Optimization Informed by Geometric Invariant Theory and Symplectic Geometry

Clayton Shonkwiler
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Recent Progress on Optimal Point Distributions and Related Fields
June 3, 2024

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this talk!

shonkwiler.org/icerm24

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Collaborators

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Florida State University

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Colorado State University

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The Ohio State University

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Take-Home Message

Symmetry + geometry sometimes tells you an optimization problem is easier than expected.

Equal-Norm Parseval Frames

A spanning set $f_1, \ldots, f_n \in \mathbb{C}^d$ is a *frame*. $\Rightarrow F = [f_1 \cdots f_n] \in \mathbb{C}^{d \times n}$

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$$\sum \|f_i\|^2 = \operatorname{tr} F^* F = \operatorname{tr} FF^* = \operatorname{tr} \mathrm{Id}_{d\times d} = d, \text{ so each } \|f_i\|^2 = \tfrac{d}{n}.$$

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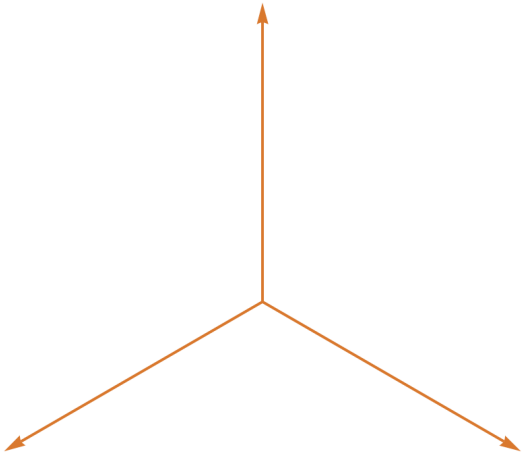

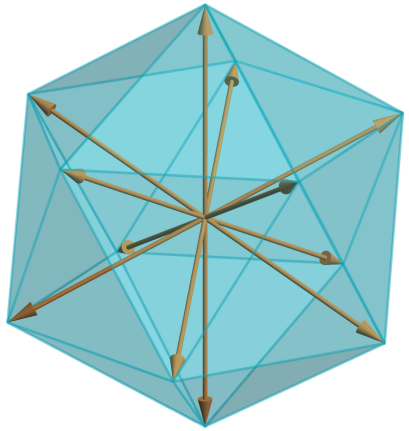

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Definition [Benedetto–Fickus, Casazza–Fickus]

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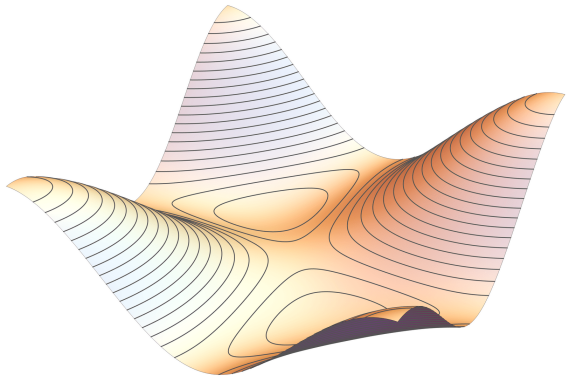
Theorem [Benedetto–Fickus]

As a function on equal-norm frames with fixed d and n, FP has no spurious local minima.



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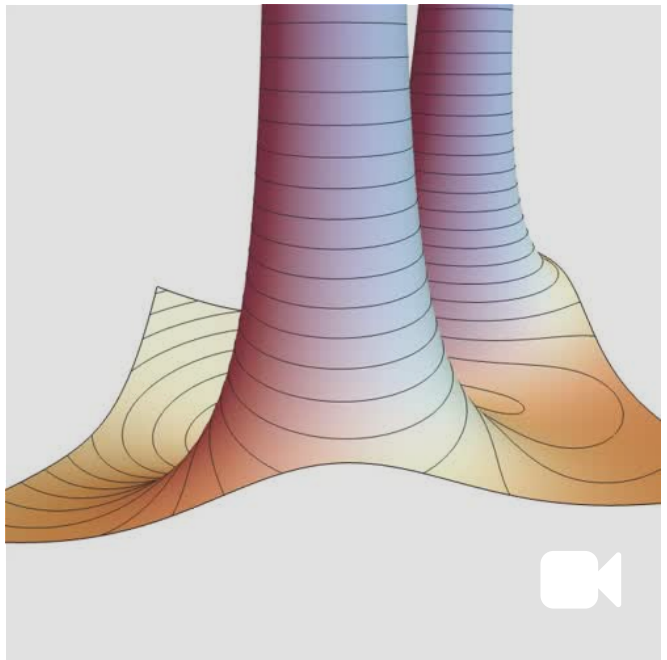
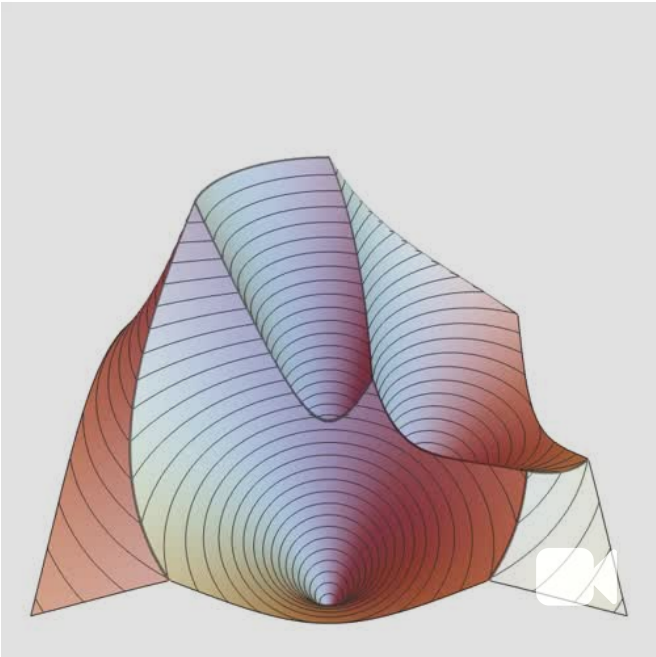
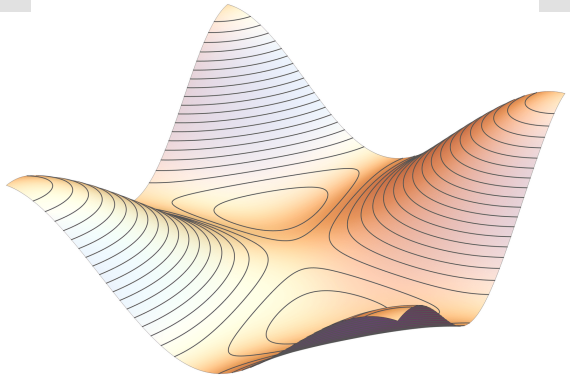






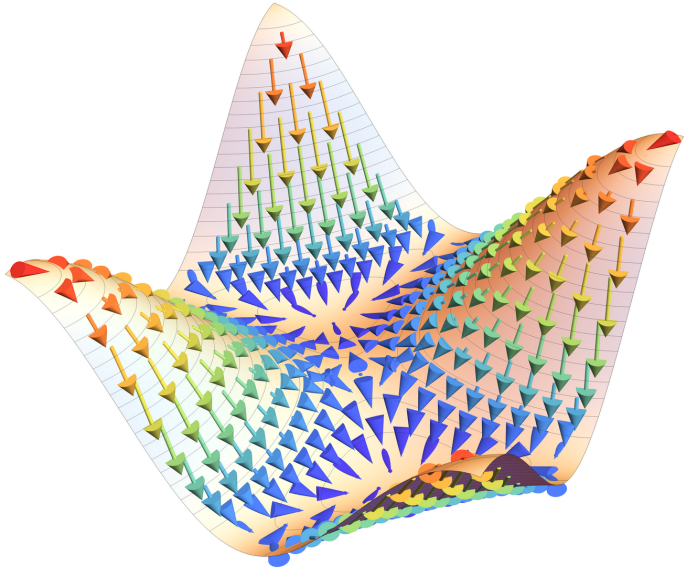
Optimization

Theorem [with Mixon, Needham, and Villar]

On the space of equal-norm frames, consider the initial value problem

$$\Gamma(F_0, 0) = F_0, \qquad \tfrac{d}{dt}\Gamma(F_0, t) = -\operatorname{grad} \mathrm{FP}(\Gamma(F_0, t)).$$

If F_0 has full spark, then $\lim_{t\to\infty} \Gamma(F_0, t)$ is an ENP frame.



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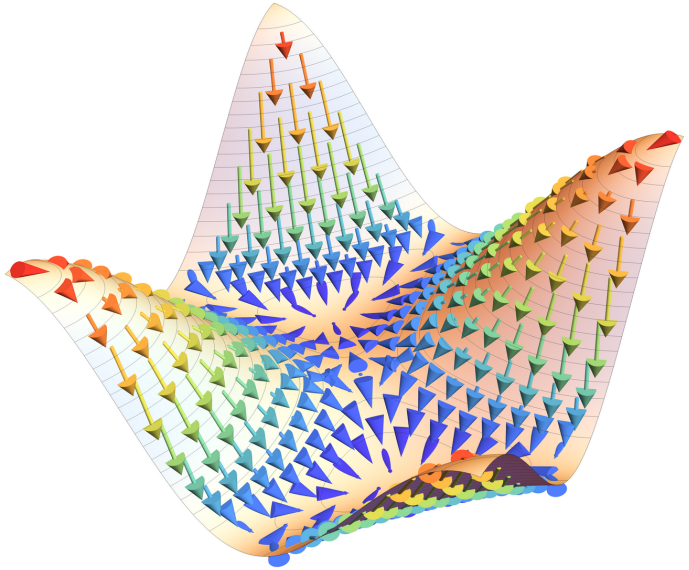

Theorem [with Needham]

Same for fusion frames.

Why Not the Other Way?

Definition [cf. Bodmann–Casazza]

The *normalizing potential* is

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Theorem [with Caine and Needham]

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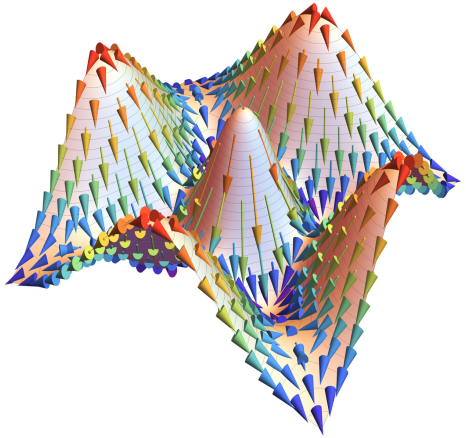

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The only critical points of E are the global minima; i.e., the normal matrices.

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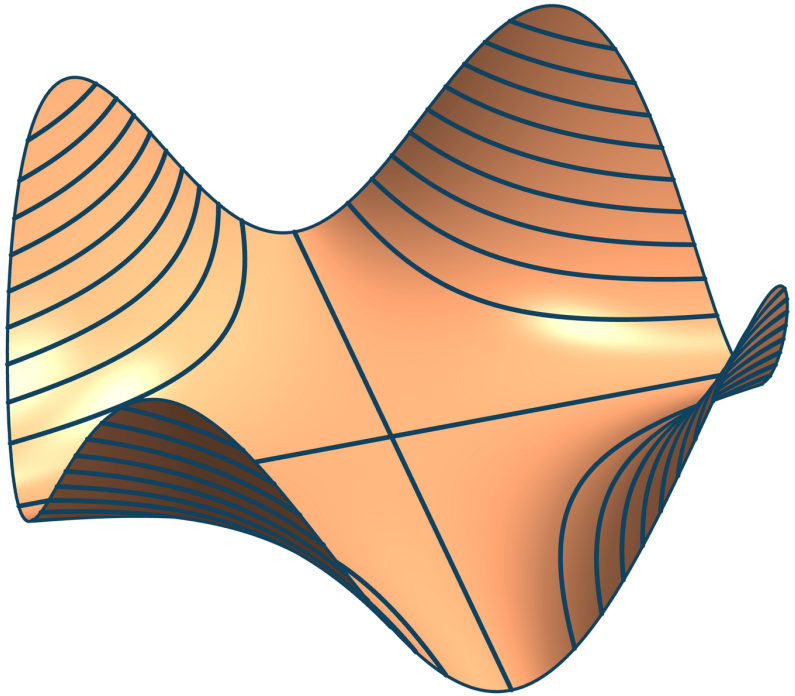

E is not quasiconvex!

Theorem [with Needham]

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Gradient Descent

Let $\mathcal{F} : \mathbb{C}^{d\times d} \times \mathbb{R} \to \mathbb{C}^{d\times d}$ be negative gradient descent of E; i.e.,

$$\mathcal{F}(A_0, 0) = A_0 \qquad \tfrac{d}{dt}\mathcal{F}(A_0, t) = -\nabla \mathrm{E}(\mathcal{F}(A_0, t)).$$

Theorem [with Needham]

For any $A_0 \in \mathbb{C}^{d\times d}$, the matrix $A_\infty := \lim_{t\to\infty} \mathcal{F}(A_0, t)$ exists, is normal, has the same eigenvalues as A_0, and is real if A_0 is.

Balancing Graphs

Define the *unbalanced energy* $\mathrm{B}(A) := \|\mathrm{diag}([A, A^*])\|^2 = \sum \left(\|A_i\|^2 - \|A^i\|^2\right)^2.$

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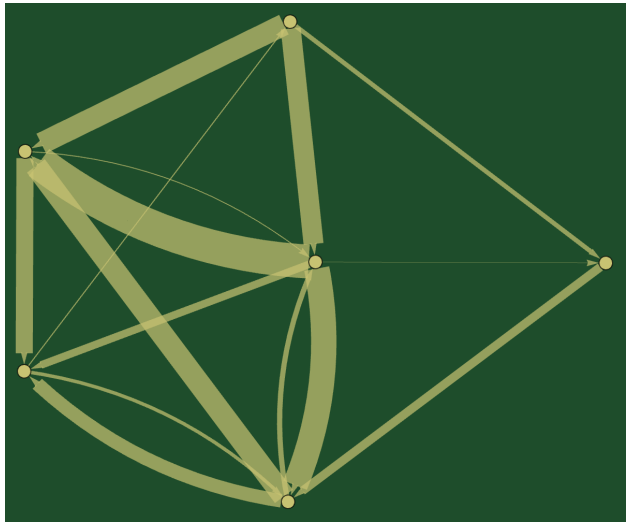
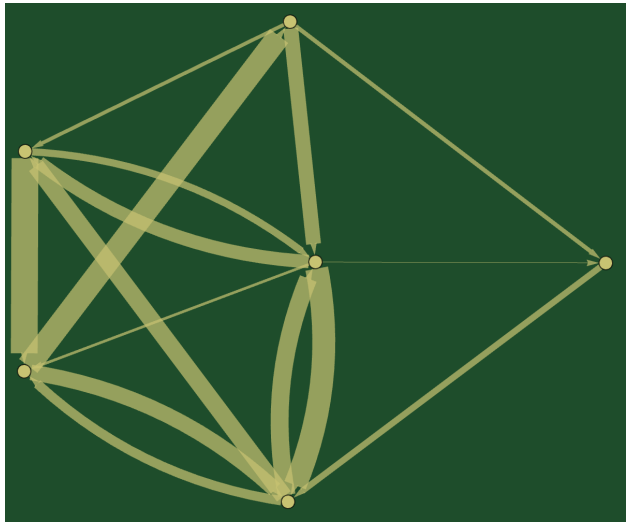
If $A = (a_{ij})_{i,j} \in \mathbb{R}^{d\times d}$ such that $\mathrm{diag}([A, A^*]) = 0$, then $\widehat{A} = \left(a_{ij}^2\right)_{i,j}$ is the adjacency matrix of a *balanced* multigraph.





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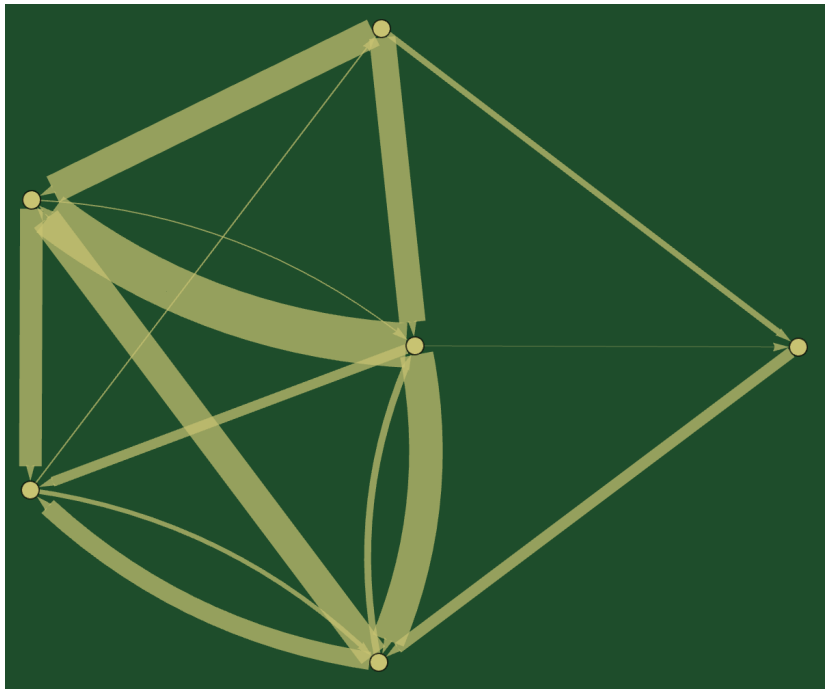

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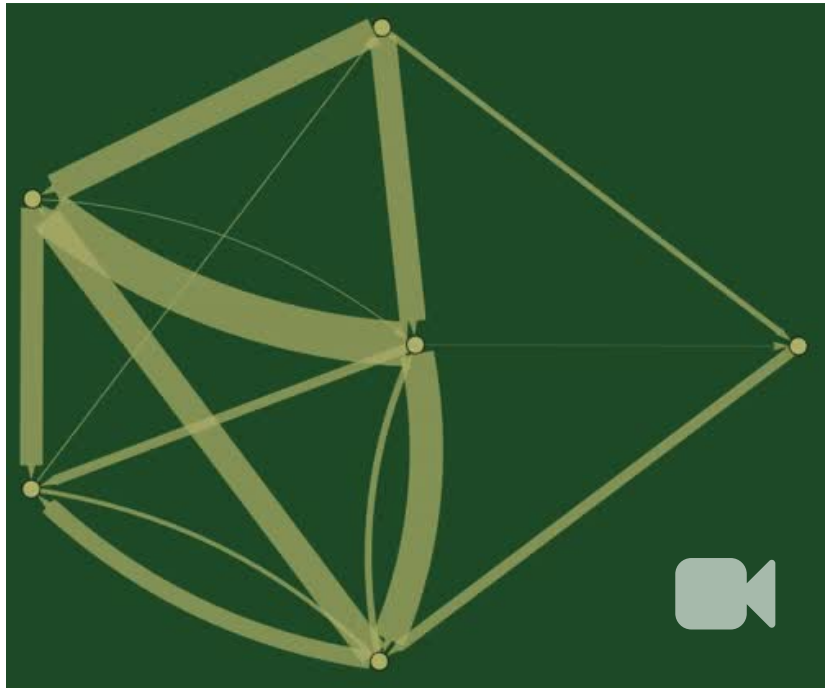

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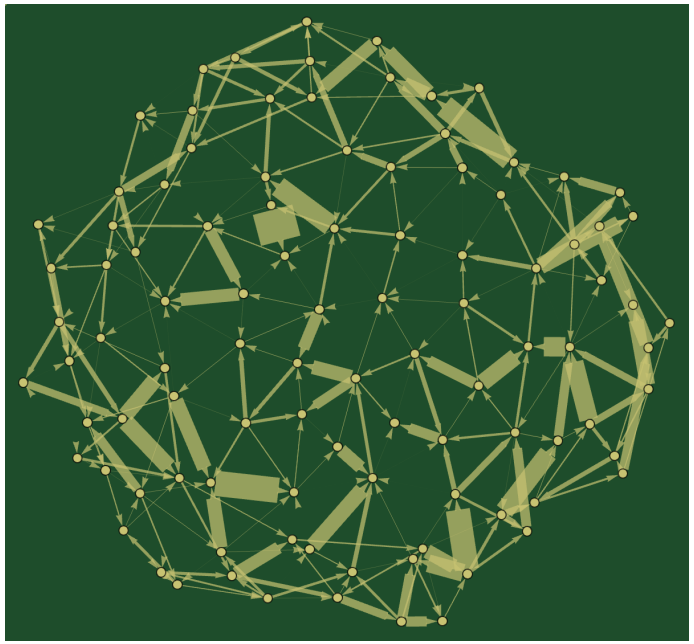

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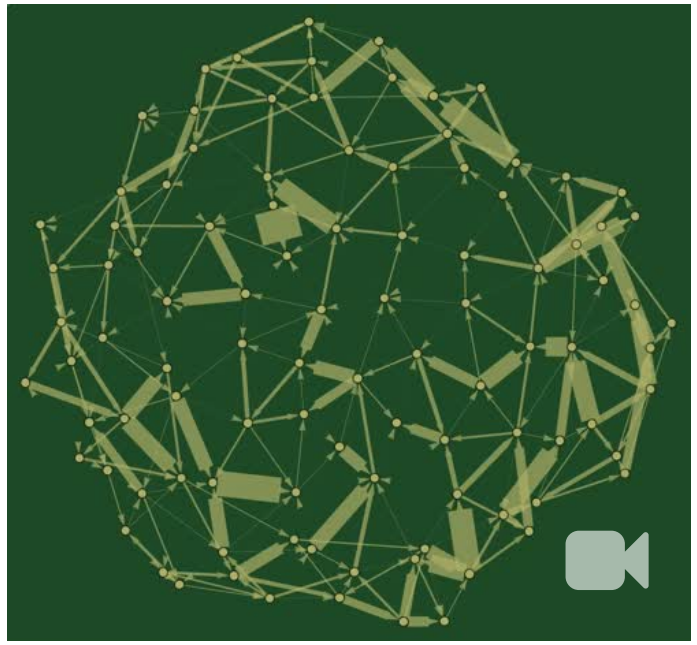

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Why?

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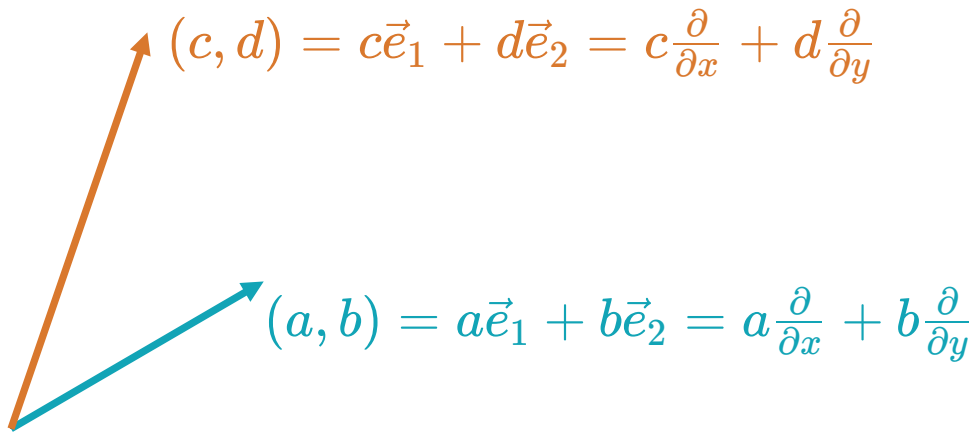
$$dx \wedge dy \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y} \right) = ad - bc$$



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$(S^2, d\theta \wedge dz)$

Functions and Symplectic Gradients

If $H : M \to \mathbb{R}$ is smooth, then there exists a unique vector field X_H so that $dH = \iota_{X_H}\omega$, i.e.,

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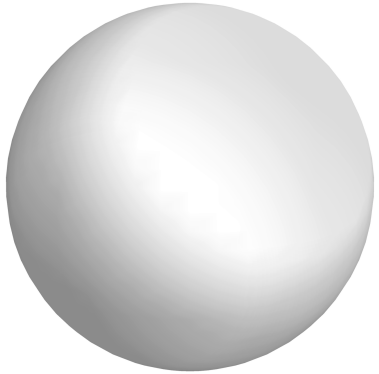

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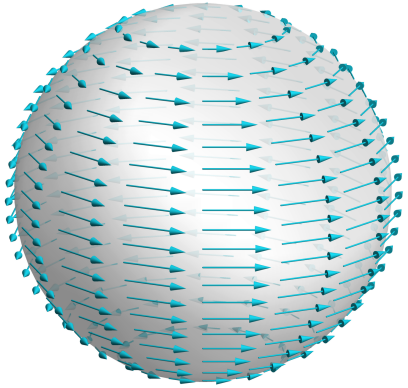

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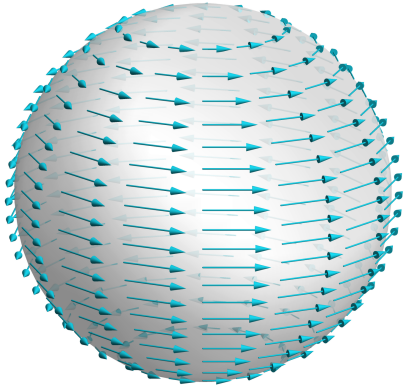

H is constant on orbits of X_H:

$$\mathcal{L}_{X_H}(H) = dH(X_H) = \omega(X_H, X_H) = 0$$

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Noether’s Theorem

“Every continuous symmetry has a corresponding conserved quantity”

Circle Actions

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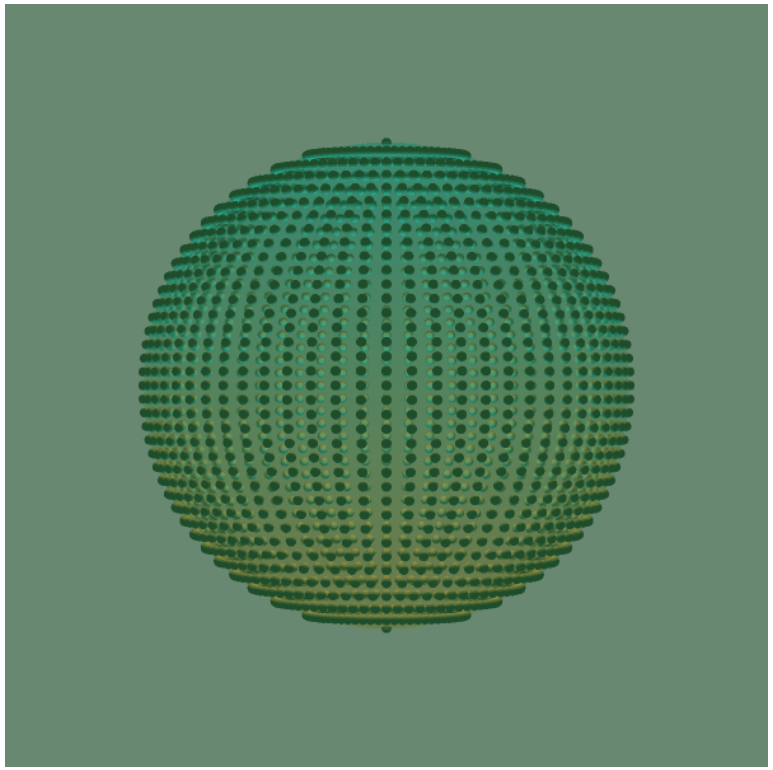

$S^1 = U(1)$ acts on $(S^2, d\theta \wedge dz)$ by

$$e^{it} \cdot (\theta, z) = (\theta + t, z).$$

So $X = \frac{\partial}{\partial \theta}$.

Symmetries and Conserved Quantities

Definition. A circle action on (M, ω) is *Hamiltonian* if there exists a *momentum map*

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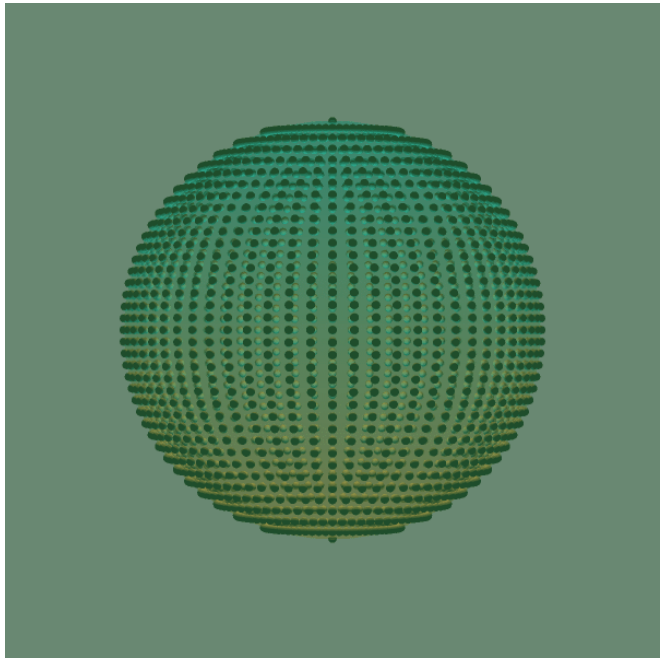

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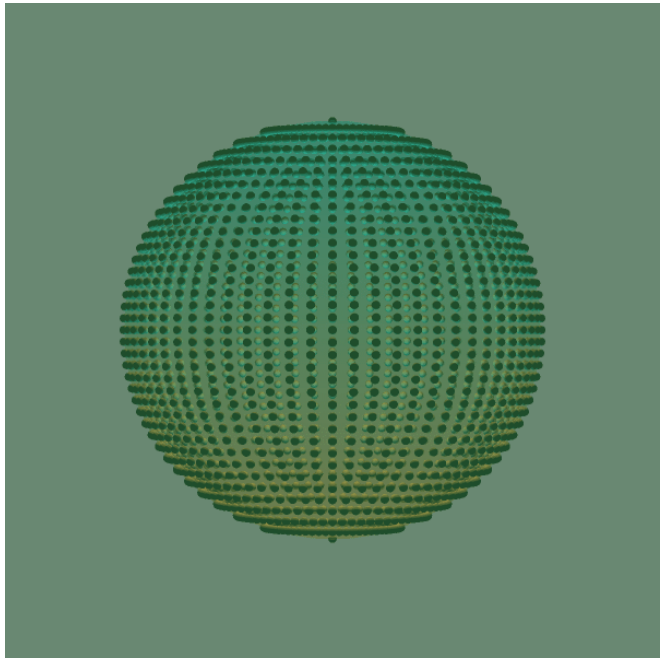

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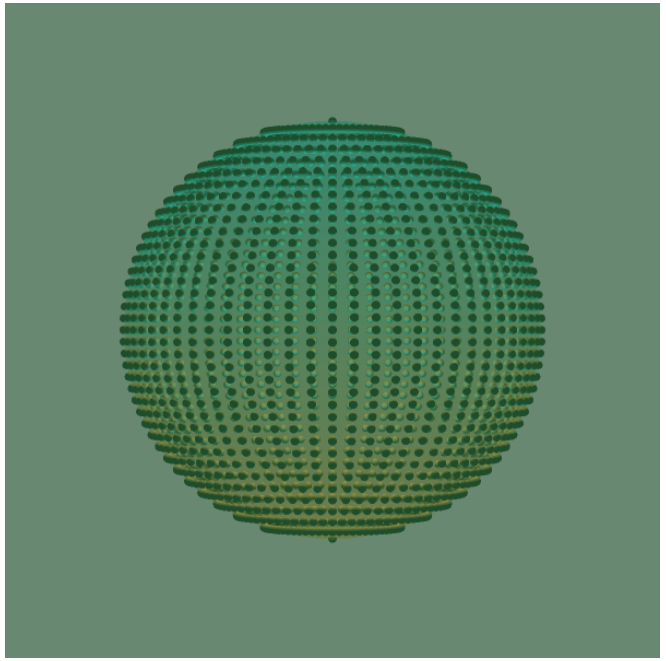

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Nice Potentials

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Theorem [Kirwan]

Reductive algebraic group action on Kähler manifold $\implies$ semistable points flow to global minima of Φ by gradient descent.

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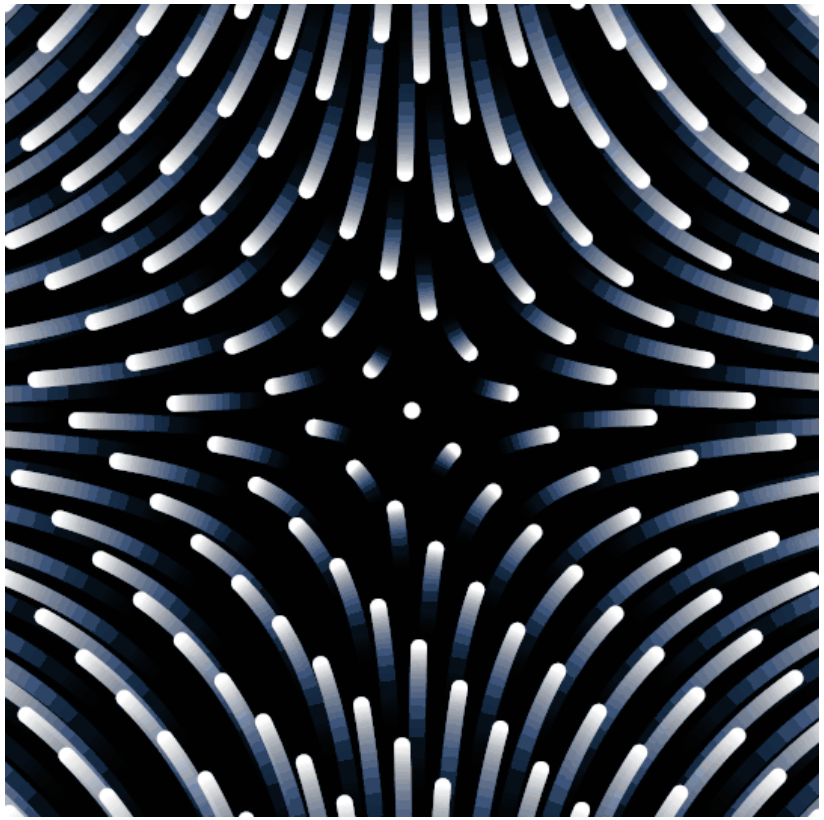

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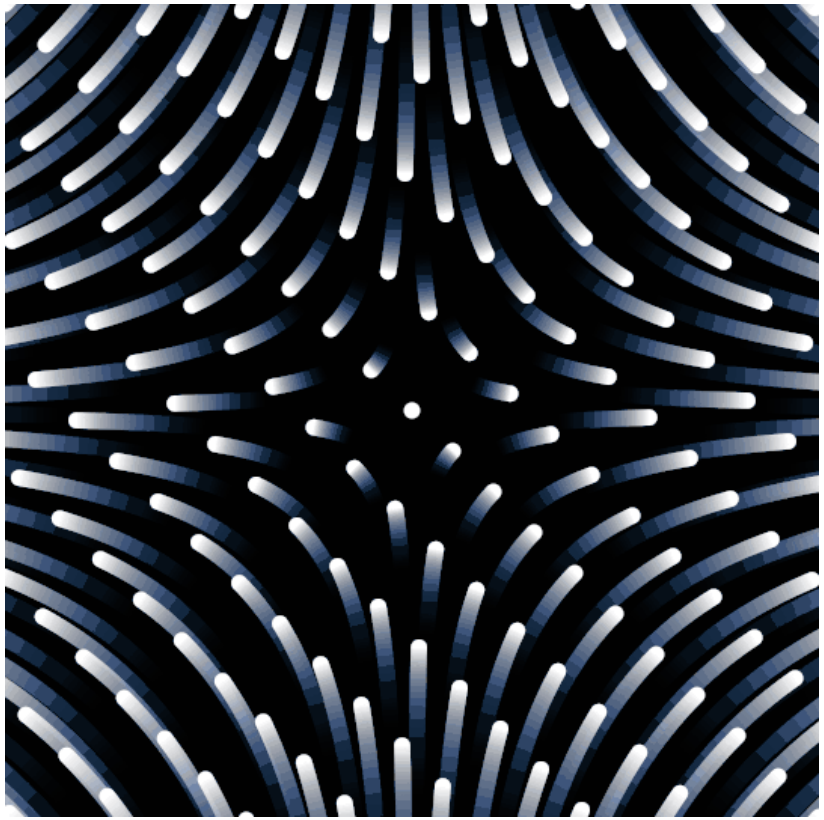

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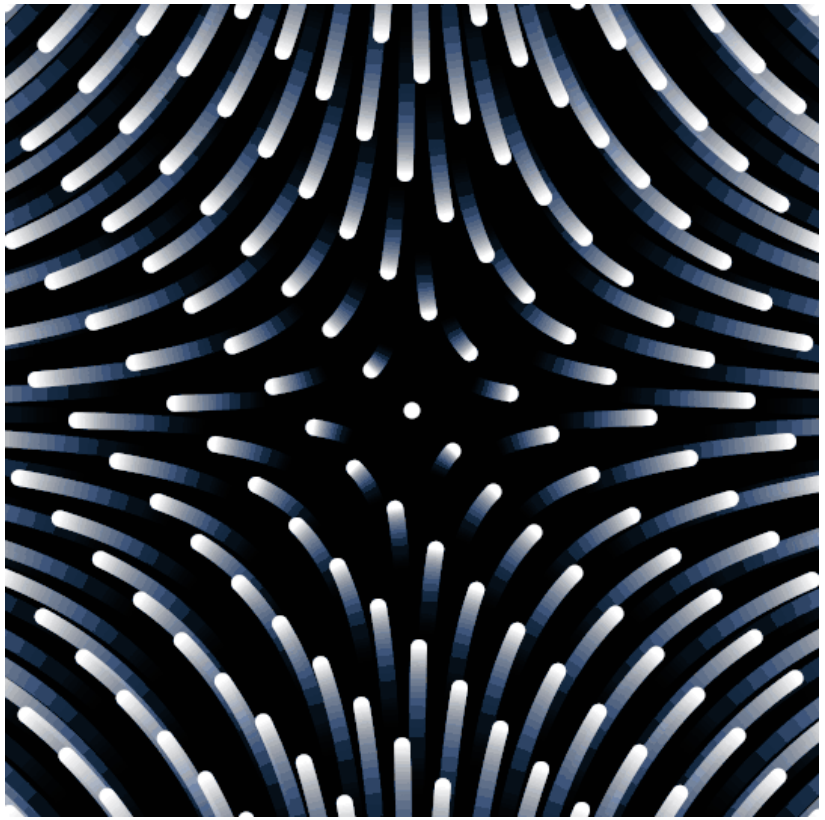

Roughly: identify orbits whose closures intersect, throw away orbits on which all G-invariant polynomials vanish.

Groups, Actions, Maps

Group	Manifold	Action	Momentum Map	Potential
$\mathrm{U}(d)$	$\left(\mathbb{CP}^{d-1}\right)^n$	$g \cdot [F] = [gF]$	$[F] \mapsto FF^*$	FP
$\mathrm{U}(1)^n$	$\mathrm{Gr}(d,n)$	$g \cdot [F] = [Fg^*]$	$[F] \mapsto \mathrm{diag}(F^*F)$	NP
$\mathrm{SU}(d)$	$\mathbb{C}^{d\times d}$	$g \cdot A = gAg^{-1}$	$A \mapsto [A, A^*]$	E
$\mathrm{S}(\mathrm{U}(1)^d)$	$\mathbb{C}^{d\times d}$	$g \cdot A = gAg^{-1}$	$A \mapsto \mathrm{diag}([A, A^*])$	B

Questions

Do similar techniques work for

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What other nice configurations are minima of potentials of this form?


Thank you!

References

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