use the following observations due to Landau '67: For all $k \geqslant 0$,

$$\lambda_k(Q,S) = \lambda_k(S,Q), \tag{3}$$

$$\lambda_k(Q,S_1) \leqslant \lambda_k(Q,S_2), \quad \text{if } S_1 \subset S_2, \tag{4}$$

$$\lambda_k(Q_1,S) \leqslant \lambda_k(Q_2,S), \quad \text{if } Q_1 \subset Q_2. \tag{5}$$

- Thus

$$\lambda_k(Q,S) \leqslant \lambda_k([0,1]^d, [-\Delta,\Delta]^d).$$

QED.

ART III Quantitative upper bound of eigenvalues for cube-convex symmetry domains.

- Whitney decomposition of the frequency domain
- construction of wave packets
- Using the wave packets for counting

Defn: A convex set $S \subset \mathbb{R}^d$ is *coordinate-wise symmetric* if

$$\forall (x_1, \cdots, x_d) \in S \implies (\sigma_1 x_1, \cdots, \sigma_d x_d) \in S, \quad \text{for all } \sigma = (\sigma_1, \cdots, \sigma_d) \in \{\pm 1\}^d.$$

Assume that $S \subset B(0,1)$, $r > 0$ and $S(r) := rS$ is the r-isotropic dilation.

$Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.

Theorem (cub-convex case - A. Israel, A.M, Acha '24)

Let $Q := [0,1]^d$ and let $S \subset \mathbb{R}^d$ be a compact convex and symmetric set.

Given $\epsilon \in (0, 1/2)$, let

$$N_\epsilon(r) := \sharp\{k : \ \lambda_k(Q, rS) > \epsilon\} \quad \text{"near 1"}$$

$Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.

Theorem (cub-convex case - A. Israel, A.M, ACHA '24)

Let $Q := [0,1]^d$ and let $S \subset \mathbb{R}^d$ be a compact convex and symmetric set.

Given $\epsilon \in (0, 1/2)$, let

$$N_\epsilon(r) := \sharp\{k : \ \lambda_k(Q, rS) > \epsilon\} \quad \text{“near 1”}$$

Then

$$|N_\epsilon(r) - (2\pi)^{-d}\mu_d(S(r))| \leqslant E_d(r, \epsilon). \tag{6}$$

where

$$E_d(r, \epsilon) \lesssim_d r^{d-1} \log(r/\epsilon)^3 + \log(r/\epsilon)^{3d}$$

$Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.

Theorem (cub-convex case - A. Israel, A.M, ACHA '24)

Let $Q := [0,1]^d$ and let $S \subset \mathbb{R}^d$ be a compact convex and symmetric set.

Given $\epsilon \in (0, 1/2)$, let

$$N_\epsilon(r) := \sharp\{k : \ \lambda_k(Q, rS) > \epsilon\} \quad \text{"near 1"}$$

Then

$$|N_\epsilon(r) - (2\pi)^{-d}\mu_d(S(r))| \leqslant E_d(r, \epsilon). \tag{6}$$

where

$$E_d(r, \epsilon) \lesssim_d r^{d-1} \log(r/\epsilon)^3 + \log(r/\epsilon)^{3d}$$

Consequently, if

$$M_\epsilon(r) := \sharp\{k : \ \epsilon < \lambda_k(Q, rS) < 1 - \epsilon\} \quad \text{"Plunge region"}$$

then

$$M_\epsilon(r) \leqslant E_d(r, \epsilon).$$

$Q = [0,1]^d$ and $S \subset \mathbb{R}^d$ is a symmetric and compact convex set.

Theorem (cub-convex case - A. Israel, A.M, ACHA '24)

Let $Q := [0,1]^d$ and let $S \subset \mathbb{R}^d$ be a compact convex and symmetric set.

Given $\epsilon \in (0, 1/2)$, let

$$N_\epsilon(r) := \sharp\{k : \ \lambda_k(Q, rS) > \epsilon\} \quad \text{“near 1”}$$

Then

$$|N_\epsilon(r) - (2\pi)^{-d}\mu_d(S(r))| \leqslant E_d(r,\epsilon). \tag{6}$$

where

$$E_d(r,\epsilon) \lesssim_d r^{d-1}\log(r/\epsilon)^3 + \log(r/\epsilon)^{3d}$$

Consequently, if

$$M_\epsilon(r) := \sharp\{k : \ \epsilon < \lambda_k(Q, rS) < 1-\epsilon\} \quad \text{“Plunge region”}$$

then

$$M_\epsilon(r) \leqslant E_d(r,\epsilon).$$

Landau's result is an **IMMEDIATECONSEQUENCE** of (6) when $Q = [0,1]^d$ and S is convex and symmetric: $\lim_{r\to\infty} r^{-d}|N_\epsilon(r) - (2\pi)^{-d}\mu_d(rS)| = 0$

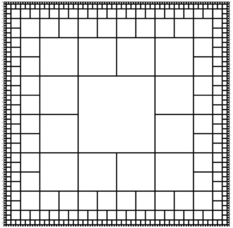
Sketch of Proof:

(1) We construct local-sine orthonormal basis, "wave packets", for $L^2(Q)$:

Sketch of Proof:

(1) We construct local-sine orthonormal basis, "wave packets", for $L^2(Q)$:

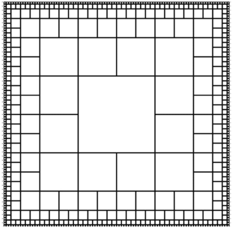
We let $\mathcal{D}$ denote a Whitney-type decomposition for $Q = [0,1]^d$, which includes cubes $L \in \mathcal{D}$.



Sketch of Proof:

(1) We construct local-sine orthonormal basis, "wave packets", for $L^2(Q)$:

We let $\mathcal{D}$ denote a Whitney-type decomposition for $Q = [0,1]^d$, which includes cubes $L \in \mathcal{D}$.



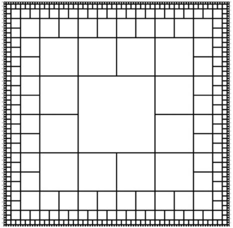
For any $L \in \mathcal{D}$, we construct "wave packets" $\{\psi_{L,\ell}\}_\ell \subset L^2(L)$ such that:

- Each $\psi_{L,\ell}$ is a C^∞
- $\bigcup_{\mathcal{I}} \{\psi_{L,\ell}\}$ is an orthonormal basis for $L^2(Q)$
- Each $\psi_{L,\ell}$ has near-exponential frequency decay
- L^2-energy of $\widehat{\psi_{L,\ell}}$ is concentrated at two points

Sketch of Proof:

(1) We construct local-sine orthonormal basis, "wave packets", for $L^2(Q)$:

We let $\mathcal{D}$ denote a Whitney-type decomposition for $Q = [0,1]^d$, which includes cubes $L \in \mathcal{D}$.



For any $L \in \mathcal{D}$, we construct "wave packets" $\{\psi_{L,\ell}\}_\ell \subset L^2(L)$ such that:

- Each $\psi_{L,\ell}$ is a C^∞
- $\bigcup_{\mathcal{I}} \{\psi_{L,\ell}\}$ is an orthonormal basis for $L^2(Q)$
- Each $\psi_{L,\ell}$ has near-exponential frequency decay
- L^2-energy of $\widehat{\psi_{L,\ell}}$ is concentrated at two points

(2) Next, we partition $\mathcal{I}$ as

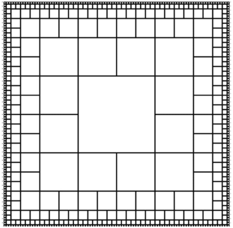
$$\mathcal{I} = \mathcal{I}_{low} \cup \mathcal{I}_{res} \cup \mathcal{I}_{hi}$$

with $\mathcal{I}_{res}$ and $\mathcal{I}_{low}$ finite sets,

Sketch of Proof:

(1) We construct local-sine orthonormal basis, "wave packets", for $L^2(Q)$:

We let $\mathcal{D}$ denote a Whitney-type decomposition for $Q = [0,1]^d$, which includes cubes $L \in \mathcal{D}$.



For any $L \in \mathcal{D}$, we construct "wave packets" $\{\psi_{L,\ell}\}_\ell \subset L^2(L)$ such that:

- Each $\psi_{L,\ell}$ is a C^∞
- $\bigcup_{\mathcal{I}} \{\psi_{L,\ell}\}$ is an orthonormal basis for $L^2(Q)$
- Each $\psi_{L,\ell}$ has near-exponential frequency decay
- L^2-energy of $\widehat{\psi_{L,\ell}}$ is concentrated at two points

(2) Next, we partition $\mathcal{I}$ as

$$\mathcal{I} = \mathcal{I}_{low} \cup \mathcal{I}_{res} \cup \mathcal{I}_{hi}$$

with $\mathcal{I}_{res}$ and $\mathcal{I}_{low}$ finite sets, such that, for some $\epsilon \in (0, 1/2)$,

$$\sum_{k \in \mathcal{I}_{hi}} \|\mathcal{T}\phi_k\|^2 + \sum_{k \in \mathcal{I}_{low}} \|(I - \mathcal{T})\phi_k\|^2 \leqslant \epsilon^2.$$

Classical construction of the local sine basis: R.R. Coifman and Y. Meyer, *Remarques sur l'analyse de Fourier à fenêtre*, In: C. R. Acad. Sci. Paris 312 (1991), pp. 259–261

We apply Functional Analysis Lemma to complete the proof.

Lemma (functional analysis lemma - israel, m.)

Let $\mathcal{H}$ be a real Hilbert space.

We apply Functional Analysis Lemma to complete the proof.

Lemma (functional analysis lemma - Israel, M.)

Let $\mathcal{H}$ be a real Hilbert space.

Let $T : \mathcal{H} \to \mathcal{H}$ be a positive semidefinite compact operator, with eigenvalues $\lambda_j(T)$, $j \geqslant 1$, be the eigenvalues of T, counted with multiplicity, and sorted in non-increasing order.

We apply Functional Analysis Lemma to complete the proof.

Lemma (functional analysis lemma - Israel, M.)

Let $\mathcal{H}$ be a real Hilbert space.

Let $T : \mathcal{H} \to \mathcal{H}$ be a positive semidefinite compact operator, with eigenvalues $\lambda_j(T)$, $j \geqslant 1$, be the eigenvalues of T, counted with multiplicity, and sorted in non-increasing order.

Let $\{\phi_k\}_{k \in \mathcal{I}}$ be an orthonormal basis for $\mathcal{H}$, with index set $\mathcal{I}$.

We apply Functional Analysis Lemma to complete the proof.

Lemma (functional analysis lemma - israel, m.)

Let $\mathcal{H}$ be a real Hilbert space.

Let $T : \mathcal{H} \to \mathcal{H}$ be a positive semidefinite compact operator, with eigenvalues $\lambda_j(T)$, $j \geqslant 1$, be the eigenvalues of T, counted with multiplicity, and sorted in non-increasing order.

Let $\{\phi_k\}_{k\in\mathcal{I}}$ be an orthonormal basis for $\mathcal{H}$, with index set $\mathcal{I}$. Let

$$\mathcal{I} = \mathcal{I}_{low} \cup \mathcal{I}_{res} \cup \mathcal{I}_{hi}$$

be a partition of $\mathcal{I}$, with $\mathcal{I}_{res}$ and $\mathcal{I}_{low}$ finite sets, such that, for some $\epsilon \in (0, 1/2)$,

$$\sum_{k\in\mathcal{I}_{hi}} \|T\phi_k\|^2 + \sum_{k\in\mathcal{I}_{low}} \|(I - T)\phi_k\|^2 \leqslant \epsilon^2. \tag{7}$$

We apply Functional Analysis Lemma to complete the proof.

Lemma (functional analysis lemma - Israel, M.)

Let $\mathcal{H}$ be a real Hilbert space.

Let $T : \mathcal{H} \to \mathcal{H}$ be a positive semidefinite compact operator, with eigenvalues $\lambda_j(T)$, $j \geqslant 1$, be the eigenvalues of T, counted with multiplicity, and sorted in non-increasing order.

Let $\{\phi_k\}_{k\in\mathcal{I}}$ be an orthonormal basis for $\mathcal{H}$, with index set $\mathcal{I}$. Let

$$\mathcal{I} = \mathcal{I}_{low} \cup \mathcal{I}_{res} \cup \mathcal{I}_{hi}$$

be a partition of $\mathcal{I}$, with $\mathcal{I}_{res}$ and $\mathcal{I}_{low}$ finite sets, such that, for some $\epsilon \in (0, 1/2)$,

$$\sum_{k\in\mathcal{I}_{hi}} \|T\phi_k\|^2 + \sum_{k\in\mathcal{I}_{low}} \|(I - T)\phi_k\|^2 \leqslant \epsilon^2. \tag{7}$$

Then

$$|M_\epsilon(T) - \#(\mathcal{I}_{low})| \leqslant \#(\mathcal{I}_{res}), \quad \text{and} \quad N_\epsilon(T) \leqslant \#(\mathcal{I}_{res})$$

where, $M_\epsilon(T) := \#\{j : \lambda_j(T) > \epsilon\}$ and $N_\epsilon(T) := \#\{j : \lambda_j(T) \in (\epsilon, 1-\epsilon)\}$.

(If (7), we say ϕ_k "mimic" the eigenfunctions.)

Now, an application of Functional Analysis Lemma to the SSLO's and the wave packets completes the proof of theorem.

Applications of our wave packets for analysis of a band-limited function $f \in L^2(\mathbb{R}^d)$:

1. Quadratures Scheme for estimating integration. The process of estimating $\int_B fd$ using quadrature rule involves finding a finite set of points $\{\xi_i\}$ in B_d and a finite set of weights $\{w_i\}$, complex numbers such that the integral can be approximated by $\sum_i w_i f(\xi_i)$ up to given machine precision ϵ:

$$\left| \int_B f(\xi) d\xi - \sum_i w_i f(\xi_i) \right| < \epsilon. \tag{8}$$

Applications of our wave packets for analysis of a band-limited function $f \in L^2(\mathbb{R}^d)$:

1. Quadratures Scheme for estimating integration. The process of estimating $\int_B fd$ using quadrature rule involves finding a finite set of points $\{\xi_i\}$ in B_d and a finite set of weights $\{w_i\}$, complex numbers such that the integral can be approximated by $\sum_i w_i f(\xi_i)$ up to given machine precision ϵ:

$$\left| \int_B f(\xi) d\xi - \sum_i w_i f(\xi_i) \right| < \epsilon. \tag{8}$$

2. Interpolation. Using our wave packets $\{g_i\}$, the main objective of the interpolation problem is to determine the coefficients $\{a_i\}_{i=1}^n$ such that

$$f(x_j) = a_1 g_1(x_j) + a_2 g_2(x_j) + \ldots + a_n g_n(x_j) \tag{9}$$

Some open questions:

1. How far is the top-eigenvalue of SSLO from 1? (sharp!)
2. Understand the distance between distinct eigenvalues.
   Our conjecture is: $|\lambda_k - \lambda_{k+1}| > c\lambda_k$
3. Identifying accumulation region of eigenvalues.

Some open questions:

1. How far is the top-eigenvalue of SSLO from 1? (sharp!)
2. Understand the distance between distinct eigenvalues.
   Our conjecture is: $|\lambda_k - \lambda_{k+1}| > c\lambda_k$
3. Identifying accumulation region of eigenvalues.
4. Take union of two disjoint intervals. How does the gap between parts of domains effect the top eigenvalue?
5. Cut the domain into finite pieces and send the parts away to infinity. Check the behaviour of eigenvalues.

Some open questions:

1. How far is the top-eigenvalue of SSLO from 1? (sharp!)
2. Understand the distance between distinct eigenvalues.
   Our conjecture is: $|\lambda_k - \lambda_{k+1}| > c\lambda_k$
3. Identifying accumulation region of eigenvalues.
4. Take union of two disjoint intervals. How does the gap between parts of domains effect the top eigenvalue?
5. Cut the domain into finite pieces and send the parts away to infinity. Check the behaviour of eigenvalues.
6. Can we choose another decomposition depending on the shape of domain and get sharper bounds?

Some open questions:

1. How far is the top-eigenvalue of SSLO from 1? (sharp!)
2. Understand the distance between distinct eigenvalues.
   Our conjecture is: $|\lambda_k - \lambda_{k+1}| > c\lambda_k$
3. Identifying accumulation region of eigenvalues.
4. Take union of two disjoint intervals. How does the gap between parts of domains effect the top eigenvalue?
5. Cut the domain into finite pieces and send the parts away to infinity. Check the behaviour of eigenvalues.
6. Can we choose another decomposition depending on the shape of domain and get sharper bounds?

Any input is welcome!

To summarize the key takeaways:

- Defined the spatio-spectral limiting operators (SSLO) for given fixed space and frequency domains, and highlighted some of the spectral properties of these operators.
- Discussed the significant role of spatio-spectral limiting operators in various applications.
- Highlighted the results in eigenvalue distribution regions in higher dimensions for three special cases of space and frequency domains:

  CUBE-CUBE; CUBE - CONVEX and symmetric; maximally AHLFORS REGULAR

References

1. I. Daubechies, *Time-Frequency Localization Operators: A Geometric Phase Space Approach*, IEEE Transactions on Information Theory, VOL. 34, NO. 4, 1988.

2. K. Hughes, A. Israel, A. Mayeli, *On the eigenvalue distribution of spatio-spectral limiting operators in higher dimensions, II*, arXiv:2403.13092

3. A. Israel and A. Mayeli, *On the Eigenvalue Distribution of Spatio-Spectral Limiting Operators in Higher Dimensions*, Applied and Computational Harmonic Analysis, Volume 70, 2024.

4. S. Karnik, J. Romberg, and M. Davenport. *Improved bounds for the eigenvalues of prolate spheroidal wave functions and discrete prolate spheroidal sequences*. In: Applied and Computational Harmonic Analysis 55.1 (2021), pp. 97 128.

5. H.J. Landau and H. Widom. *Eigenvalue distribution of time and frequency limiting*. In: Journal of Mathematical Analysis and Applications 77.2 (1980), pp. 469481.

6. D. Slepian. *Prolate spheroidal wave functions, Fourier analysis, and uncertainty IV: Extensions to many dimensions; generalized prolate spheroidal functions*. In: Bell Systems Tech. J. 43.6 (1964), pp. 3009 3058.

Thank you for listening!