Optimization Informed by Geometric Invariant Theory and Symplectic Geometry

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Recent Progress on Optimal Point Distributions and Related Fields
June 3, 2024
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Take-Home Message

Symmetry + geometry sometimes tells you an optimization problem is easier than expected.
A spanning set $f_1, \ldots, f_n \in \mathbb{C}^d$ is a frame. \Rightarrow F = [f_1 \cdots f_n] \in \mathbb{C}^{d \times n}$
Equal-Norm Parseval Frames

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Definition.

\( \{f_1, \ldots, f_n\} \subset \mathbb{C}^d \) is a Parseval frame if \( \text{Id}_{d \times d} = FF^* = f_1 f_1^* + \cdots + f_n f_n^* \).
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An *equal-norm Parseval frame* (ENP frame) is a Parseval frame $f_1, \ldots, f_n$ with $\|f_i\|^2 = \|f_j\|^2$ for all $i$ and $j$. 
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\[
\sum \|f_i\|^2 = \text{tr} F^*F = \text{tr} FF^* = \text{tr} \text{Id}_{d \times d} = d, \text{ so each } \|f_i\|^2 = \frac{d}{n}.
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**Equal-Norm Parseval Frames**
Frame Potential

**Definition** [Benedetto–Fickus, Casazza–Fickus]

The *frame potential* is

$$FP(F) = \|FF^*\|_{F}^{2}$$
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**Proposition** [cf. Welch]

The equal-norm Parseval frames are exactly the global minima of \( \text{FP}_{\text{equal norm}} \).
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The equal-norm Parseval frames are exactly the global minima of \( \text{FP}_{\text{equal norm}} \).

**Theorem** [Benedetto–Fickus]
As a function on equal-norm frames with fixed \( d \) and \( n \), \( \text{FP} \) has no spurious local minima.
Frame Potential
Optimization

**Theorem [with Mixon, Needham, and Villar]**

On the space of equal-norm frames, consider the initial value problem

\[
\Gamma(F_0, 0) = F_0, \quad \frac{d}{dt}\Gamma(F_0, t) = -\text{grad} \text{FP}(\Gamma(F_0, t)).
\]

If \(F_0\) has full spark, then \(\lim_{t \to \infty} \Gamma(F_0, t)\) is an ENP frame.
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If \( F_0 \) has full spark, then \( \lim_{t \to \infty} \Gamma(F_0, t) \) is an ENP frame.

**Theorem** [with Needham]

Same for fusion frames.
Why Not the Other Way?

**Definition** [cf. Bodmann–Casazza]

The *normalizing potential* is

\[
\text{NP}(f_1, \ldots, f_n) = \sum_{i=1}^{n} \|f_i\|^4.
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**Definition** [cf. Bodmann–Casazza]
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**Theorem [with Caine and Needham]**
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\[ \tilde{\Gamma}(F_0, 0) = F_0 \quad \frac{d}{dt} \tilde{\Gamma}(F_0, t) = -\text{grad} \text{NP}(\tilde{\Gamma}(F_0, t)). \]

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Normal Matrices

Definition.

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Define the *non-normal energy* $E : \mathbb{C}^{d \times d} \to \mathbb{R}$ by

$E(A) := \|[A, A^*]\|^2$. 
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A ∈ \mathbb{C}^{d \times d} is normal if \( AA^* = A^* A \).

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\[
0 = AA^* - A^* A = [A, A^*].
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Obvious Fact.
The normal matrices are the global minima of \( E \).
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Theorem [with Needham]
The only critical points of $E$ are the global minima; i.e., the normal matrices.
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$E$ is not quasiconvex!
Gradient Descent

Let $\mathcal{F} : \mathbb{C}^{d \times d} \times \mathbb{R} \to \mathbb{C}^{d \times d}$ be negative gradient descent of $E$; i.e.,

$$\mathcal{F}(A_0, 0) = A_0 \quad \frac{d}{dt} \mathcal{F}(A_0, t) = -\nabla E(\mathcal{F}(A_0, t)).$$

**Theorem** [with Needham]

For any $A_0 \in \mathbb{C}^{d \times d}$, the matrix $A_\infty := \lim_{t \to \infty} \mathcal{F}(A_0, t)$ exists, is normal, has the same eigenvalues as $A_0$, and is real if $A_0$ is.
Balancing Graphs

Define the *unbalanced energy* $B(A) := \|\text{diag}([A, A^*])\|^2 = \sum (\|A_i\|^2 - \|A^i\|^2)^2$. 
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If $A = (a_{ij})_{i,j} \in \mathbb{R}^{d \times d}$ such that $\text{diag}([A, A^*]) = 0$, then $\hat{A} = (a^2_{ij})_{i,j}$ is the adjacency matrix of a *balanced* multigraph.
Balancing Graphs

Let $\mathcal{F}(A_0, 0) = A_0$, $\frac{d}{dt} \mathcal{F}(A_0, t) = -\nabla B(\mathcal{F}(A_0, t))$ be negative gradient flow of $B$. 
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**Theorem [with Needham]**

For any $A_0 \in \mathbb{C}^{d \times d}$, the matrix $A_\infty := \lim_{t \to \infty} \mathcal{F}(A_0, t)$ exists, is balanced, has the same eigenvalues and principal minors as $A_0$, and has zero entries wherever $A_0$ does.

If $A_0$ is real, so is $A_\infty$, and if $A_0$ has all non-negative entries, then so does $A_\infty$. 
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Why?
A symplectic manifold is a smooth manifold $M$ together with a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$. 
Symplectic Geometry

A *symplectic manifold* is a smooth manifold $M$ together with a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$.

**Example:** $(\mathbb{R}^2, dx \wedge dy) = (\mathbb{C}, \frac{i}{2} dz \wedge d\bar{z})$
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Example: $(\mathbb{R}^2, dx \wedge dy) = (\mathbb{C}, \frac{i}{2}dz \wedge d\bar{z})$

\[dx \wedge dy \left( a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}, c \frac{\partial}{\partial x} + d \frac{\partial}{\partial y} \right) = ad - bc\]

\[(c, d) = c\bar{e}_1 + d\bar{e}_2 = c \frac{\partial}{\partial x} + d \frac{\partial}{\partial y}\]

\[(a, b) = a\bar{e}_1 + b\bar{e}_2 = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}\]
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\((S^2, d\theta \wedge dz)\)
Functions and Symplectic Gradients

If $H : M \to \mathbb{R}$ is smooth, then there exists a unique vector field $X_H$ so that $dH = \iota_{X_H} \omega$, i.e.,

$$dH(\cdot) = \omega(X_H, \cdot)$$

($X_H$ is called the Hamiltonian vector field for $H$, or sometimes the symplectic gradient of $H$)
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**Example.** $H : (S^2, d\theta \wedge dz) \to \mathbb{R}$ given by $H(\theta, z) = z$.

$H$ is constant on orbits of $X_H$:

$$\mathcal{L}_{X_H}(H) = dH(X_H) = \omega(X_H, X_H) = 0$$

$$dH = dz = \iota_{\frac{\partial}{\partial \theta}} (d\theta \wedge dz), \text{ so } X_H = \frac{\partial}{\partial \theta}.$$
Noether’s Theorem

“Every continuous symmetry has a corresponding conserved quantity”
Circle Actions

A circle action on $(M, \omega)$ determines a vector field $X$ by

$$X(p) = \frac{d}{dt} \bigg|_{t=0} e^{it} \cdot p$$
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\]

\(S^1 = U(1)\) acts on \((S^2, d\theta \wedge dz)\) by

\[
e^{it} \cdot (\theta, z) = (\theta + t, z).
\]

So \(X = \frac{\partial}{\partial \theta} \cdot \).
Symmetries and Conserved Quantities

**Definition.** A circle action on \((M, \omega)\) is *Hamiltonian* if there exists a *momentum map* \(\mu : M \rightarrow \mathbb{R}\) so that
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d\mu = \iota_X \omega = \omega(X, \cdot),
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where \(X\) is the vector field generated by the circle action. In other words, \(X = X_\mu\).
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$$\mu(\theta, z) = z$$
Nice Potentials

Suppose $\mu : (M, \omega) \to g^*$ is the momentum map of a Hamiltonian $G$ action.
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This kind of function is really nice!

Frances Kirwan
Nice Potentials

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Define \( \Phi : M \to \mathbb{R} \) by \( \Phi(p) = \|\mu(p)\|^2 \).

Theorem [Kirwan]

Reductive algebraic group action on Kähler manifold \( \iff \) semistable points flow to global minima of \( \Phi \) by gradient descent.

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Frances Kirwan
Geometric Invariant Theory (GIT)

The GIT quotient consists of group orbits which can be distinguished by $G$-invariant (homogeneous) polynomials.
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\[
\mathbb{C}^* \actsleft \mathbb{C} \mathbb{P}^2
\]
\[
t \cdot [z_0 : z_1 : z_2] = [z_0 : tz_1 : \frac{1}{t} z_2]
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Geometric Invariant Theory (GIT)

The GIT quotient consists of group orbits which can be distinguished by $G$-invariant (homogeneous) polynomials.

$$\mathbb{C}^* \curvearrowright \mathbb{C}P^2$$

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$$\mathbb{C}P^2 // \mathbb{C}^* \cong \mathbb{C}P^1$$
Geometric Invariant Theory (GIT)

The GIT quotient consists of group orbits which can be distinguished by $G$-invariant (homogeneous) polynomials.

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\mathbb{C}^* \sim \mathbb{C}P^2 \\
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$$

$$
\mathbb{C}P^2/\mathbb{C}^* \cong \mathbb{C}P^1
$$

Roughly: identify orbits whose closures intersect, throw away orbits on which all $G$-invariant polynomials vanish.
### Groups, Actions, Maps

<table>
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<tr>
<th>Group</th>
<th>Manifold</th>
<th>Action</th>
<th>Momentum Map</th>
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<tbody>
<tr>
<td>U(d)</td>
<td>$\left(\mathbb{CP}^{d-1}\right)^n$</td>
<td>$g \cdot [F] = [gF]$</td>
<td>$[F] \mapsto FF^*$</td>
<td>FP</td>
</tr>
<tr>
<td>U(1)$^n$</td>
<td>Gr($d, n$)</td>
<td>$g \cdot [F] = [Fg^*]$</td>
<td>$[F] \mapsto \text{diag}(F^*F)$</td>
<td>NP</td>
</tr>
<tr>
<td>SU(d)</td>
<td>$\mathbb{C}^{d\times d}$</td>
<td>$g \cdot A = gAg^{-1}$</td>
<td>$A \mapsto [A, A^*]$</td>
<td>E</td>
</tr>
<tr>
<td>S(U(1)$^d$)</td>
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Questions

Do similar techniques work for

1. Tightening (or normalizing) probabilistic frames?
2. Constructing doubly-stochastic matrices?
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Does this machinery tell us anything about the Paulsen problem?
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1. Tightening (or normalizing) probabilistic frames?
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Does this machinery tell us anything about the Paulsen problem?

What other nice configurations are minima of potentials of this form?
Thank you!
References

Three proofs of the Benedetto–Fickus theorem
Dustin Mixon, Tom Needham, Clayton Shonkwiler, and Soledad Villar
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Fusion frame homotopy and tightening fusion frames by gradient descent
Tom Needham and Clayton Shonkwiler
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arXiv:2208.11045

Geometric approaches to matrix normalization and graph balancing
Tom Needham and Clayton Shonkwiler
arXiv:2405.06190