TOPOLOGY OF SPACES OF STRUCTURED VECTOR CONFIGURATIONS

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Workshop on Recent Progress on Optimal Point Distributions and Related Fields
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Spaces of Point Configurations

We consider structural aspects of certain spaces of vector configurations and structured matrices:

Unit norm, Tight frames: \[
\left\{ F = [f_1 | f_2 | \ldots | f_N] \in \mathbb{C}^{d \times N} \mid \|f_j\| = 1 \ \forall \ j \text{ and } FF^* = \frac{N}{d}I_d \right\}
\]

Normal Matrices: \[
\left\{ A \in \mathbb{C}^{d \times d} \mid AA^* = A^*A \right\}
\]

Weighted Adjacency Matrices for Balanced Digraphs:
\[
\left\{ A = (a_{ij})_{ij} \in \mathbb{R}_{\geq 0}^{d \times d} \mid \sum_i a_{ik} = \sum_j a_{kj} \ \forall \ k \right\}
\]

Tight Frame Fields on Vector Bundles:
\[
\left\{ \sigma = (\sigma_1, \ldots, \sigma_n) : M \to E^N \mid \pi \circ \sigma_j = \text{Id}_M, \sigma(p) \text{ tight for all } p \in M \right\}
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Main Idea:

Prove theorems about the coarse structure of these spaces using tools from symplectic geometry and algebraic topology.
Spaces of Frames
Unit Norm Tight Frames

An $N$-frame in $\mathbb{C}^d$ is a full rank matrix $F \in \mathbb{C}^{d \times N}$. The space of Unit norm, Tight frames is

$$\text{UNTF}(d, N) = \left\{ F = [f_1 | f_2 | \ldots | f_N] \in \mathbb{C}^{d \times N} \mid \|f_j\| = 1 \ \forall \ j \text{ and } FF^* = \frac{N}{d} I_d \right\}$$
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Frames are used to give redundant representations of signals $v \in \mathbb{C}^d$

$$\mathbb{C}^d \ni v \mapsto F^*v = (\langle v, f_j \rangle)_{j=1}^N \in \mathbb{C}^N$$
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“frame operator for $F$”
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Theorem (Casazza–Kovačevic, Goyal–Kovačevic–Kelner, Holmes–Paulsen). Among $N$-frames in $\mathbb{C}^d$, unit norm, tight frames give optimal reconstruction error under white noise or measurement erasures.

Unit norm, tight frames generalize orthonormal bases: $\text{UNTF}(d, d) = U(d)$
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The space of UNTFs

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is a potentially singular real algebraic variety with potentially complicated topology.
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Frame Homotopy Conjecture - Larson, ’02: The space \( \text{UNTF}(d, N) \) is connected \( \forall \ N \geq d \geq 1 \).

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Theorem (N-Shonkwiler, ’21). Any space of frames of the following form is connected:

$$\mathcal{F}(r, S) = \left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \ \forall j \ \text{and} \ FF^* = S \right\}$$

- $r = (r_1, \ldots, r_N) \in \mathbb{R}^N$ with $r_1 \geq r_2 \geq \ldots \geq r_N \geq 0$ is a collection of vector norms and
- $S$ is a positive-definite Hermitian frame operator
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- \( S \) is a positive-definite Hermitian frame operator

frame operator can be tuned for “colored noise”

allows variable “measurement power”
A symplectic manifold \((M, \omega)\) is an even-dimensional manifold \(M\) endowed with a closed, nondegenerate 2-form \(\omega\). A symplectic manifold \((M, \omega)\) locally looks like \((\mathbb{C}^d, - \text{Im}\langle \cdot , \cdot \rangle)\).
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**Examples.**

- \((\mathbb{C}^d, - \text{Im} \langle \cdot, \cdot \rangle)\)
- \((\mathbb{C}^{d \times N}, - \text{Im} \langle \cdot, \cdot \rangle_{\text{Fro}})\)
- \((S^2, \omega = \text{signed area})\)

\[
\omega_x(u, v) = x \cdot (u \times v)
\]
Concepts from Symplectic Geometry

Let $G$ be a Lie group with an action on $M$ which preserves $\omega$. A momentum map for this action is a smooth map

$$\mu : M \to \mathfrak{g}^* \approx \mathfrak{g}$$

which is equivariant with respect to the co-adjoint action $G \curvearrowright \mathfrak{g}^*$ and which satisfies

$$d_p\mu(X)(\xi) = \omega_p(Y_\xi|_p, X)$$

for $X \in T_pM$, $\xi \in \mathfrak{g}$, $Y_\xi$ the associated infinitesimal vector field.
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Intuitively, $\mu$ encodes **conserved quantities** of the action:

To each $\xi \in \mathfrak{g}$, define a function $\mu_\xi : M \rightarrow \mathbb{R}$ by $\mu_\xi(p) = \mu(p)(\xi)$. Then the flow of $Y_\xi$ preserves level sets of $\mu_\xi$.

**Example.**

$S^1 \curvearrowright S^2$ by rotation around $z$-axis

$\mu =$ height (identifying $\text{Lie}(S^1) \approx \mathbb{R}$)
Connectivity of Frame Spaces

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**Proof Idea.** Given \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0) \), the space

\[ \left\{ F \in \mathbb{C}^{d \times N} \mid \text{spec}(FF^*) = \lambda \right\} / \text{U}(d) \]

has a natural symplectic structure (isomorphic to a complex flag manifold; Grassmannian if \( \lambda = 1 \)).
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It has a Hamiltonian action by the torus \( U(1)^N \) (right multiplication) with momentum map

\[ [F] \mapsto \mu([F]) = \left(-\frac{1}{2}\|f_j\|^2\right)_{j=1}^N \in \mathbb{R}^N \]
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\[ [F] \mapsto \mu([F]) = \left( -\frac{1}{2} \|f_j\|^2 \right)_{j=1}^N \in \mathbb{R}^N \]

Theorem (Atiyah ’82). Level sets of momentum maps of torus actions are connected.

Connectivity of \( \mathcal{F}(r, S) \), with \( \text{spec}(S) = \lambda \), follows easily from connectivity of \( \mu^{-1} \left( -\frac{1}{2} (r_j^2) \right) \).
Geometry of Frame Spaces

Question: When is a frame space a smooth manifold?
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Theorem (N-Shonkwiler, ’22). Given vectors of norms $r$ and eigenvalues $\lambda$, the space

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is a smooth manifold $\iff$ there are no partitions $r = r' \sqcup r''$ and $\lambda = \lambda' \sqcup \lambda''$ with $r' < \lambda'$ and $r'' < \lambda''$.

For, $r = (r_1 \geq \cdots \geq r_N)$ and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$, write $r < \lambda$ if $\sum_{j=1}^{k} r_j \leq \sum_{j=1}^{k} \lambda_j \ \forall k = 1,\ldots,N$ (pad $\lambda$ with zeros).
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**Theorem (Shur-Horn Theorem, Casazza-Leon, ‘10).**

$$\left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \ \forall \ j \text{ and } \text{spec}(FF^*) = \lambda \right\} \neq \emptyset \iff r < \lambda.$$
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If it has singularities, they occur exactly at orthodecomposable frames, and singularities locally look like products of a quadratic cone and a manifold.

Description of singularities uses a result of Arms-Marsden-Moncrief ’81.

Generalizes a result of Dykema-Strawn ’06: The space $\text{UNTF}(d, N)$ is a smooth manifold if $d$ and $N$ are relatively prime. Answers open questions of Cahill-Mixon-Strawn ’17.
Full Spark Frames

Rough idea of **Compressed Sensing**: “A random matrix $F \in \mathbb{C}^{d \times N}$ is good at compressing sparse vectors in $\mathbb{C}^N$, via $\mathbb{C}^N \ni v \mapsto Fv \in \mathbb{C}^d$, with high probability.”

Can the quantitative version be improved if we choose a random unit norm tight frame? Empirical evidence suggests that the answer is “yes” [Chen-Rodrigues-Wassell, ’12, ’13].
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A first step:

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- **Question:** What is the probability that a random UNTF is full spark?
  - **Cahill-Mixon-Strawn:** full spark frames are open and dense in UNTFs
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$$\left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \forall j \text{ and } \text{spec}(FF^*) = \lambda \right\}$$

satisfies exactly one of three conditions:

- It is empty (when Schur-Horn condition $r < \lambda$ fails)
- It is nonempty and contains only frames which are not full spark (S-H condition non-strict)
- It is nonempty and full spark frames are a subset of full Hausdorff measure (otherwise)
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Proof Ingredients.

$$\left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \ \forall \ j \text{ and } \text{spec}(FF^*) = \lambda \right\} / (U(d) \times U(1)^N)$$

is a symplectic stratified space [Sjamaar-Lerman, ’91] with Hamiltonian torus action with momentum map

$$[F] \mapsto (\mu_{j,k})_{j,k}$$

where $\mu_{j,k}$ is the $k^{th}$ eigenvalue of the partial frame operator $\sum_{\ell=1}^j f_\ell f_\ell^*$. These are known as eigensteps in the frame theory community.
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Proof Ingredients.

The eigensteps satisfy the Gelfand-Tsetlin pattern.

Defines a convex polytope whose Lebesgue measure can be used to compute Hausdorff measure on frame space (Duistermaat-Heckmann Theorem).
Normal Matrices and Balancing Directed Graphs
Normal Matrices

A matrix $A \in \mathbb{C}^{d \times d}$ is normal if $AA^* = A^*A$.

Normal matrices the general setting for the Spectral Theorem

$$\left\{ A \in \mathbb{C}^{d \times d} \mid AA^* = A^*A \right\} = \left\{ UDU^* \mid U \text{ unitary}, D \text{ diagonal} \right\}$$
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Normal matrices have spectra which are Lipschitz stable under perturbations [Bauer-Fike Theorem, 1960]

$\Rightarrow$ applications in control theory

Normality plays a role in dynamics on networks [Asllani-Carletti, 2018]

$\Rightarrow$ applications in mathematical biology

This motivates algorithms for finding the nearest normal matrix to a given $A \in \mathbb{C}^{d \times d}$. 
Normal Matrices via Gradient Flow

A classical, natural measure of non-normality of a matrix $A$ is $E(A) := \|AA^* - A^*A\|_{\text{Fro}}^2$.

Recall: The function $E : \mathbb{C}^{d \times d} \to \mathbb{R}$ is not quasi-convex, but for an arbitrary initialization $A_0 \in \mathbb{C}^{d \times d}$, we have:

Theorem [N-Shonkwiler, '24]. Gradient descent of the functional $E : A \mapsto \|AA^* - A^*A\|_{\text{Fro}}^2$ converges to a normal matrix $A_\infty$. $A_\infty$ has the same eigenvalues as $A_0$ and if $A_0$ is real, then so is $A_\infty$. Moreover, there exist $c, \epsilon > 0$ such that, if $E(A_0) < \epsilon$ then $\|A_0 - A_\infty\|_{\text{Fro}}^2 \leq c\sqrt{E(A_0)}$.

This can be adapted to preserve total weight $\|A_0\|_{\text{Fro}}^2$. 
Application: Topology of Unit Norm Normal Matrices

The space of normal matrices is contractible.

The space of unit norm normal matrices

\[ \mathcal{U} \mathcal{N}_F(d) = \{ A \in F^{d \times d} \mid AA^* = A^*A \text{ and } \|A\|_{\text{Fro}} = 1 \}, \ F = \mathbb{R} \text{ or } \mathbb{C} \]

can have interesting topology.
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Example. \( \{ A \in \mathbb{R}^{2 \times 2} \mid \|A\|_{\text{Fro}} = 1 \} \) stereographically projected to \( \mathbb{R}^3 \).

Image of unit norm nilpotent matrices in blue.

Image of \( \mathcal{UN}_{\mathbb{R}}(2) \) in pink.
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Image of \( \mathcal{U} \mathcal{N}_{\mathbb{R}}(2) \) in pink.

Theorem [N-Shonkwiler, ’24].

- \( \pi_k(\mathcal{U} \mathcal{N}_\mathbb{C}(d)) \) is trivial for all \( k \leq 2d - 2 \).
- \( \pi_k(\mathcal{U} \mathcal{N}_\mathbb{R}(d)) \) is trivial for all \( k \leq d - 2 \).

Proof. \( \mathcal{U} \mathcal{N}_F(d) \) is homotopy equivalent to \{ non-nilpotent \( d \times d \) matrices \}, via gradient descent of \( E \).

The space of nilpotent matrices is a stratified space with high codimension strata. Use transversality.
Balancing Digraphs

Let $A = (a_{ij})_{ij} \in \mathbb{R}^{d \times d}$ be the adjacency matrix of a weighted, directed graph.

We say that the graph is balanced if

$$\sum_i a_{ik} = \sum_j a_{kj} \forall k$$

Balancing is necessary for, e.g., traffic flow problems [Hooi-Tong, 1970].
Balancing Graphs by Gradient Descent

Let $A_0 \in \mathbb{R}^{d \times d}$ be the entry-wise square of an adjacency matrix of a weighted digraph.

Theorem [N-Shonkwiler, '24]. Gradient descent of the functional $A \mapsto \|\text{diag}(AA^* - A^*A)\|_{\text{Frob}}^2$ converges to the entry-wise square of the adjacency matrix of a balanced digraph. It has the same eigenvalues and principal minors as $A_0$, and has zero entries whenever $A_0$ does. This can be adapted to preserve total weight $\|A_0\|_{\text{Frob}}^2$.

Also partially follows by symplectic principles.

Corollary [N-Shonkwiler, '24]. The space of balanced, unit norm adjacency matrices is homotopy equivalent to the space of unit norm normal matrices.
Tight frames on Vector Bundles
Vector Bundles

A rank-\(d\) real vector bundle over an \(n\)-dimensional smooth manifold \(M\) is a smooth manifold \(E\) with projection \(E \xrightarrow{\pi} M\) whose fibers \(E_p = \pi^{-1}(p)\) are isomorphic to \(\mathbb{R}^d\) such that \(E \approx M \times \mathbb{R}^d\), locally.

A Riemannian structure on \(E\) is a smooth choice of inner product \(\langle \cdot, \cdot \rangle_p\) on each fiber \(E_p\).

A vector field for \(E\) is a smooth map \(\sigma : M \to E\) such that \(\pi \circ \sigma = \text{Id}_M\).

Examples.

- Trivial bundle \(M \times \mathbb{R}^d \to M\)
- Tangent bundle over a manifold \(TM \to M\)
- Tautological bundle \(E \to \text{Gr}(k, \mathbb{R}^d)\)
Frames on Vector Bundles

Let $E \to M$ be a rank-$d$ vector bundle over an $n$-dimensional manifold. An $N$-frame for $E$ is a collection of $N$ vector fields $(\sigma_1, \ldots, \sigma_N)$ such that $\{\sigma_1(p), \ldots, \sigma_N(p)\}$ is spanning for all $p$.

The frame is a tight frame if $[\sigma_1(p), \ldots, \sigma_N(p)]$ defines a tight frame on the fiber for all $p$. (Requires a choice of Riemannian structure for this to make sense.)

Prior Work: [Kuchment, ’08], [Freeman-Poore-Wei-Wyse, ’14], [Freeman-Hotovy-Martin, ’14], [Kuchment, ’16], [Auckly-Kuchment, ’18]
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Motivation: Robust representation of “signals” on vector bundles; i.e., representing vector fields.

Measurements are coefficients of the signal w.r.t. the frame. This gives a map $M \rightarrow \mathbb{R}^N$. 

signal = vector field
measurement system = $N$-frame
Frames on Vector Bundles

Let $E \to M$ be a rank-$d$ vector bundle over an $n$-dimensional manifold. An $N$-frame for $E$ is a collection of $N$ vector fields $(\sigma_1, \ldots, \sigma_N)$ such that $\{\sigma_1(p), \ldots, \sigma_N(p)\}$ is spanning for all $p$.

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Motivation: Robust representation of “signals” on vector bundles; i.e., representing vector fields.

Experiment: Signal reconstruction error for noisy signals on $S^2$ for tight versus random 3-frames.

**Figure 1.** Left: plots of the empirical CDFs of the MSE distributions for the Parseval frame (dark blue) and the random frames (light orange). Right: histograms of the MSE distributions for the Parseval frame (dark blue) and for the best of the random frames (lighter orange).
Existence of Tight Frames

**Question.** When do (tight) $N$-frames on $E \to M$ exist?
Existence of Tight Frames

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Examples.

• 3-frame on $TS^2$
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Such a frame would require a nonvanishing vector field on $S^2$, which DNE by the Hairy Ball Theorem.
Existence of Tight Frames

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Examples.

- 3-frame on $TS^2$

- A 2-frame on $TS^2$ does not exist

Such a frame frame would require a nonvanishing vector field on $S^2$, which DNE by the Hairy Ball Theorem.

- A tight $N$-frame for $TM$ exists for sufficiently large $N$.

Follows by Whitney embedding argument, see [Freeman-Poore-Wei-Wyse, ’14].
Theorem (Ballas-N-Shonkwiler,’23). Let $E \to M$ be a vector bundle. An $N$-frame exists if and only if a tight $N$-frame exists.
Existence of Tight Frames

Theorem (Ballas-N-Shonkwiler,’23). Let $E \to M$ be a vector bundle. An $N$-frame exists if and only if a tight $N$-frame exists.

\[
\begin{align*}
\pi_{i+1}(M) &\xrightarrow{\text{Id}} \pi_i(\mathcal{P}^n(\mathbb{R}^k)) \xrightarrow{\iota_*} \pi_i(\mathcal{P}^n(E)) \xrightarrow{\overline{\iota}_*} \pi_i(M) \xrightarrow{\text{Id}} \pi_{i-1}(\mathcal{P}^n(\mathbb{R}^k)) \\
\pi_{i+1}(M) &\xrightarrow{\text{Id}} \pi_i(\mathcal{F}^n(\mathbb{R}^k)) \xrightarrow{\iota_*} \pi_i(\mathcal{F}^n(E)) \xrightarrow{\overline{\iota}_*} \pi_i(M) \xrightarrow{\text{Id}} \pi_{i-1}(\mathcal{F}^n(\mathbb{R}^k))
\end{align*}
\]

Theorem (Ballas-N-Shonkwiler,’23). Let $E \to M$ be a rank-$d$ vector over an $n$-manifold. Then there exists a tight frame for $E$ if $N \geq d + n$. 
Existence of Tight Frames

Theorem (Ballas-N-Shonkwiler,'23). Let $E \to M$ be a vector bundle. An $N$-frame exists if and only if a tight $N$-frame exists.

\[ \begin{array}{cccccc}
\pi_{i+1}(M) & \to & \pi_i(\mathcal{P}^n(\mathbb{R}^k)) & \to & \pi_i(\mathcal{P}^n(E)) & \to & \pi_i(M) & \to & \pi_{i-1}(\mathcal{P}^n(\mathbb{R}^k)) \\
\downarrow \text{Id} & & \downarrow \iota_* & & \downarrow \bar{\iota}_* & & \downarrow \text{Id} & & \downarrow \iota_* \\
\pi_{i+1}(M) & \to & \pi_i(\mathcal{F}^n(\mathbb{R}^k)) & \to & \pi_i(\mathcal{F}^n(E)) & \to & \pi_i(M) & \to & \pi_{i-1}(\mathcal{F}^n(\mathbb{R}^k))
\end{array} \]

Theorem (Ballas-N-Shonkwiler,'23). Let $E \to M$ be a rank-$d$ vector over an $n$-manifold. Then there exists a tight frame for $E$ if $N \geq d + n$.

Sufficient, but not necessary!

Theorem (Ballas-N-Shonkwiler,'23). There exists a closed, orientable $d$-manifold $M$ such that $TM$ does not admit a tight $(d + 1)$-frame if and only if $d > 3$. 
Open Questions

• What about higher homotopy/(co)homology of frame spaces?

• What about the corresponding question for spaces of real frames?

• Can symplectic methods be applied to frames in infinite-dimensional Hilbert spaces?

• Can geometry of the Gelfand-Tsetlin polytope be used to get quantitative statements about compressed sensing properties of random frames?

• Can we efficiently generate random frames using Markov chain sampling in G-T polytope?

• Applications of frame theory on vector bundles?
Thanks for Listening!

References:


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