Quantitative Propagation of Chaos for 2D Viscous Vortex Model on the Whole Space

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Mean Field Limit for Newton Dynamics

Consider the classical Newton dynamics for N indistinguishable point particles in the mean filed scaling in the classical regime. Denote (q_i, p_i) the position and the velocity of particle number i. Then

$$\dot{q}_i = p_i, \quad \dot{p}_i = \frac{1}{N}\sum_{j\neq i} K(q_i - q_j), \quad i = 1, 2, \cdots, N,$$

where $q_i, p_i \in \mathbb{R}^3$.
As $N \to \infty$, the conjectured PDE is the famous Vlasov(-Poisson) equation

$$\partial_t f + p\cdot\nabla_q f + K \star_q \rho_f \cdot \nabla_p f = 0,$$

where $\rho_f(t,q) = \int_{\mathbb{R}^3} f(t,q,p)\,\mathrm{d}p$.
Gravitational or Coulomb force: $K(q) = \pm\frac{q}{|q|^3}$, i.e. the inverse-square law.
The case with repulsive electrostatic interactions with diffusion on $\Pi_q^2 \times \mathbb{R}_p^2$ was recently obtained by Bresch, Jabin & Soler ('22). Refer to P.-E. Jabin's talk.

The viscous vortex model

Consider the weakly interacting particle system for N indistinguishable point particles. Denote $x_i \in \mathbb{R}^2$ the position of particle number i. The dynamics reads

$$\mathrm{d}x_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^i, \quad i = 1, 2, \cdots, N, \quad (IPS)$$

where $x_i \in \mathbb{R}^2$, and W^i are N independent Brownian motions which may model random collisions on particles with rate σ. Let us assume that $\sigma > 0$. The interaction kernels K model 2−body interaction forces between particles. In the point vortex model, one takes $K(x) = \frac{1}{2\pi}\frac{(x_2, -x_1)}{|x|^2}$.
The (proved) limit PDE reads (as $N \to \infty$)

$$\partial_t f + \mathrm{div}_x (f K \star_x f) = \frac{\sigma^2}{2}\Delta_x f. \qquad (MFD)$$

Goal: Establish and quantify the convergence: Any $k-$marginal $F_{N,k}(t)$ converges weakly to $f_t^{\otimes k}$, or the empirical measure $\mu_N^t = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t}$ converges in law to δ_{f_t}.

How large is N?

- Cosmology/astrophysics: N ranges from 10^{10} to $10^{20} - 10^{50}$; some models of dark matter even predict up to 10^{60} particles.
- In plasma physics, N is typically of order $10^{20} - 10^{25}$. This is the typical order of magnitude for physics settings.
- When used for numerical purposes (particles' method), the number is of order $10^{9} - 10^{12}$.
- In biology or life sciences, typical population of micro-organisms is typically of order 10^{6} to 10^{12}.
- In other applications such as collective dynamics, social sciences or economics, N can be much lower of order 10^{3}.

Whenever possible, it is critical to quantify how fast the convergence to the continuous limit holds in terms of N and also in terms of time t.

Examples of Kernels

Classical Results: McKean ('67), Braun& Hepp ('77), Dobrusin ('79), Sznitmann ('91) ... $K \in W^{1,\infty}$ (K is Lipschitz!) (Coupling Method).

The classical methods fail for systems with some singular kernels. But they are still very useful in many applications.

Examples of Singular Kernels:

- Biot-Savart Law with $K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$. More general, conservative flows, Hamiltonian systems.
- The Poisson kernels $K(x) = \pm C_d \frac{x}{|x|^d}$. (Repulsive or attractive.) Gradient flows.

The limit behavior of $\eta^N(t) = \sqrt{N}(\mu_N(t) - f)$ for systems with singular kernels is quite less understood. Fernandez and Méléard ('97): $K \in C_b^{2+d/2}$. W., Zhao and Zhu ('23): $K \in L^\infty$ or $|x|K \in L^\infty$ if K is anti-symmetric.

Recent Results (1st order systems)

- 2D Euler: Goodman, Hou and Lowengrub ('90). Schochet ('96), Hauray ('09). Well-prepared initial data. Point vortex system.
- 2D Navier-Stokes: Osada ('87), Fournier, Hauray and Mischer ('14). (The Biot-Savart kernel $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$. Compactness argument.)
- Patlak-Keller-Segel: Haskovec and Schmeiser ('11), Fournier and Jourdain ('15) (very sub-critical regime, no rate...) Cattiaux-Pédèches ('16). Similar setting: Liu & Yang ('16), Bolley, Chafaï, & Fontbona ('18), Li, Liu & Yu ('19). Fournier and Tardy ('23).
- 1st order systems with singular kernels: Jabin and W. ('18) ($K \in W^{-1,\infty}$ (but also $\mathrm{div} K \in W^{-1,\infty}$)). Bresch, Jabin and W. ('20) (general singular kernels). Coulomb (like) flows or conservative flows, deterministic case: Duerinckx('16), Serfaty ('20), Rosenzweig ('20-'21). Guillin, Le Bris & Monmarché ('21).
- Feng-Yu Wang's group on change-of-measure coupling, see Xing Huang etc's recent works.

The Liouville Equation

Statistical approach: The coupled law of $N-$particle $F_N(t, x_1, \cdots, x_N)$ is governed by the Liouville equation

$$\partial_t F_N + \sum_{i=1}^{N} \mathrm{div}_{x_i}\Big(F_N \frac{1}{N}\sum_{j\neq i} K(x_i - x_j)\Big) = \frac{\sigma^2}{2}\sum_{i=1}^{N} \Delta_{x_i} F_N.$$

The coupled law F_N is symmetric/exchangeable, i.e. $F_N \in \mathcal{P}_{sym}(\mathbb{R}^{2N})$, since those particles are indistinguishable.

The observable (statistical information: temperature, pressure for instance) is contained in the marginals $F_{N,k}$ of F_N as

$$F_{N,k}(t, x_1, \cdots, x_k) = \int_{\mathbb{R}^{2(N-k)}} F_N(t, x_1, \cdots, x_N)\, \mathrm{d}x_{k+1} \cdots \mathrm{d}x_N,$$

for fixed $k = 1, 2, \cdots$.

BBGKY hierarchy: The evolution of $F_{N,k}$ involves $F_{N,k+1}$.

Formal Derivation assuming Molecular Chaos

BBGKY hierarchy: The evolution of $F_{N,k}$ involves $F_{N,k+1}$. For example, integrating the Liouville Eq. w.r.t. $x_2, \cdots, x_N$ and using the symmetry of F_N, one obtains that

$$\partial_t F_{N,1} + \frac{N-1}{N} \int_E \mathrm{div}_x(F_{N,2}\, K(x-y))\,\mathrm{d}y = \frac{\sigma^2}{2} \Delta_x F_{N,1}.$$

If we assume that $F_{N,2}(t,x,y) = F_{N,1}(t,x) F_{N,1}(t,y)$ (Molecular Chaos) for any $t \geq 0$, then we recover the limit PDE (MFD) as $N \to \infty$.
But for fixed N, $F_{N,2}(t) \neq F_{N,1}(t)^{\otimes 2}$, even we start from i.i.d. initial data $F_N(0) = F_{N,1}(0)^{\otimes N}$, since correlation does exist since particles do interact!

Relaxation: **Kac's chaos** ('56). To derive the space homogeneous Boltzmann equation.

Propagation of Chaos

Definition 1 (Kac's chaos)

Let $E = \mathbb{R}^d$ or any Polish space. A sequence $(F_N)_{N\geq 2}$ of symmetric probability measures, *i.e.* $F_N \in \mathcal{P}_{Sym}(E^N)$, is said to be $f-$chaotic for a probability measure f on E, if for any fixed $k = 1, 2, 3, \cdots$, $F_{N,k} \rightharpoonup f^{\otimes k}$, as $N \to \infty$.

"Asymptotic independence" for a finite group.

Definition 2 (Propagation of (Kac's) chaos)

The diagram commutes.

$$\begin{array}{ccc} F_{N,k}(0) & \rightharpoonup & f^{\otimes k}(0) \\ \Downarrow_{\text{IPS}} & & \Downarrow_{\text{MFD}} \\ F_{N,k}(t) & \rightharpoonup & f^{\otimes k}(t) \end{array}$$

See recent work by D. Lacker ('21), Jabin, Poyato & Soler ('21) and Bresch, Jabin & Soler ('22) based on BBGKY.

From Relative Entropy to Propagation of Chaos

We use the (scaled) relative entropy to quantify chaos

$$0 \leq \mathcal{H}_N(F_N|f^{\otimes N})(t) = \frac{1}{N}\int_{E^N} F_N \log \frac{F_N}{f^{\otimes N}}\,\mathrm{d}x_1\cdots\mathrm{d}x_N.$$

Thanks to the monotonicity of the (scaled) relative entropy

$$\mathcal{H}_k(F_{N,k}|f^{\otimes k}) := \frac{1}{k}\int_{E^N} F_{N,k}\log\frac{F_{N,k}}{f^{\otimes k}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \leq \mathcal{H}_N(F_N|f^{\otimes N})$$

and the classical Csiszár-Kullback-Pinsker inequality

$$\|F_{N,k} - f^{\otimes k}\|_{L^1} \leq \sqrt{2k\mathcal{H}_k(F_{N,k}|f^{\otimes k})},$$

one can obtain *propagation of chaos* given a vanishing sequence of $H_N(F_N|f^{\otimes N})$. See also Ben Arous & Zeitouni ('99).

2D Vortex Model on Π^2

Theorem (Jabin & W. ('18))

Assume that $K \in \dot{W}^{-1,\infty}(\Pi^d)$ *with* $\operatorname{div} K \in \dot{W}^{-1,\infty}$*. Assume further that* $f \in L^\infty([0,\ T],\ W^{2,p}(\Pi^d))$ *for any* $p < \infty$ *solves (MFD) with* $\inf f > 0$ *and* $\int_{\Pi^d} f = 1$*. Then*

$$\mathcal{H}_N(F_N^t \,|\, f_t^{\otimes N}) \leq e^{\bar{M}(\|K\|+\|K\|^2)\,t} \left(\mathcal{H}_N(F_N^0 \,|\, f_0^{\otimes N}) + \frac{1}{N} \right),$$

where we denote $\|K\| = \|K\|_{\dot{W}^{-1,\infty}} + \|\operatorname{div} K\|_{\dot{W}^{-1,\infty}}$ *and* $\bar{M}$ *is a universal constant.*

This result applies to the Biot-Savart law, i.e. $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$, since $K = \operatorname{div} V$ with

$$V = \frac{1}{2\pi}\begin{bmatrix} -\arctan\frac{x_1}{x_2} & 0 \\ 0 & \arctan\frac{x_2}{x_1} \end{bmatrix}.$$

Uniform-in-time propagation of chaos by **Guillin, Le Bris & Monmarché** ('21).

2D Vortex Model on $\mathbb{R}^2$

Theorem (Feng & W. ('23))

Assume that F_N is an entropy solution to the Liouville equation and that $f \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^2))$ solves (MFD) with $f \geq 0$ and $\int_{\mathbb{R}^2} f(t,x)\,\mathrm{d}x = 1$. Assume that the initial data $f_0 \in W^{2,\infty}(\mathbb{R}^2)$ satisfies the growth condition

$$|\nabla \log f_0(x)|^2 \leq C_1(1 + |x|^2), \tag{1}$$

$$|\nabla^2 \log f_0(x)| \leq C_2(1 + |x|^2), \tag{2}$$

and the Gaussian-type bound from above that there exists some $C_3 > 0$ such that

$$f_0(x) \leq C_3 \exp(-C_3^{-1}|x|^2). \tag{3}$$

Then we have

$$H_N(F_N^t | f_t^{\otimes N}) \leq M e^{Mt}\Big(H_N(F_N^0 | f_0^{\otimes N}) + \frac{1}{N}\Big),$$

where M is some universal constant that only depends on those initial bounds.

Ideas of the proof

Let us focus on the torus case first. We compute the time evolution of the relative entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(F_N|f^{\otimes N})(t) \le -\frac{\sigma^2}{2N}\int_{\Pi^{dN}} |\nabla \log \frac{F_N}{f^{\otimes N}}|^2 \,\mathrm{d}F_N + \int_{\Pi^{dN}} \Big(\frac{1}{N^2}\sum_{i,j=1}^{N} \phi(x_i, x_j)\Big)\,\mathrm{d}F_N,$$

where

$$\phi(x,y) = \nabla \log f(x)\cdot (K\star f(x) - K(x-y)) + (\,\mathrm{div}K\star f(x) - \,\mathrm{div}K(x-y)).$$

Using symmetrization, i.e. taking $\frac{1}{2}(\phi(x,y)+\phi(y,x))$ as the new $\phi(x,y)$, one writes

$$\phi(x,y) = -\frac{1}{2}K(x-y)\cdot(\nabla \log f(x) - \nabla \log f(y)) - \,\mathrm{div}K(x-y)$$
$$+ \text{Bounded Terms}.$$

Consider the 2D Navier-Stokes and the 2D Euler case. Then the kernel K is the Biot-Savart kernel, which is divergence free, i.e. $\mathrm{div}_x K = 0$. Dropping the Fisher information term (which could be useful),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(F_N|f^{\otimes N}) \leq \int_{\Pi^{dN}} \Big(\frac{1}{N^2}\sum_{i,j=1}^{N}\phi(x_i,x_j)\Big)\,\mathrm{d}F_N \quad (\sim O(1) \text{ a prior!})$$

where after symmetrization, $\phi \in L^\infty$ and more importantly

$$\int_E \phi(x,y)f(y)\,\mathrm{d}y = 0, \forall x, \quad \int_E \phi(x,y)f(x)\,\mathrm{d}x = 0, \forall y.$$

Recall a Jensen-type inequality, i.e. for any parameter $\eta > 0$,

$$\int F_N\Phi_N \leq \frac{1}{\eta}\frac{1}{N}\int F_N \log\frac{F_N}{f^{\otimes N}} + \frac{1}{\eta}\frac{1}{N}\log\int f^{\otimes N}\exp\left(\eta N\Phi_N\right).$$

GOAL: Show the 2nd term is $o(1)$ as $N \to \infty$.

Theorem (Uniform in N large deviation type estimate)

We have

$$\sup_{N\geq 2}\int_{\Pi^{dN}} f^{\otimes N}\exp\left(\frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j)\right)\mathrm{d}X^N$$
$$=\sup_{N\geq 2}\int_{\Pi^{dN}} f^{\otimes N}\exp\left(N\int_{\Pi^{2d}}\phi(x,y)(\,\mathrm{d}\mu_N-\,\mathrm{d}f)^{\otimes 2}(x,y)\right)\mathrm{d}X^N\leq C<\infty,$$

provided that $\|\phi\|_{L^\infty}\leq c_0$ *and*

$$\int_E \phi(x,y)f(y)\,\mathrm{d}y=0,\forall x,\quad \int_E \phi(x,y)f(x)\,\mathrm{d}x=0,\forall y.$$

Ben Arous and Brunaud ('90): with ϕ continuous.
We need the estimate directly for discontinuous ϕ.
Carefully use two cancellation rules. Law of Large Numbers but for "Double Indices". A recent proof using martingales by Lim, Lu and Nolen ('19).

Now we turn to the whole space case. Using symmetrization, i.e. taking $\frac{1}{2}(\phi(x,y)+\phi(y,x))$ as the new $\phi(x,y)$, one writes

$$\phi(x,y) = -\frac{1}{2}K(x-y)\cdot(\nabla\log f(x) - \nabla\log f(y)) - \operatorname{div}K(x-y) + \frac{1}{2}\nabla\log f(x)\cdot K*f(x) + \frac{1}{2}\nabla\log f(y)\cdot K*f(y).$$

One can propagate the initial assumptions on f_0 to get similar bounds on f_t, for $t\in[0,T]$. For simplicity, $\nabla\log f_t(x)$ is linearly growing in x, and $\nabla^2\log f_t(x)$ is quadraticlly growing in x. Hence

- $\nabla\log f_t(\cdot)\cdot K\star f(\cdot)\in L^\infty$;
- $\sup_y|K(x-y)\cdot(\nabla\log f(x)-\nabla\log f(y))|\leq C(1+|x|^2)$;

Using the upper Gaussian tail estimate for f_t, one can still recover the Large Deviation type bounds as in the torus case.

Relative Entropy and Modulated (Potential) Energy

Now we focus on gradient flows, i.e. $K = -\nabla V$.

- Relative entropy (Jabin and W., ('18)) : Less structure and less singularity.

  Recall that there is a term

$$-\frac{1}{N^2}\sum_{i\neq j}\int_{\Pi^{dN}} \mathrm{div} K(x_i - x_j)\,\mathrm{d}F_N$$

  in the time evolution of the relative entropy.

- Modulated Energy (Duerinckx ('16) and Serfaty (with an appendix with Duerinckx) ('20)): More structure and also more singular (Riesz potentials+ possible perturbation!). Deterministic flows (now also compatible with some multiplicative/additive noises as in Rosenzweig and Serfaty ('21))

  Recall the modulated (potential) energy is defined as

$$F(X^N, f) = \int_{x\neq y} V(x-y)(\,\mathrm{d}\mu_N(x) - f(x))(\,\mathrm{d}\mu_N(y) - f(y)).$$

Modulated Free Energy

Idea: introducing weights G_N and $G_{f^{\otimes N}}$ in the relative entropy to cancel the term $\mathrm{div} K$ in its time evolution

$$E_N(F_N|f^{\otimes N}) = \frac{1}{N}\int_{\Pi^{dN}} F_N \log\left(\frac{F_N/G_N}{f^{\otimes N}/G_{f^{\otimes N}}}\right) \mathrm{d}x_1\cdots \mathrm{d}x_N,$$

where G_N is the Gibbs measure, $G_{f^{\otimes N}}$ is a tilted Gibbs measure by the limit f.
In an equivalent way

$$E_N(F_N|f^{\otimes N}) = \mathcal{H}_N(F_N|f^{\otimes N}) + \mathcal{K}_N(F_N|f^{\otimes N}),$$

with

$$\mathcal{K}_N(F_N|f^{\otimes N}) = \frac{1}{\sigma^2}\mathbb{E}_{F_N}\int_{x\neq y} V(x-y)(\mathrm{d}\mu_N(x) - \mathrm{d}f(x))(\mathrm{d}\mu_N(y) - \mathrm{d}f(y)).$$

Note that $\sigma^2 E_N = \sigma^2\mathcal{H}_N + \mathbb{E}_{F_N}(F(X^N, f))$, where σ^2 is the temperature.

Further Discussion

1 Propagation of Chaos: Uniform-in-time estimates in particular for kinetic Vlasov equations; Derivation of Vlasov-Poisson/Landau. The underlying space for the velocity variable is on the whole space.

2 Central limit theory/Gaussian fluctuation for interacting particle systems with singular interactions. Can we quantify the convergence rate as well?

3 (Path/dynamical) Large deviation principles for kinetic theories (Boltzmann, Landau, Vlasov-Poisson, 2D Euler equation...).

4 Non-exchangeable particle system.

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Thank you!