TOPOLOGY OF SPACES OF STRUCTURED VECTOR CONFIGURATIONS

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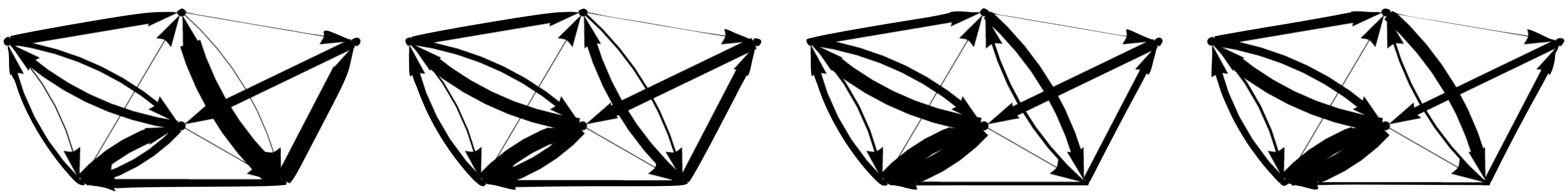
Joint work with Clayton Shonkwiler (Colorado State University)



Workshop on Recent Progress on Optimal Point Distributions and Related Fields
ICERM
June 3, 2024

Spaces of Point Configurations

We consider structural aspects of certain spaces of vector configurations and structured matrices:

Unit norm, Tight frames: $\left\{ F = [f_1 \mid f_2 \mid \ldots \mid f_N] \in \mathbb{C}^{d\times N} \mid \|f_j\| = 1 \ \forall\, j \text{ and } FF^* = \frac{N}{d}\mathrm{I}_d \right\}$

Normal Matrices: $\left\{ A \in \mathbb{C}^{d\times d} \mid AA^* = A^*A \right\}$

Weighted Adjacency Matrices for Balanced Digraphs:

$$\left\{ A = (a_{ij})_{ij} \in \mathbb{R}^{d\times d}_{\geq 0} \mid \sum_i a_{ik} = \sum_j a_{kj} \ \forall\, k \right\}$$

Tight Frame Fields on Vector Bundles:

$$\left\{ \sigma = (\sigma_1, \ldots, \sigma_n) : M \to E^N \mid \pi \circ \sigma_j = \mathrm{Id}_M,\ \sigma(p) \text{ tight for all } p \in M \right\}$$

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Main Idea:

Prove theorems about the coarse structure of these spaces using tools from symplectic geometry and algebraic topology.

Spaces of Frames

Unit Norm Tight Frames

An N-frame in $\mathbb{C}^d$ is a full rank matrix $F \in \mathbb{C}^{d \times N}$. The space of Unit norm, Tight frames is

$$\mathrm{UNTF}(d, N) = \left\{ F = [f_1 \mid f_2 \mid \ldots \mid f_N] \in \mathbb{C}^{d \times N} \mid \|f_j\| = 1 \ \forall\, j \text{ and } FF^* = \frac{N}{d}\mathrm{I}_d \right\}$$

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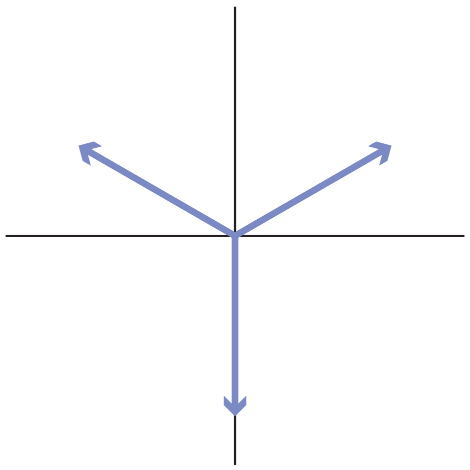
Frames are used to give redundant representations of signals $v \in \mathbb{C}^d$

$$\mathbb{C}^d \ni v \mapsto F^* v = \left(\langle v, f_j \rangle \right)_{j=1}^{N} \in \mathbb{C}^N$$



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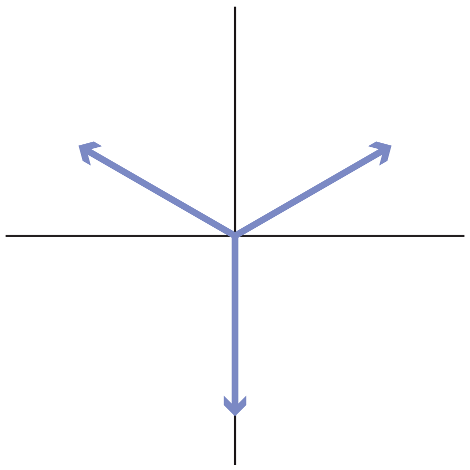

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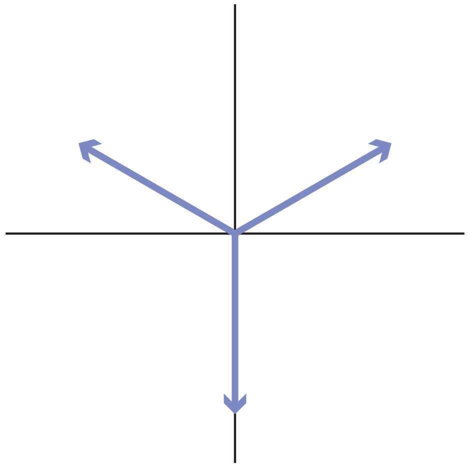

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"frame operator for F"

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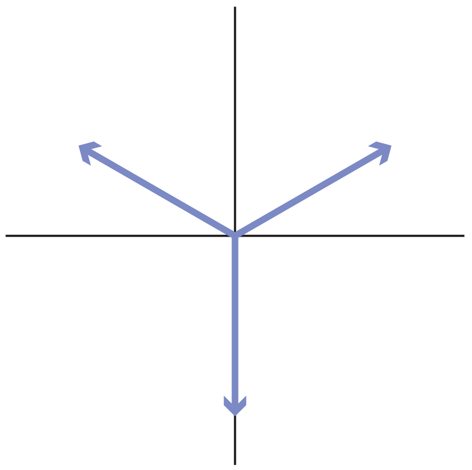

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Theorem (Casazza–Kovačevic, Goyal–Kovačevic–Kelner, Holmes–Paulsen). Among N-frames in $\mathbb{C}^d$, unit norm, tight frames give optimal reconstruction error under white noise or measurement erasures.

Unit norm, tight frames generalize orthonormal bases: $\mathrm{UNTF}(d, d) = \mathrm{U}(d)$

Structure of UNTFs

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The space of UNTFs

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- $r = (r_1, \ldots, r_N) \in \mathbb{R}^N$ with $r_1 \geq r_2 \geq \ldots \geq r_N \geq 0$ is a collection of vector norms and
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frame operator can be tuned for “colored noise”

allows variable “measurement power”

Concepts from Symplectic Geometry

A symplectic manifold (M, ω) is an even-dimensional manifold M endowed with a closed, nondegenerate 2-form ω. A symplectic manifold (M, ω) locally looks like $\big(\mathbb{C}^d, -\operatorname{Im}\langle \cdot\,, \cdot \rangle\big)$

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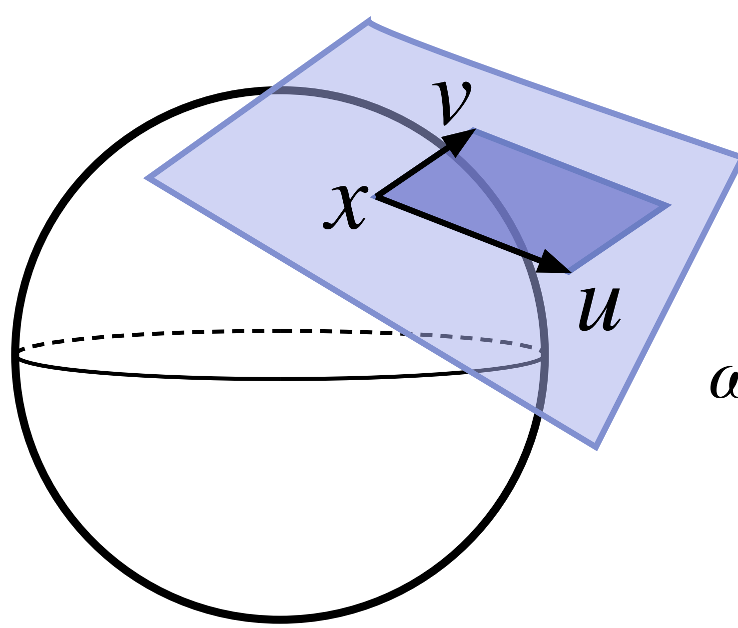
Examples.

- $(\mathbb{C}^d, -\operatorname{Im}\langle \cdot, \cdot \rangle)$
- $(\mathbb{C}^{d\times N}, -\operatorname{Im}\langle \cdot, \cdot \rangle_{\text{Fro}})$
- $(S^2, \omega =$ signed area$)$



$$\omega_x(u, v) = x \cdot (u \times v)$$

Concepts from Symplectic Geometry

Let G be a Lie group with an action on M which preserves ω. A momentum map for this action is a smooth map

$$\mu : M \to \mathfrak{g}^* \approx \mathfrak{g}$$

which is equivariant with respect to the co-adjoint action $G \curvearrowright \mathfrak{g}^*$ and which satisfies

$$d_p\mu(X)(\xi) = \omega_p(Y_\xi|_p, X)$$

for $X \in T_pM$, $\xi \in \mathfrak{g}$, Y_ξ the associated infinitesimal vector field.

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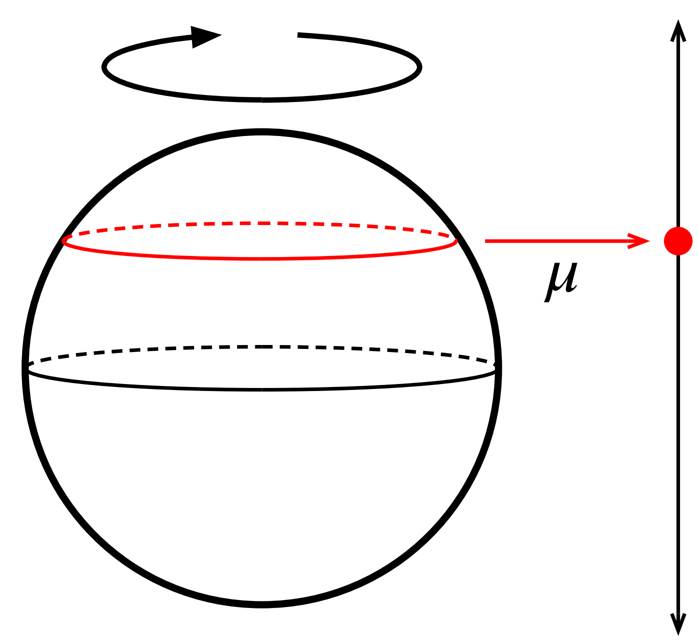
Intuitively, μ encodes conserved quantities of the action:

To each $\xi \in \mathfrak{g}$, define a function $\mu_\xi : M \to \mathbb{R}$ by $\mu_\xi(p) = \mu(p)(\xi)$. Then the flow of Y_ξ preserves level sets of μ_ξ

Example.

$S^1 \curvearrowright S^2$ by rotation around z-axis

$\mu =$ height (identifying $\mathrm{Lie}(S^1) \approx \mathbb{R}$)



Connectivity of Frame Spaces

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Proof Idea. Given $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0)$, the space

$$\left\{ F \in \mathbb{C}^{d \times N} \mid \operatorname{spec}(FF^*) = \lambda \right\} / \mathrm{U}(d)$$

has a natural symplectic structure (isomorphic to a complex flag manifold; Grassmannian if $\lambda = \mathbf{1}$).

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It has a Hamiltonian action by the torus $\mathrm{U}(1)^N$ (right multiplication) with momentum map

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Theorem (Atiyah '82). Level sets of momentum maps of torus actions are connected.

Connectivity of $\mathscr{F}(r,S)$, with $\operatorname{spec}(S) = \lambda$, follows easily from connectivity of $\mu^{-1}\left(-\frac{1}{2}\left(r_j^2\right)_j\right)$.

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is a smooth manifold $\Leftrightarrow$ $\nexists$ partitions $r = r' \sqcup r''$ and $\lambda = \lambda' \sqcup \lambda''$ with $r' \prec \lambda'$ and $r'' \prec \lambda''$.

For, $r = (r_1 \geq \cdots \geq r_N)$ and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$, write $r \prec \lambda$ if $\sum_{j=1}^{k} r_j \leq \sum_{j=1}^{k} \lambda_j \; \forall k = 1,\ldots,N$ (pad λ with zeros).

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Theorem (Shur-Horn Theorem, Casazza-Leon, '10).
$\left\{ F \in \mathbb{C}^{d\times N} \mid \|f_j\| = r_j \, \forall\, j \text{ and } \operatorname{spec}(FF^*) = \lambda \right\} \neq \varnothing \Leftrightarrow r \prec \lambda$.

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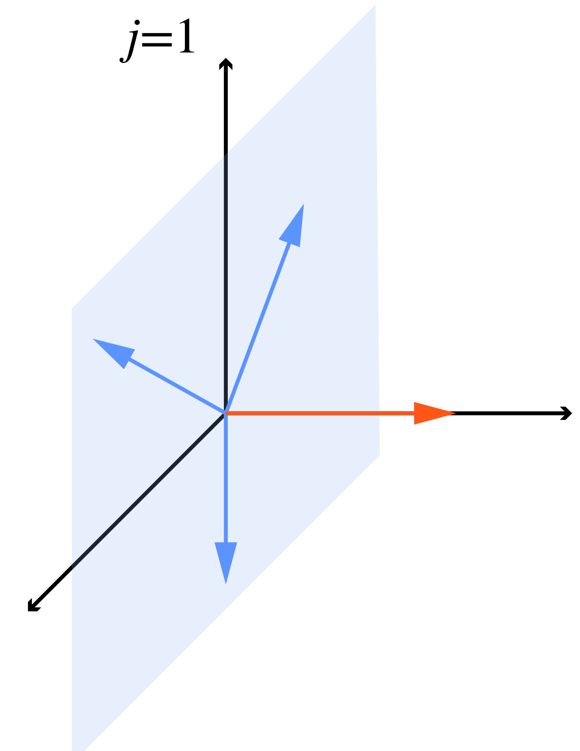

If it has singularities, they occur exactly at orthodecomposable frames, and singularities locally look like products of a quadratic cone and a manifold.

Description of singularities uses a result of Arms-Marsden-Moncrief ’81.

Generalizes a result of Dykema-Strawn ’06: The space $\mathrm{UNTF}(d,N)$ is a smooth manifold if d and N are relatively prime. Answers open questions of Cahill-Mixon-Strawn ’17.

Full Spark Frames

Rough idea of Compressed Sensing: "A random matrix $F \in \mathbb{C}^{d \times N}$ is good at compressing sparse vectors in $\mathbb{C}^N$, via $\mathbb{C}^N \ni v \mapsto Fv \in \mathbb{C}^d$, with high probability."

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- We say a frame $F \in \mathbb{C}^{d \times N}$ is full spark if any choice of d columns is spanning.
- Question: What is the probability that a random UNTF is full spark?
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Theorem (N-Shonkwiler, '22). Given vectors of norms r and eigenvalues λ, the space

$$\left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \ \forall\, j \text{ and } \operatorname{spec}(FF^*) = \lambda \right\}$$

satisfies exactly one of three conditions:

- It is empty (when Schur-Horn condition $r \prec \lambda$ fails)
- It is nonempty and contains only frames which are not full spark (S-H condition non-strict)
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Proof Ingredients.

$$\left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \;\forall\, j \text{ and } \operatorname{spec}(FF^*) = \lambda \right\} / \left(\mathrm{U}(d) \times \mathrm{U}(1)^N \right)$$

is a symplectic stratified space [Sjamaar-Lerman, '91] with Hamiltonian torus action with momentum map

$$[F] \mapsto (\mu_{jk})_{j,k}$$

where μ_{jk} is the k^{th} eigenvalue of the partial frame operator $\sum_{\ell=1}^{j} f_\ell f_\ell^*$.

These are known as eigensteps in the frame theory community.

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Proof Ingredients.

The eigensteps satisfy the Gelfand-Tsetlin pattern.

Defines a convex polytope whose Lebesgue measure can be used to compute Hausdorff measure on frame space (Duistermaat-Heckmann Theorem).

$$\begin{array}{ccccccccc}
\lambda_1 & & \geq & & \lambda_2 & & \geq & & \lambda_3 \\
 & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} \\
 & & \mu_{31} & & \geq & & \mu_{32} & & \geq & & \mu_{33} \\
 & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} \\
 & & & & \mu_{21} & & \geq & & \mu_{22} \\
 & & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} \\
 & & & & & & \mu_{11}
\end{array}$$

$d = 3$ G-T pattern

Normal Matrices and Balancing Directed Graphs

Normal Matrices

A matrix $A \in \mathbb{C}^{d \times d}$ is normal if $AA^* = A^*A$.

Normal matrices the general setting for the Spectral Theorem

$$\left\{A \in \mathbb{C}^{d \times d} \mid AA^* = A^*A\right\} = \left\{UDU^* \mid U \text{ unitary, } D \text{ diagonal}\right\}$$

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A matrix $A \in \mathbb{C}^{d \times d}$ is normal if $AA^* = A^*A$.

Normal matrices the general setting for the Spectral Theorem

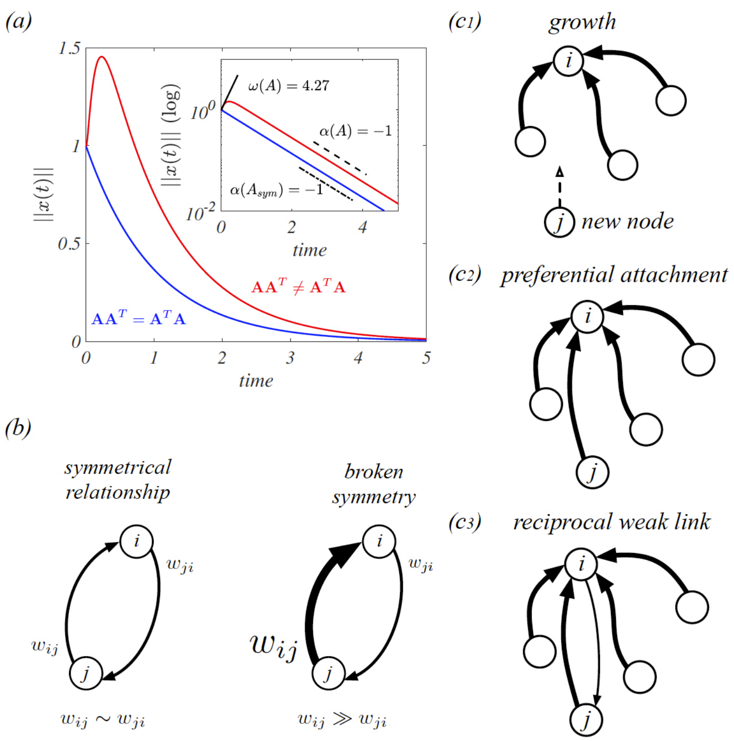
$$\{A \in \mathbb{C}^{d \times d} \mid AA^* = A^*A\} = \{UDU^* \mid U \text{ unitary, } D \text{ diagonal}\}$$

Normal matrices have spectra which are Lipschitz stable under perturbations [Bauer-Fike Theorem, 1960]

$\Rightarrow$ applications in control theory

Normality plays a role in dynamics on networks [Asllani-Carletti, 2018]

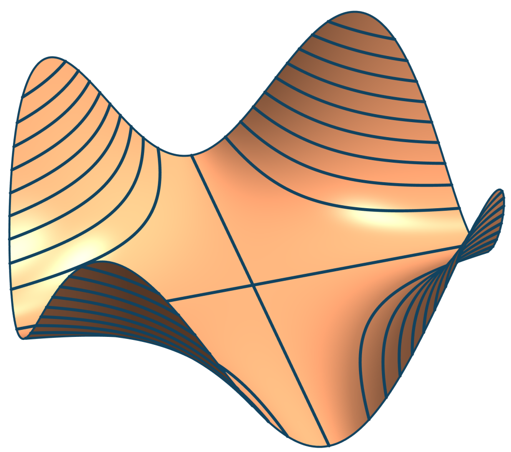
$\Rightarrow$ applications in mathematical biology



This motivates algorithms for finding the nearest normal matrix to a given $A \in \mathbb{C}^{d \times d}$.

Normal Matrices via Gradient Flow

A classical, natural measure of non-normality of a matrix A is $\mathrm{E}(A) := \|AA^* - A^*A\|^2_{\mathrm{Fro}}$.



Recall: The function $\mathrm{E} : \mathbb{C}^{d\times d} \to \mathbb{R}$ is not quasi-convex, but for an arbitrary initialization $A_0 \in \mathbb{C}^{d\times d}$, we have:

Theorem [N-Shonkwiler, '24]. Gradient descent of the functional $\mathrm{E} : A \mapsto \|AA^* - A^*A\|^2_{\mathrm{Fro}}$ converges to a normal matrix A_∞. A_∞ has the same eigenvalues as A_0 and if A_0 is real, then so is A_∞. Moreover, there exist $c, \epsilon > 0$ such that, if $\mathrm{E}(A_0) < \epsilon$ then $\|A_0 - A_\infty\|^2_{\mathrm{Fro}} \leq c\sqrt{\mathrm{E}(A_0)}$.

This can be adapted to preserve total weight $\|A_0\|^2_{\mathrm{Fro}}$.

Application: Topology of Unit Norm Normal Matrices

The space of normal matrices is contractible.

The space of unit norm normal matrices

$$\mathscr{U}\mathscr{N}_{\mathbb{F}}(d) = \left\{ A \in \mathbb{F}^{d \times d} \mid AA^* = A^*A \text{ and } \|A\|_{\mathrm{Fro}} = 1 \right\}, \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

can have interesting topology.

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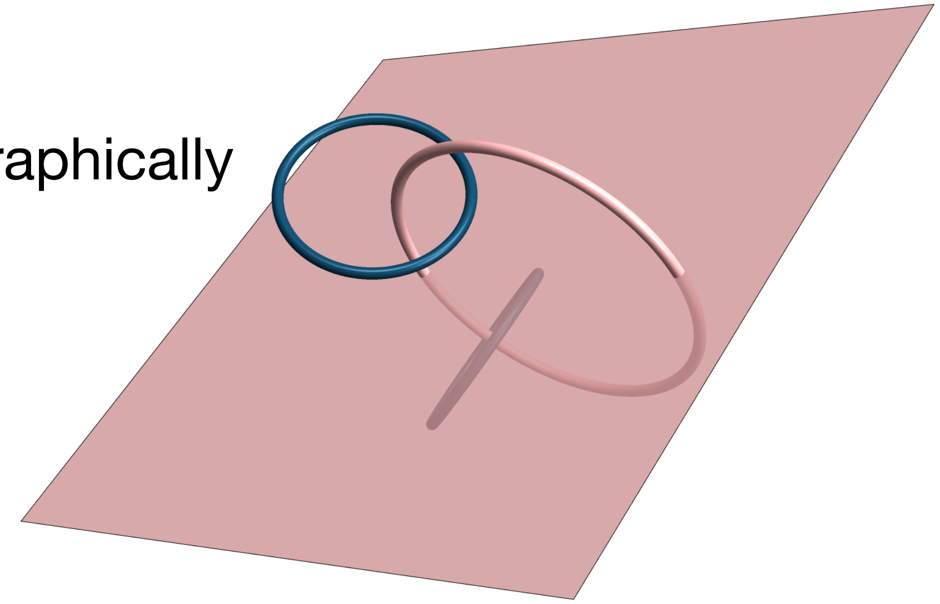
can have interesting topology.

Example. $\{A \in \mathbb{R}^{2\times 2} \mid \|A\|_{\mathrm{Fro}} = 1\}$ stereographically projected to $\mathbb{R}^3$.

Image of unit norm nilpotent matrices in blue.

Image of $\mathscr{U}\mathscr{N}_{\mathbb{R}}(2)$ in pink.



Application: Topology of Unit Norm Normal Matrices

The space of normal matrices is contractible.

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$$\mathcal{UN}_{\mathbb{F}}(d) = \left\{A \in \mathbb{F}^{d\times d} \mid AA^* = A^*A \text{ and } \|A\|_{\mathrm{Fro}} = 1\right\},\ \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

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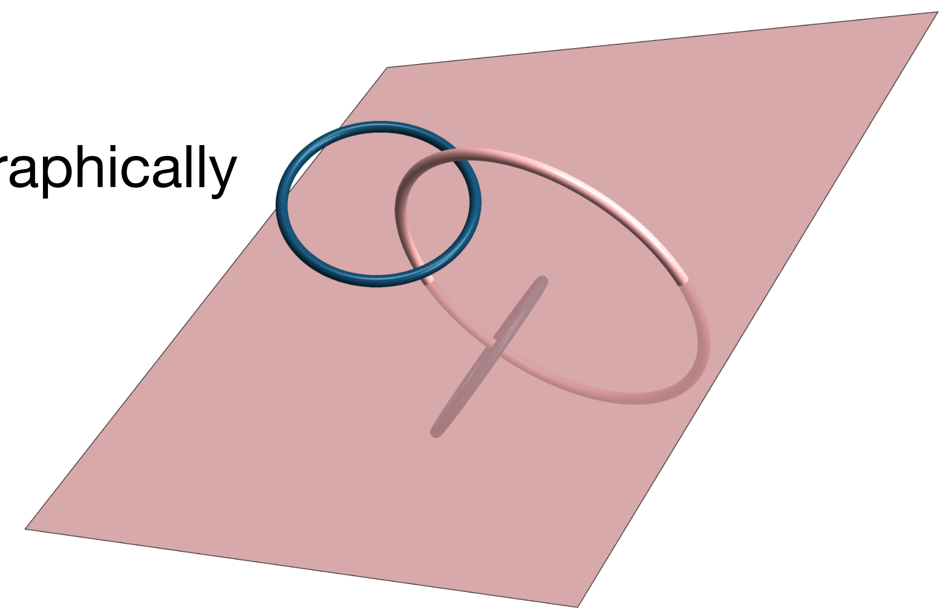
Image of $\mathcal{UN}_{\mathbb{R}}(2)$ in pink.



Theorem [N-Shonkwiler, '24].

- $\pi_k(\mathcal{UN}_{\mathbb{C}}(d))$ is trivial for all $k \leq 2d-2$.
- $\pi_k(\mathcal{UN}_{\mathbb{R}}(d))$ is trivial for all $k \leq d-2$.

Proof. $\mathcal{UN}_{\mathbb{F}}(d)$ is homotopy equivalent to $\{$non-nilpotent $d \times d$ matrices$\}$, via gradient descent of E.

The space of nilpotent matrices is a stratified space with high codimension strata. Use transversality.

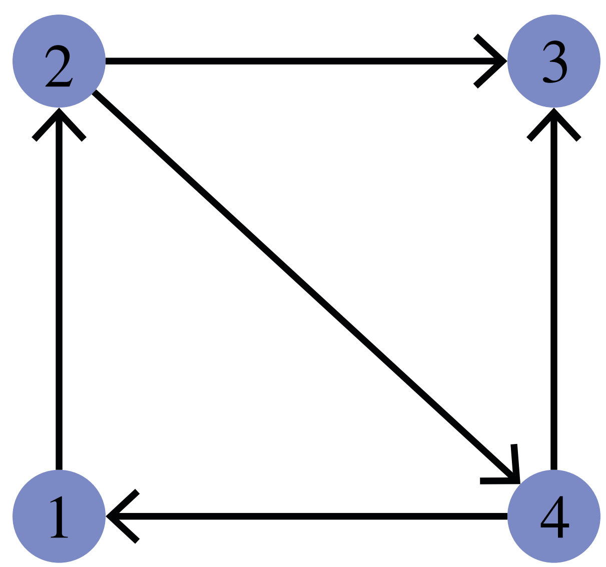
Balancing Digraphs

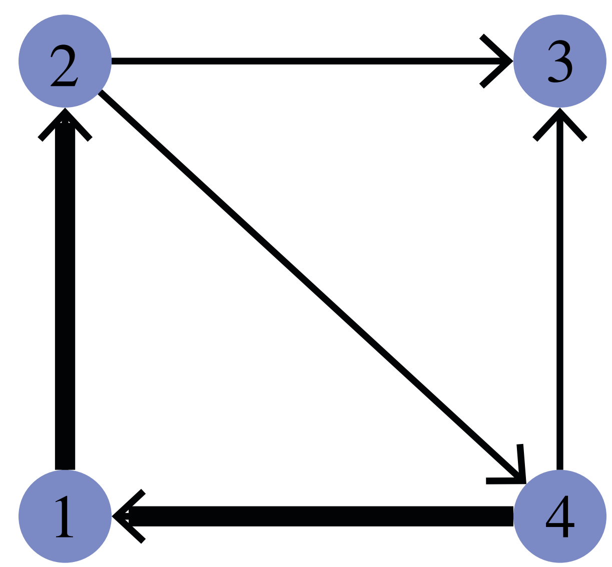
Let $A = (a_{ij})_{ij} \in \mathbb{R}^{d \times d}$ be the adjacency matrix of a weighted, directed graph.

We say that the graph is balanced if

$$\sum_i a_{ik} = \sum_j a_{kj} \ \forall\ k$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$





$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

Unbalanced

Balanced

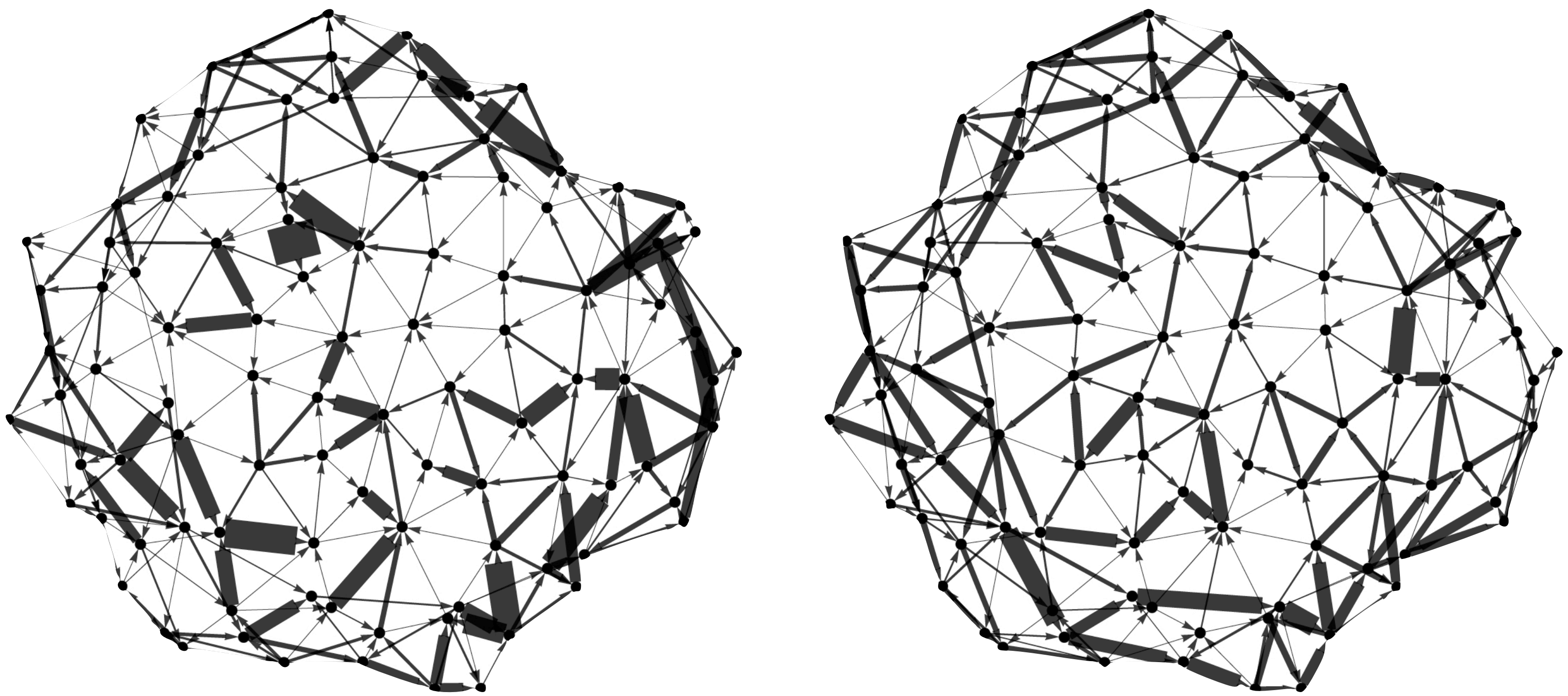
Balancing is necessary for, e.g., traffic flow problems [Hooi-Tong, 1970].

Balancing Graphs by Gradient Descent

Let $A_0 \in \mathbb{R}^{d \times d}$ be the entry-wise square of an adjacency matrix of a weighted digraph.

Theorem [N-Shonkwiler, '24]. Gradient descent of the functional $A \mapsto \|\mathrm{diag}(AA^* - A^*A)\|^2_{\mathrm{Frob}}$ converges to the entry-wise square of the adjacency matrix of a balanced digraph. It has the same eigenvalues and principal minors as A_0, and has zero entries whenever A_0 does.

This can be adapted to preserve total weight $\|A_0\|^2_{\mathrm{Frob}}$.



Also partially follows by symplectic principles.

Corollary [N-Shonkwiler, '24]. The space of balanced, unit norm adjacency matrices is homotopy equivalent to the space of unit norm normal matrices.

Tight frames on Vector Bundles

Vector Bundles

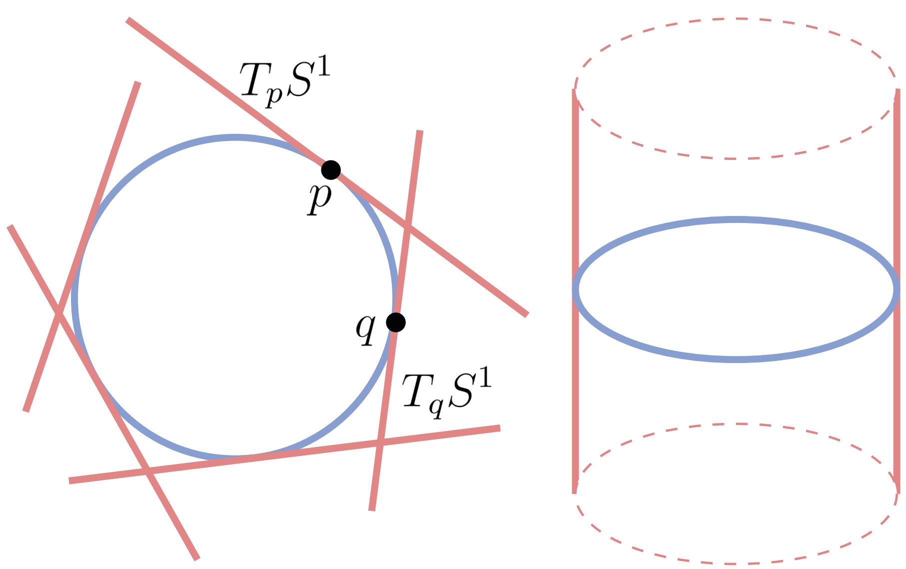
A rank-d real vector bundle over an n-dimensional smooth manifold M is a smooth manifold E with projection $E \xrightarrow{\pi} M$ whose fibers $E_p = \pi^{-1}(p)$ are isomorphic to $\mathbb{R}^d$ such that $E \approx M \times \mathbb{R}^d$, locally.

A Riemannian structure on E is a smooth choice of inner product $\langle \cdot , \cdot \rangle_p$ on each fiber E_p.

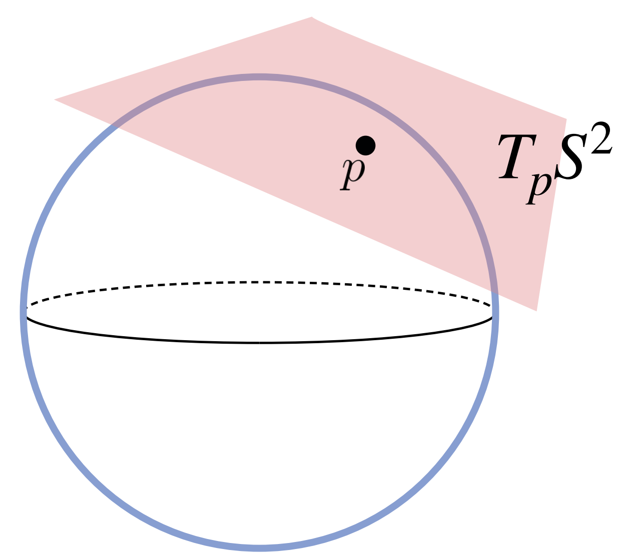
A vector field for E is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$.

Examples.

- Trivial bundle $M \times \mathbb{R}^d \to M$
- Tangent bundle over a manifold $TM \to M$



$TS^1 \approx S^1 \times \mathbb{R}$



$TS^2 \not\approx S^2 \times \mathbb{R}^2$

- Tautological bundle $E \to \mathrm{Gr}(k, \mathbb{R}^d)$

Frames on Vector Bundles

Let $E \to M$ be a rank-d vector bundle over an n-dimensional manifold. An N-frame for E is a collection of N vector fields $(\sigma_1, \ldots, \sigma_N)$ such that $\{\sigma_1(p), \ldots, \sigma_N(p)\}$ is spanning for all p.

The frame is a tight frame if $[\sigma_1(p), \ldots, \sigma_N(p)]$ defines a tight frame on the fiber for all p.
(Requires a choice of Riemannian structure for this to make sense.)

Prior Work: [Kuchment, '08], [Freeman-Poore-Wei-Wyse, '14], [Freeman-Hotovy-Martin, '14], [Kuchment, '16], [Auckly-Kuchment, '18]

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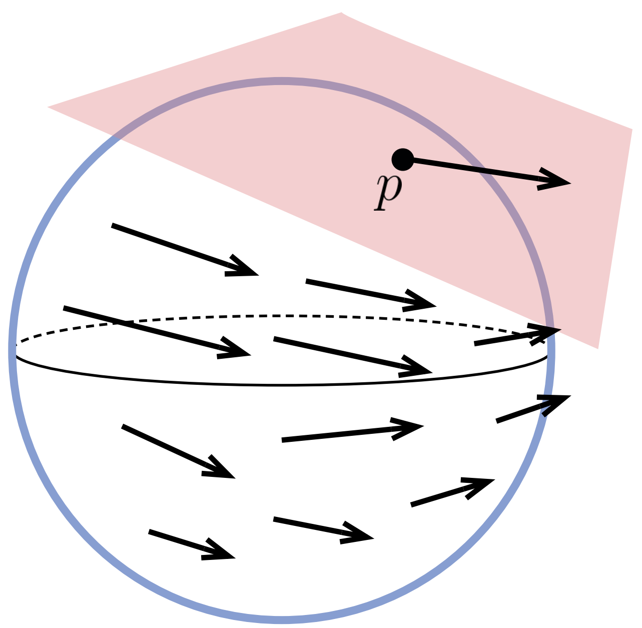
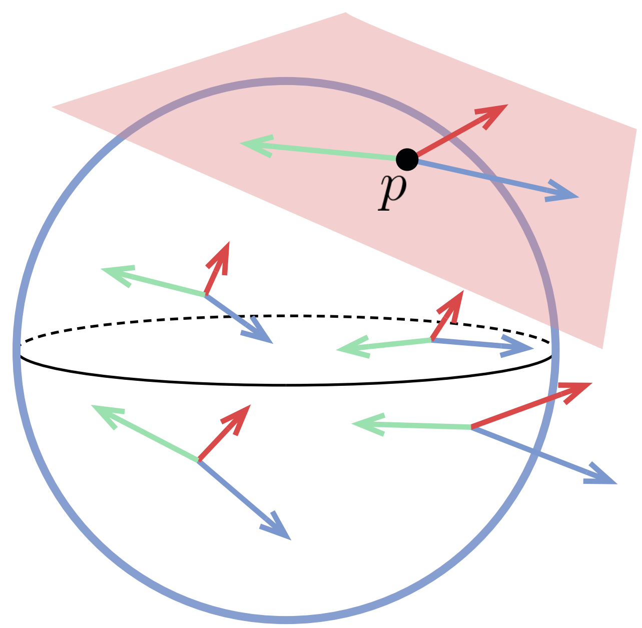
Motivation: Robust representation of "signals" on vector bundles; i.e., representing vector fields.



signal = vector field



measurement system = N-frame

Measurements are coefficients of the signal w.r.t. the frame. This gives a map $M \to \mathbb{R}^N$.

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Motivation: Robust representation of "signals" on vector bundles; i.e., representing vector fields.

Experiment: Signal reconstruction error for noisy signals on S^2 for tight versus random 3-frames.

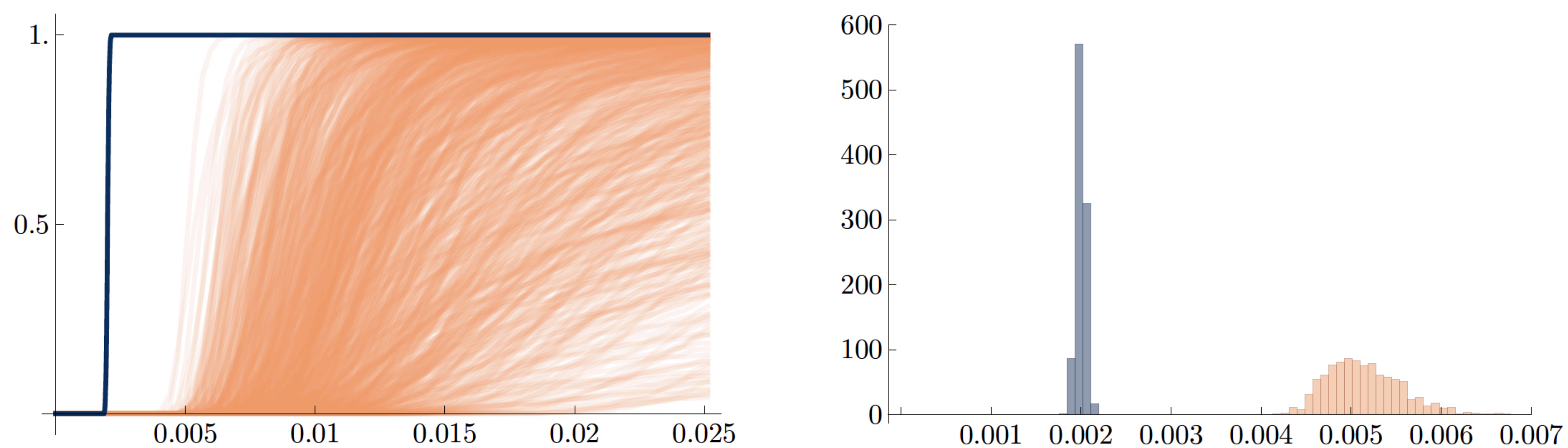

FIGURE 1. Left: plots of the empirical CDFs of the MSE distributions for the Parseval frame (dark blue) and the random frames (light orange). Right: histograms of the MSE distributions for the Parseval frame (dark blue) and for the best of the random frames (lighter orange).

Existence of Tight Frames

Question. When do (tight) N-frames on $E \to M$ exist?

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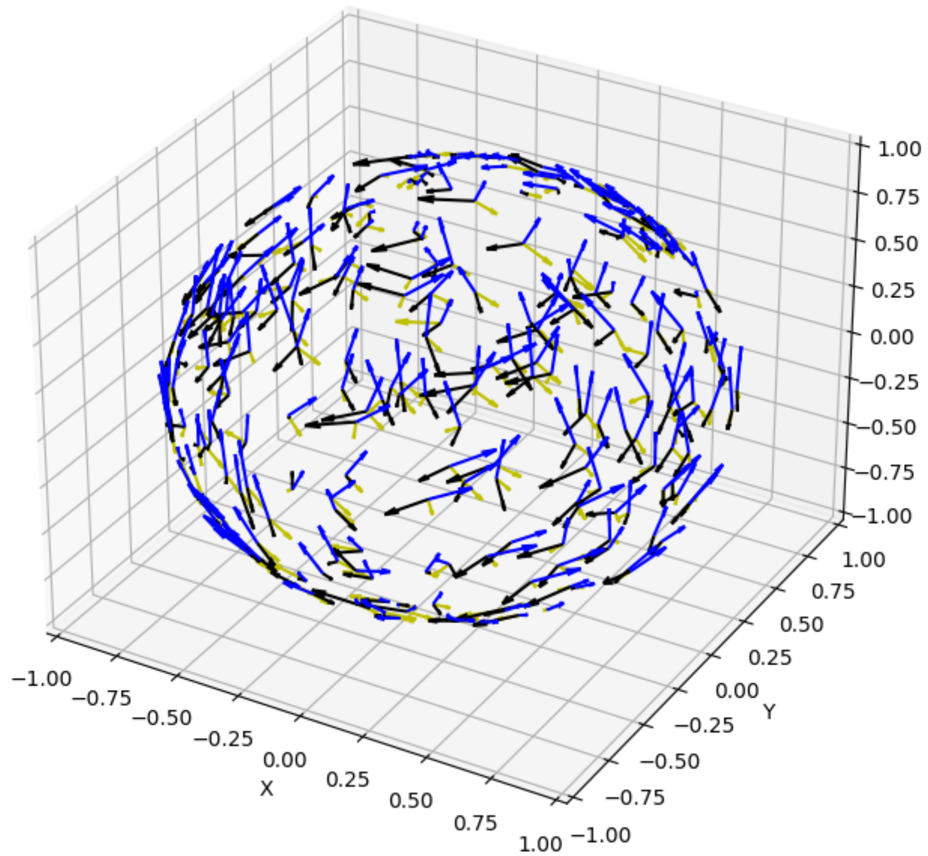
Examples.

- 3-frame on TS^2



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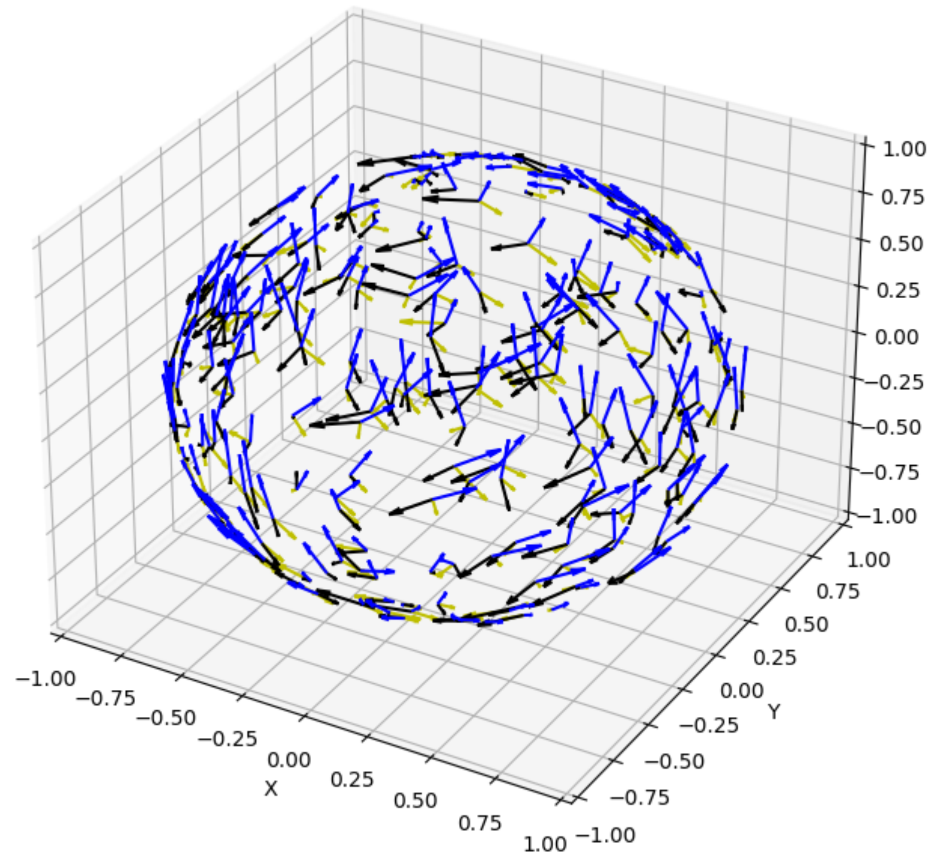

- A 2-frame on TS^2 does not exist

Such a frame frame would require a nonvanishing vector field on S^2, which DNE by the Hairy Ball Theorem.

From Wikipedia article on Hairy Ball Theorem

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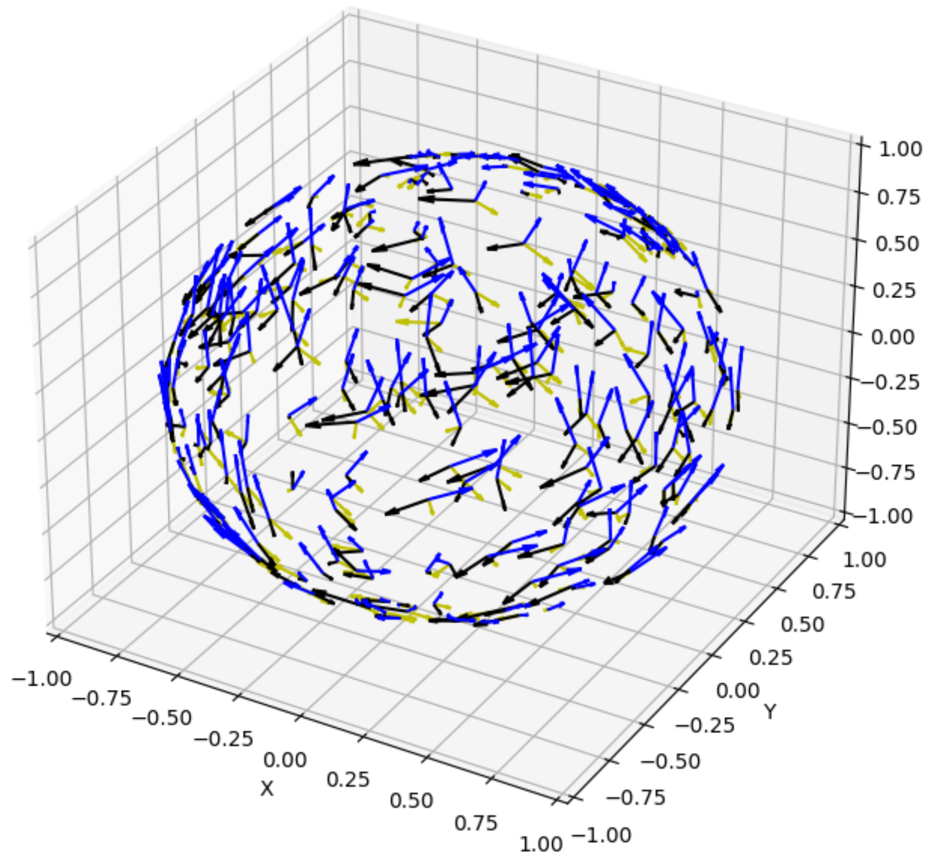

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- A tight N-frame for TM exists for sufficiently large N.

Follows by Whitney embedding argument, see [Freeman-Poore-Wei-Wyse, '14].

Existence of Tight Frames

Theorem (Ballas-N-Shonkwiler,'23). Let $E \to M$ be a vector bundle. An N-frame exists if and only if a tight N-frame exists.

$$\begin{array}{ccccccccc}
\pi_{i+1}(M) & \longrightarrow & \pi_i(\mathcal{P}^n(\mathbb{R}^k)) & \longrightarrow & \pi_i(\mathcal{P}^n(E)) & \longrightarrow & \pi_i(M) & \longrightarrow & \pi_{i-1}(\mathcal{P}^n(\mathbb{R}^k)) \\
\updownarrow \text{Id} & & \downarrow \iota_* & & \downarrow \bar{\iota}_* & & \updownarrow \text{Id} & & \downarrow \iota_* \\
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Theorem (Ballas-N-Shonkwiler,'23). Let $E \to M$ be a rank-d vector over an n-manifold. Then there exists a tight frame for E if $N \geq d + n$.

Sufficient, but not necessary!

Theorem (Ballas-N-Shonkwiler,'23). There exists a closed, orientable d-manifold M such that TM does not admit a tight $(d+1)$-frame if and only if $d > 3$.

Open Questions

- What about higher homotopy/(co)homology of frame spaces?
- What about the corresponding question for spaces of real frames?
- Can symplectic methods be applied to frames in infinite-dimensional Hilbert spaces?
- Can geometry of the Gelfand-Tsetlin polytope be used to get quantitative statements about compressed sensing properties of random frames?
- Can we efficiently generate random frames using Markov chain sampling in G-T polytope?
- Applications of frame theory on vector bundles?

Thanks for Listening!

References:

Tom Needham and Clayton Shonkwiler, *Symplectic Geometry and Connectivity of Spaces of Frames*, Advances in Computational Mathematics, 2021.

Tom Needham and Clayton Shonkwiler, *Toric Symplectic Geometry and Full Spark Frames*, Applied and Computational Harmonic Analysis, 2022.

Tom Needham and Clayton Shonkwiler, *Geometric Approaches to Matrix Normalization and Graph Balancing*, arXiv preprint 2405.06190, 2024.

Samuel Ballas, Tom Needham and Clayton Shonkwiler, *On the Existence of Parseval Frames for Vector Bundles*, arXiv preprint 2312.13488, 2023.

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