Opinion-dynamics Models with Random-Time Interactions

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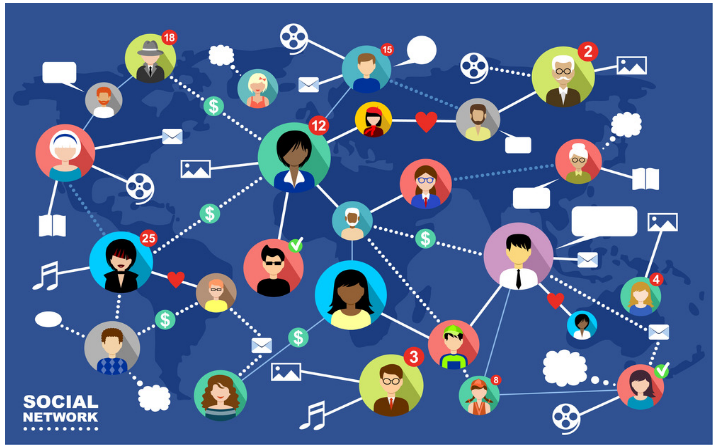
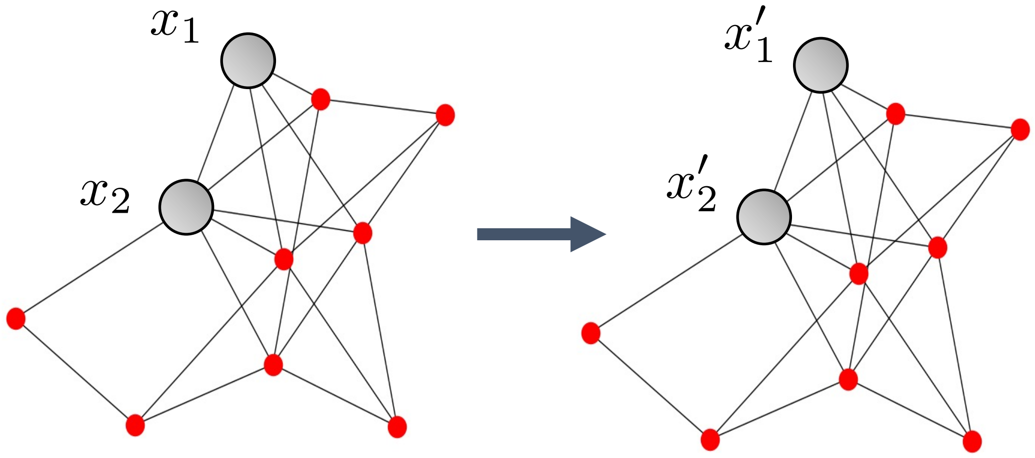
Opinion dynamics on social networks

- Opinion dynamics provide a qualitative approach to study opinion evolution and formation as a dynamical process on networks.





Models of opinion dynamics

- ***Discrete***-valued models

  voter models [S. Redner 2019]; contagion model;
  approximation theory (node-based; degree-based; mean-field) [Gleeson PRX 2013]

- ***Continuous***-valued models

  DeGroot model [DeGroot 1974]; Friedkin–Johnsen model [Friedkin & Jehnsen 2011];
  Bounded-confidence models [Deffuant 2000] [Hegselmann & Krause 2002] [Blondel, Hendrickx, Tsitsiklis 2021]

- Time can be either discrete or continuous

  Gómez-Serrano, Graham, Boudec 2021; Liu, Chen, Basar, Belabbas, 2017; C. & Porter SIAP 2023

- Time is treated as a ***deterministic*** value!

Outline

I. ***Modeling*** of opinion dynamics

  - Temporal stochasticity

II. ***Analysis*** of the opinion-dynamics models

  - Network structures, waiting-time distributions

III. Summary and extensions

How to model the stochastic nature of human interactions?

- Random ***networks***: Erdős–Rényi, small-world, tie-decay, etc.

  Sugishita, Porter (2021); Fennell, Burke, Quayle, Gleeson (2021)
- ***Interaction rules*** that involve randomness

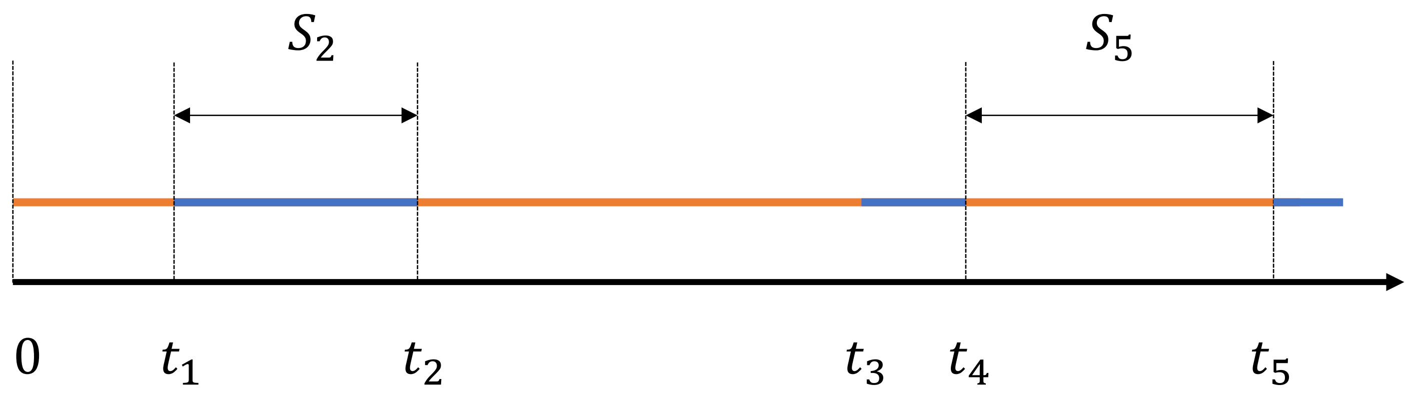
  Pineda, Toral, Hernández-García (2013); Deffuant, Neau, Amblard, Weisbuch (2000); Fernández–Gracia, Eguíluz, San Miguel (2011)
- Temporal network

  heavy-tailed dynamics in human dynamics: Web browsing, email exchange, trade transition, etc.
  Barabási Nature (2005); Vázquez, Oliveira, Dezsö, Goh, Kondor, Barabási (2006); Iribarren, Moro (2009)

Does the distribution type influence the dynamics occurring on networks?

When do agents interact?

- Renewal process $\mathrm{R}(t)$ contains a series of events
- Time between events are i.i.d and $S_i \sim \psi(t)$ waiting-time distribution



- Interactions occur at random times when events occur
  1. node-based process $\psi_i(t)$
  2. edge-based process $\psi_{ij}(t)$

How do agents interact?

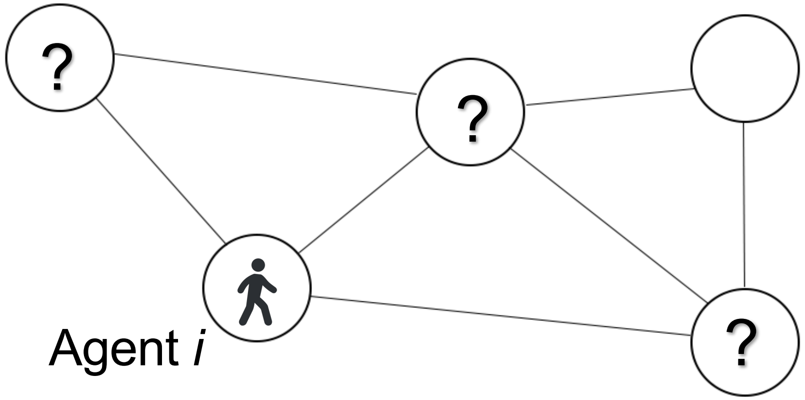
1. Adoption rule
   - when interactions occur, agent i adopts agent j's opinion
   - $(x_i, x_j) \to x_i' = x_j$
2. Compromise rule
   - Averaged opinion: $x_i' = \sum_{j \in N_i} x_j / |N_i|$
   - Bounded-confidence type:

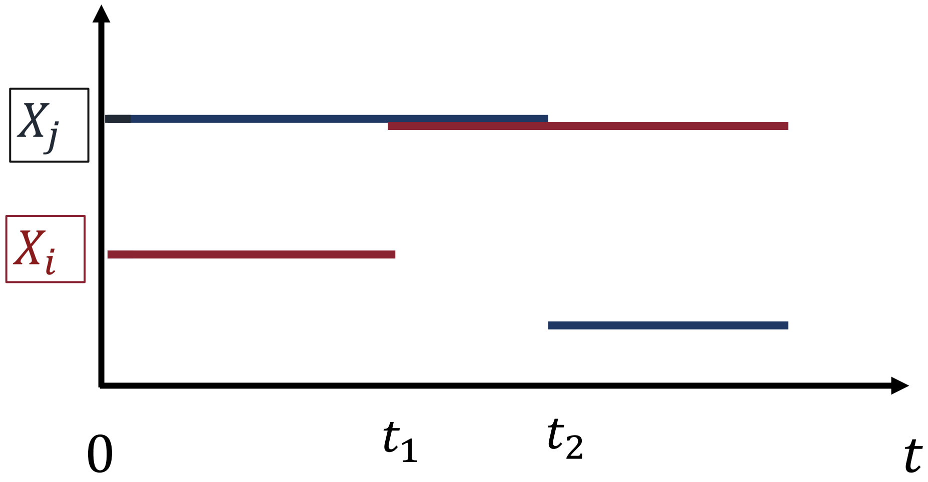
$$x_i' = x_i + 0.5 \times 1_{|x_i - x_j| < c}(x_j - x_i)$$

3. Asynchronous or synchronous interactions
4. Nonhomogeneous consideration (on weighted, directed graphs)

Model I: node-based + opinion adoption + heterogeneity

- A weighted graph $G = (V, E)$: $V = \{1,2,\dots,N\}$, and $E = \{A_{ij}\}$
- Agent i adopts a new opinion
  - after waiting for $\tau \sim \psi_i(t)$ (waiting-time distribution)
  - which comes from agent j with probability $\widetilde{A}_{ij} = A_{ij}/\sum_j A_{ij}$
  - Reset τ to 0 after an update





[1] C. & Porter SIADS 2023

Model I: expectation

(1) The governing equation of expected opinion $\boldsymbol{x_i(t)} = \mathbb{E}[\boldsymbol{X_i(t)}]$ satisfies

$$x_i(t) = \sum_j \widetilde{A}_{ij} \left[\phi_i \star (\theta_i x_j)\right](t) + \phi_i(t) x_i(0)$$

(2) When $\psi_i(t)$ is continuous w.r.t time, $\boldsymbol{x_i(t)}$ satisfies

$$\dot{x}_i(t) = \sum_j \widetilde{A}_{ij} \theta_i(t) x_j(t) - \left[\chi_i \star x_i\right](t)$$

Here, ϕ_i , θ_i , χ_i are related to the waiting-time distribution ψ_i by:

$$\phi_i(t) = 1 - \int_0^t \psi_i(t')dt', \quad \hat{\theta}_i = \left(1 - \hat{\psi}_i\right)^{-1} \hat{\psi}_i, \quad \hat{\chi}_i(s) = s\hat{\theta}_i(s)$$

Model I: examples of Markovian opinion dynamics

- All nodes follow ***Dirac delta*** WTD $\boldsymbol{\psi_i(t) = \delta(t - \Delta_i)}$

$$x_i(t) = x_i(0)\,, \qquad t \in [0, \triangle_i)\,,$$
$$x_i(t) = \sum_j \widetilde{A}_{ij} x_j(t - \triangle_j)\,, \qquad t \in [\triangle_i, \infty)\,.$$

$\Longrightarrow$ DeGroot model

$$x(n+1) = Px(n), \quad P = \widetilde{A}$$

- All nodes follow ***exponential*** WTD $\boldsymbol{\psi_i(t) \propto \exp(-\lambda_i t)}$

$$\dot{x}_i(t) = \lambda_i \sum_j \widetilde{A}_{ij} x_j(t) - \lambda_i x_i(t)$$

$\Longrightarrow$ continuous-time DeGroot

$$\dot{x}(t) = Px(t), \qquad P = \Lambda(\widetilde{A} - I)$$

- Models are ***Markovian***!

Model I: examples of non-Markovian opinion dynamics

- All nodes follow ***Gamma*** WTD $\boldsymbol{\psi_i(t) \propto t\exp(-\lambda_i t)}$

$$\dot{x}(t) = \mathcal{K}(t)\left[\widetilde{A}x(t) - \tilde{x}\{t\}\right]$$

$$\mathcal{K}_i(t) = \int_0^t \kappa_i(t')\,dt'\,, \quad \tilde{x}_i\{t\} = \frac{\int_0^t \kappa_i(t-t')x_i(t')\,dt'}{\mathcal{K}_i(t)}\,.$$

- All nodes follow ***heavy-tailed*** WTDs, such as
    - log-normal $\quad \psi_i(t) = \frac{1}{\sqrt{2\pi}\sigma_i t}\exp\left[-\frac{(\ln t - \mu_i)^2}{2\sigma_i^2}\right]$
    - power-law $\quad \psi_i(t) = \lambda_i(t+1)^{-\lambda_i - 1}$

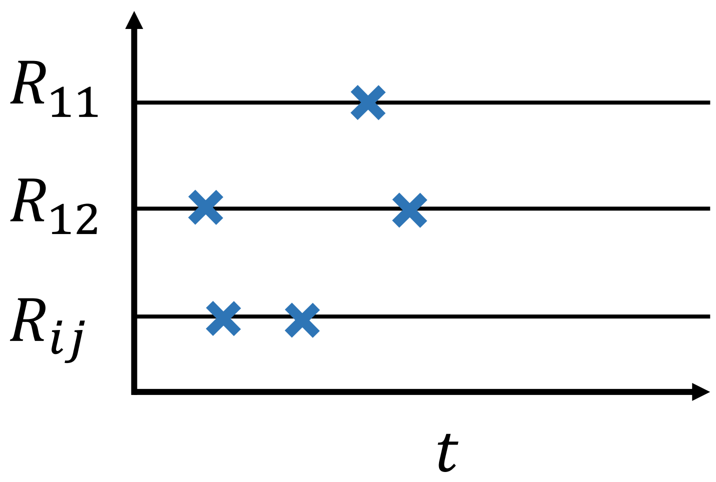
- Unable to compute inverse Laplace transform terms explicitly!

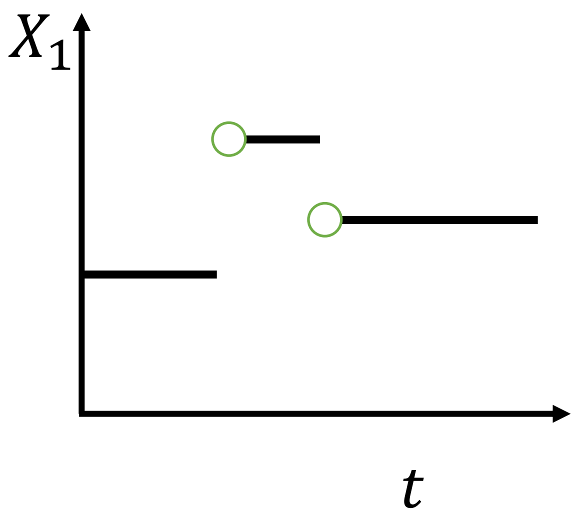
- ***We can approximate $\psi_i(t)$ with a sum of Dirac delta or exponential distributions***

Model II: edge-based + opinion compromise + heterogeneity

- Each edge e_{ij} corresponds to an independent renewal process $R_{ij}(t)$
- If an event of R_{ij} occurs at time t, agent X_i updates its opinion at time t_+ in a ***bounded-confidence*** style

$$X_i(t_+) = X_i(t) + 0.5\mathbb{I}_{|X_i(t)-X_j(t)|<c}(X_j(t) - X_i(t))$$





Model II: approximate expectation of Markovian

- Let Z be the number of events on the interval $[t, t + dt)$

$$Z = \begin{cases} 0 & \text{with probability} \quad e^{-\Lambda dt} \\ 1 & \text{with probability} \quad \Lambda dt e^{-\Lambda dt} \\ \geq 2 & \text{with probability} \quad \sum_{k=2}^{\infty} \frac{(\Lambda dt)^k}{k!} e^{-\Lambda dt} \end{cases}$$

- The governing equation of expected opinion $x_i(t) = \mathbb{E}[X_i(t)]$ satisfies

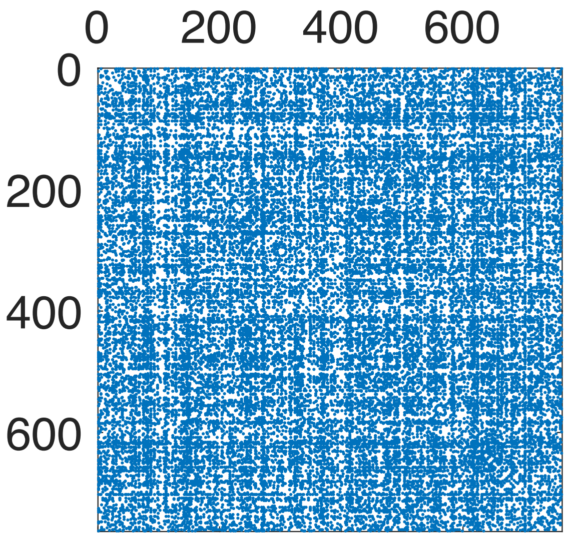
$$x_i(t + dt) = x_i(t) + \frac{1}{|E|} \sum_{j:e_{ij} \in E} \Lambda dt e^{-\Lambda dt} \mathbb{E} \left\{ \frac{1}{2} \mathbf{1}_{|X_i(t) - X_j(t)| < c} \left[X_j(t) - X_i(t) \right] \right\} + O(dt^2)$$

- With $\Lambda = \lambda |E|$ and $dt \to 0$

$$\dot{x}_i(t) \approx \frac{\lambda}{2} \sum_{j:(i,j) \in E} \mathbf{1}_{|x_i(t) - x_j(t)| < c} \left[x_j(t) - x_i(t) \right] .$$

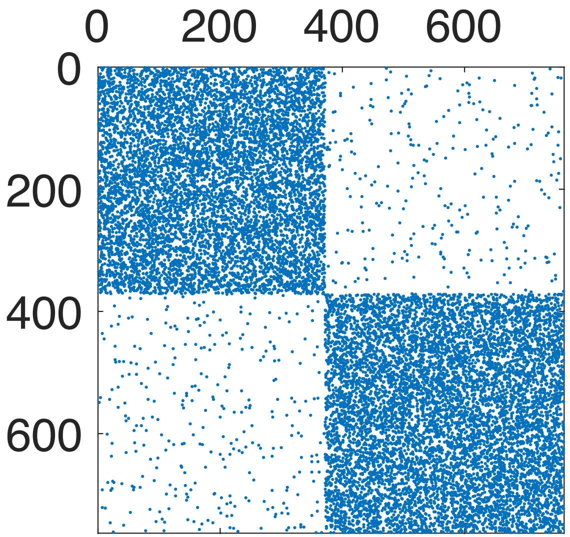
How network structures affect dynamics?

Caltech Facebook Network



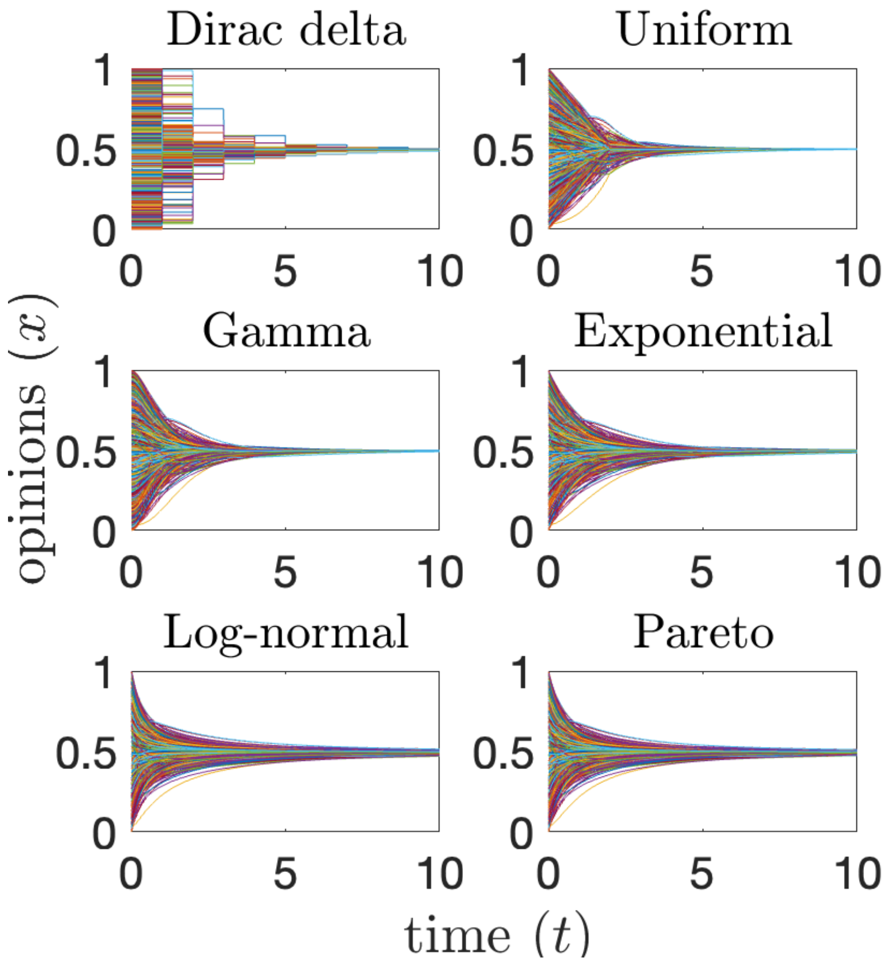
N = 762
Total number of edges = 16,651

Stochastic Block Model

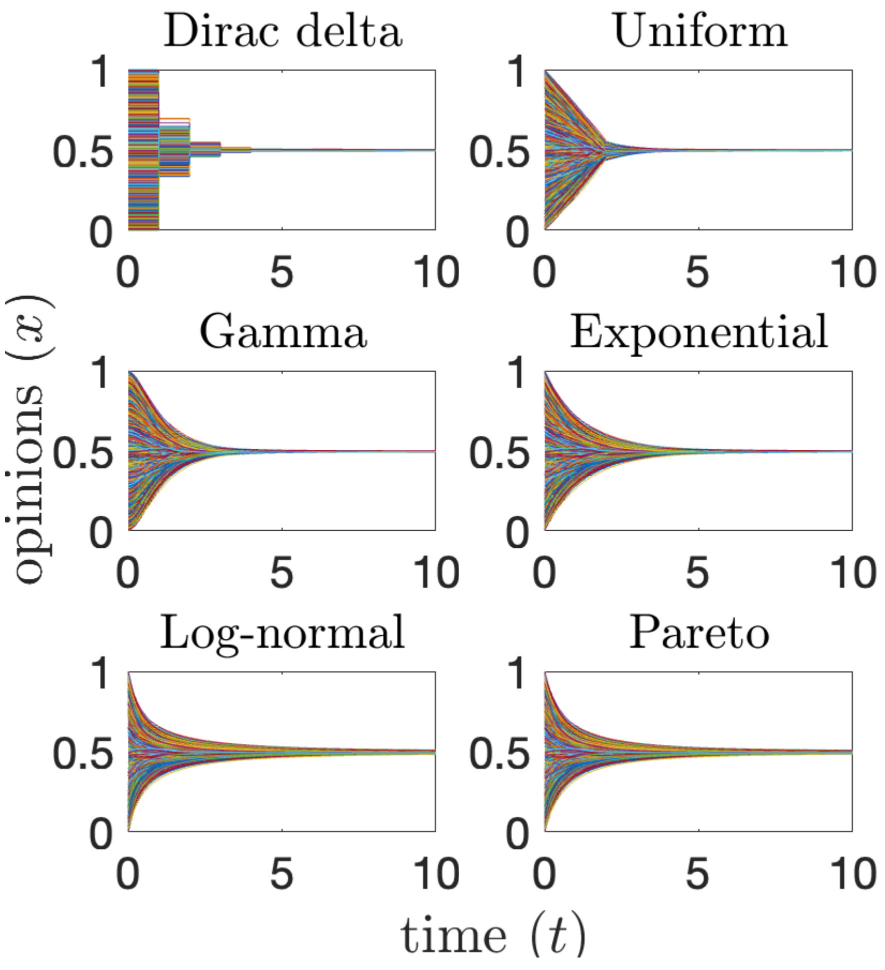


N = 762
P(between communities) = 0.002
P(within communities) = 0.0554

Caltech Facebook Network



Stochastic Block Model



Model I: convergence analysis

- Conditions: all nodes follow the same WTD $\psi(t)$; $\widetilde{A}$ is diagonalizable and -1 is not an eigenvalue of $\widetilde{A}$
- Result: Given x_0, ***for any WTD,*** the opinion converges

$$\lim_{t\to\infty} x(t) = x^* = \sum_{\{d:\ \nu_d=1\}} c_d^0 v_d$$

- Highlight of the proof: $\phi(t)$ is the probability survival function

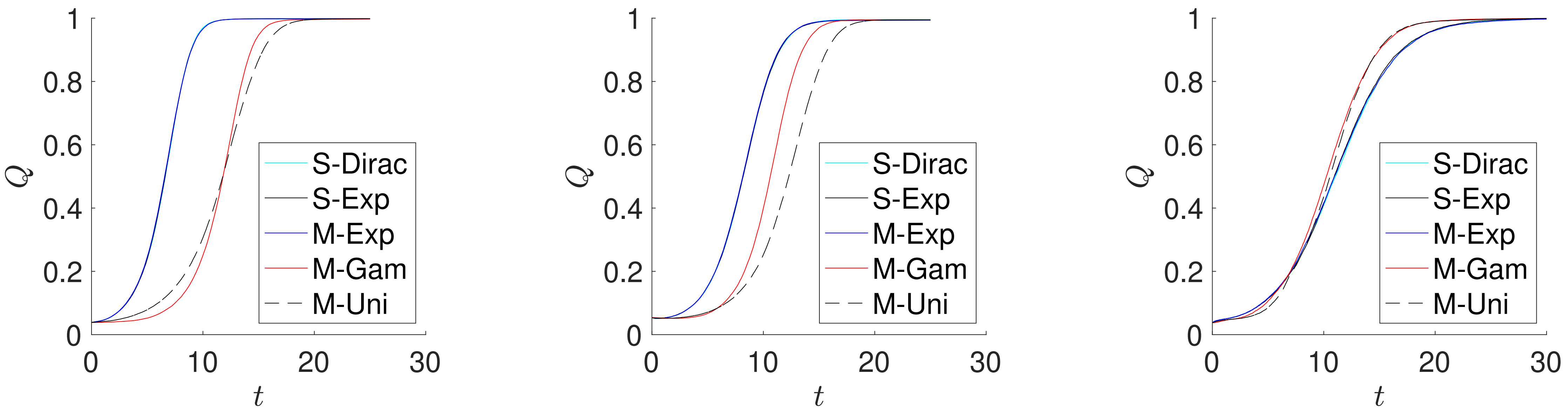
$$|c_d(t)| \leqslant |\nu_d| \|c_d\|_{L^\infty} + \phi(t)|(1-\nu_d)c_d^0|$$

- $\phi(t)$ is linear when the WTD is uniform.

Model II: order parameter

$$Q(t) = \frac{1}{N^2} \sum_{e_{ij} \in E} 1_{|X_i(t) - X_j(t)| < C}$$

(a) Complete graph (b) Stochastic block model (c) Random graph



Model III: node-based + compromise

- Let $\psi(t)$ be the waiting-time distribution
- Let $u_k(t)$ be the probability that *k* events occur during $[0, t)$

$$u_0(t) = 1 - \int_0^t \psi(\tau) d\tau \,, \qquad u_{k+1}(t) = \int_0^t \psi(\tau) u_k(t-\tau) d\tau \,, k \geq 0$$

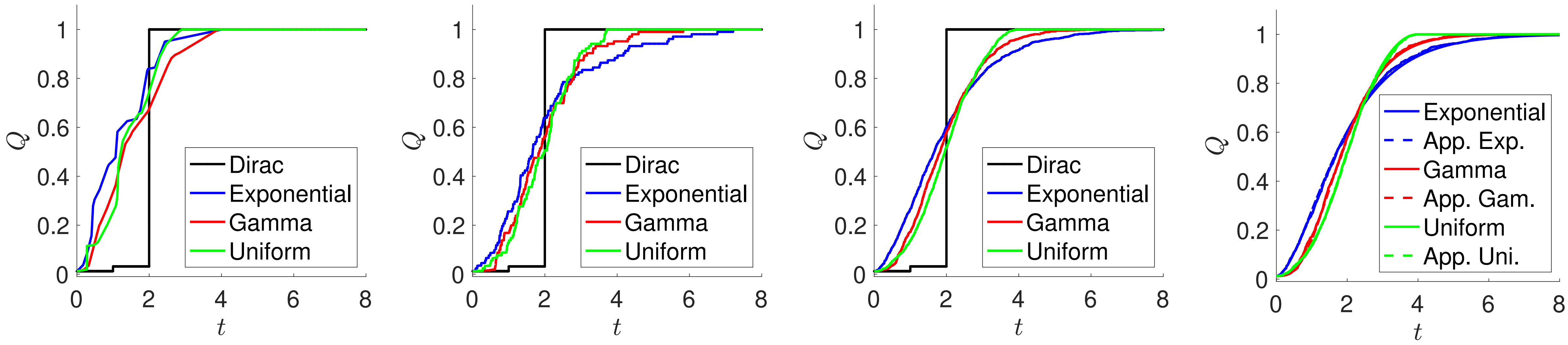
- For any function $f: \mathbb{R}^N \to \mathbb{R}$, we have

$$E\big[f\big(X(t)\big)\big] = \sum_{k=0}^{\infty} E\left[f(X[k])\right] u_k(t)$$

where $X[k]$ is the opinion after k updates.

Model III: node-based + compromise

(a) $N_{copy} = 10$ (b) $N_{copy} = 100$ (c) $N_{copy} = 1000$ (d) $Approx.$



Summary and outlook

- ❖ Summary
  - Waiting-time distribution governs when agents interact
  - Adoption or compromise rule governs how agents interact
  - Network structures decide steady states and WTDs decide transient
- ❖ Extension
  - Mean-field description of models?
  - Variance dynamics or measure evolution?
  - Inference from non-Markovian data?

Thank you!