A new duality method for the mean-field limit

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Summarzing our main goal

- Prove the mean-field limit for classical Hamiltonian systems with or without diffusion.
- The approach should be compatible with a broad array of possible structure and singularities.
- We only require the interaction kernel to be in L^2, which corresponds to the scaling of the law of large numbers.
- We introduce a duality method that captures the structure of correlations in the dynamics.

A complete toolbox for astronomical calculations

In 1687, Newton published in his famous Principia his even more famous laws of motion. Newton's laws allow to calculate the trajectories of any number of celestial objects. In Newton's days, that mostly meant the solar system but such astronomical calculations are nowadays also performed at much larger scales.

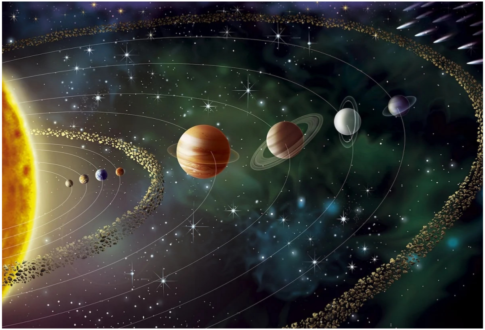

Figure: Credits: NASA.

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Figure: Hubble's view of the galaxy NGC 5291. Credits: NASA.

The first many-particle system

Consider N celestial objects and attribute to each of them a number i, $i = 1, \ldots N$. Denote

$$
\begin{aligned}
m_i &= \text{Total mass of object } \# i, \\
X_i(t) &= \text{position of the center of mass at time } t, \\
V_i(t) &= \text{velocity of the center of mass at time } t.
\end{aligned}
$$

Then we have the following system of coupled ODE's

$$
\frac{d}{dt} X_i(t) = V_i(t), \quad m_i \frac{d}{dt} V_i(t) = \sum_{j \neq i} m_i \, m_j \, K(X_i - X_j), \qquad (1)
$$

with the square inverse gravitation law

$$
K(x) = -\frac{x}{|x|^3} \quad \text{in dimension 3}, \qquad K(x) = -\frac{x}{|x|^d} \quad \text{in dimension } d.
$$

Plasmas: inside the future Tokamak at ITER

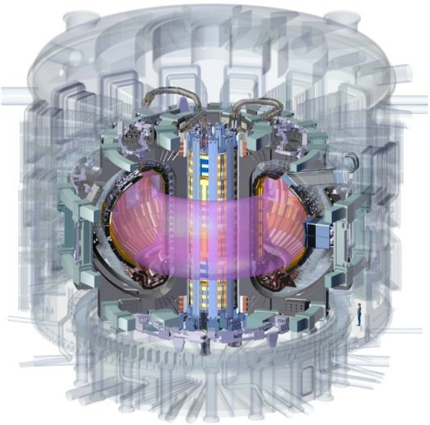

Figure: Credits: ITER, France.

The dynamics of point charges in a plasma

The dynamics of charged objects, such as ions or electrons in a plasma, obey very similar laws when their velocities is small enough w.r.t. the speed of light. Denote by

m_i = Total mass of object $\#i$, q_i = Total charge of object $\#i$,
$X_i(t)$ = position of the center of mass at time t,
$V_i(t)$ = velocity of the center of mass at time t.

Then we have the following system of coupled ODE's

$$\frac{d}{dt}X_i(t) = V_i(t), \quad m_i \frac{d}{dt}V_i(t) = \sum_{j\neq i} q_i\, q_j\, K(X_i - X_j), \tag{2}$$

with the electrostatic force derived by Coulomb in 1785

$$K(x) = \frac{x}{|x|^3} \quad \text{in dimension } 3, \qquad K(x) = \frac{x}{|x|^d} \quad \text{in dimension } d.$$

Particles or agent are everywhere

Many-particle or multi-agent systems are now used in a widespread range of applications, with usually a very large number of particles

- Plasmas: Particles are ions or electrons. $\longrightarrow N \sim 10^{20} - 10^{25}$.
- Astrophysics: Particles are dark matter particles, galaxies or galaxy clusters... $\longrightarrow N \sim 10^{10} - 10^{25}$ (10^{60} for some models of dark matter).
- Fluids: Point vortices, suspensions...
- Bio-mechanics: Medical aerosols, suspensions in the blood...
- Bio-Sciences: Collective behaviors of animals, swarming or flocking, but also dynamics of micro-organisms, chemotaxis, cell migration, neural networks... $\longrightarrow$ for typical population of micro-organisms $N \sim 10^{6} - 10^{12}$.
- Social Sciences and Economics: Opinion dynamics, consensus formation, mean-field games... $\longrightarrow N \sim 10^{3} - 10^{4}$.

Many-particle at large scales: AbacusSummit

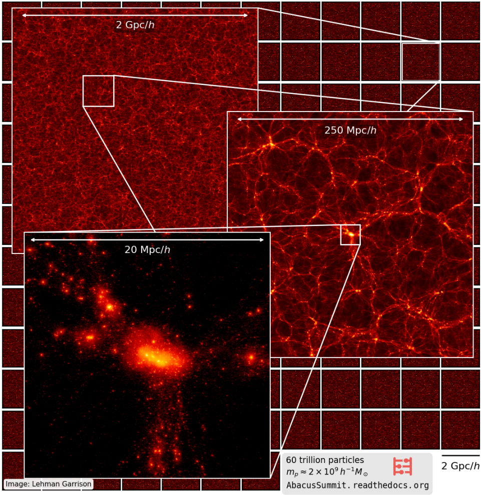

Figure: Credits: The AbacusSummit Team.

AbacusSummit in a few words

- Up 6.10^{13} particles!
- See more at https://abacussummit.readthedocs.io/en/latest/.
- Direct simulations requires re $O(N^2)$ calculations per time step to estimate for each i

$$\sum_{j\neq i} G\, m_i \, \frac{X_j - X_i}{|X_j - X_i|^d}.$$

- AbacusSummit is based on the method introduced in Metchnik (2009) (see also Maksimova et al. (2021), Garrison et al. (2021), Garrison et al. (2019), and Garrison et al. (2018)). The approach is based on a so-called Fast Multipole Method, introduced originally by Greengard and Rokhlin Jr., to have only $O(N)$ calculations.

A simpler system at larger scales

- It is striking that the behavior of the system appears to be simpler when observed at large scales and for a large number of particles.
- This corresponds to a (now!) natural idea that continuum mechanics or other coherent limits may emerge from many-particle systems.
- This notion was first formalized by Maxwell in 1866 and Boltzmann in 1872 for the kinetic theory of gases. Its rigorous derivation is at the heart of the famous 6th problem of Hilbert.
- The specific mean-field limit was first introduced by Vlasov in the 1930's.

Mean-field limits for kinetic equations dimension $d \geq 2$

The rigorous derivation of mean-field limit for Vlasov-Poisson is still fully open in multi-dimension, in spite of many efforts:

- The case of Lipschitz interactions $K(x)$ was handled by McKean in for the stochastic setting and by Braun and Hepp, and Dobrushin in the deterministic case.
- Mild singularities $K(x) << |x|^{-1}$ in Hauray-Jabin 09 and 15.
- Truncated kernels (essential for numerics) in Boers-Pickl 16, Lazarovici-Pickl 17, Pickl 19.
- Swarming models: Carrillo-Choi-Hauray-Salem 18 for cones of vision, Mucha-Peszek 18 for mild singular communication weights in Cucker-Smale.
- So-called Monokinetic limits with the full singularity were obtained in Duerinckx-Serfaty 20.
- Repulsive $2d$ Vlasov-Poisson-Fokker-Planck in Bresch-Jabin-Soler 22, with a partial result in $3d$.

Our general model

We consider the system of N identical particles or agents

$$\frac{d}{dt}X_i(t) = \frac{1}{N-1} \sum_{j=1\ldots N,\, j\neq i} K(X_i - X_j),$$

in the mean-field scaling, for a given two-body interaction given by the kernel K. For simplicity, we assume that $\operatorname{div} K = 0$ but that could be relaxed.

We typically consider X_i in a domain Ω that is either in the whole space, $\Omega = \mathbb{R}^d$, or the torus $\Omega = \Pi^d$. But the method would apply to any combination, periodic in some directions, or to a bounded domain with "easy" boundary conditions.

Our general model

We consider the system of N identical particles or agents

$$dX_i(t) = S(X_i)\, dt + \frac{1}{N-1} \sum_{j=1\ldots N,\, j\neq i} K(X_i - X_j)\, dt + \varepsilon_N\, dW_i,$$

in the mean-field scaling, for a given two-body interaction given by the kernel K. For simplicity, we assume that $\operatorname{div} K = 0$ but that could be relaxed.

The talk focuses on the case without diffusion. But the analysis is identical if diffusion is included, or for vanishing diffusion. The W_i are then assumed to be N independent Wiener processes, with possibly $\varepsilon_N \to 0$ as $N \to \infty$. Smooth self-interactions could also easily be included.

An example: 2nd order Newton's dynamics

We consider very general form of interactions which contain, as a particular example, the classical second order dynamics. Take $X_i = (Q_i, V_i)$ with $d = 2n$ and $Q_i,\ V_i \in \mathbb{R}^n$ solving

$$\frac{d}{dt} Q_i(t) = V_i(t),$$
$$\frac{d}{dt} V_i(t) = \frac{1}{N-1} \sum_{j=1}^{N} K(Q_i - Q_j).$$

Classical examples of kernels K are Coulombian interactions $K(x) = K(q) = \alpha \frac{q}{|q|^d}$ corresponding to electrostatic ($\alpha > 0$) or gravitational ($\alpha < 0$) interactions.

But there exists a large variety of interesting kernels, such as Stokeslets $K(x) = -\left(\frac{Id}{|q|} + \frac{q \otimes q}{|q|^3}\right) \cdot v$.

The mean-field limit

In many applications, the number of particles N is very large: $N \sim 10^{20} - 10^{25}$ is typical in physics for example (around the Avogadero number for plasmas).

For this reason, we would prefer to replace the exact many-particle system by a continuous PDE on the 1-particle distribution $f(t,x)$.

The conjectured mean-field limit is the Vlasov (or McKean-Vlasov) equation,

$$\partial_t f + K \star_x f(x) \cdot \nabla_x f = 0,$$

or in the 2nd order kinetic case, as formally derived originally by Vlasov,

$$\partial_t f(t,q,v) + v \cdot \nabla_q f + K \star_q \rho(q) \cdot \nabla_v f = 0, \;\; \rho = \int_{\mathbb{R}^n} f \, dv.$$

A statistical description

There are many ways to formulate the limit of many-particle systems. We follow here a statistical description by introducing the full joint law,

$F_N(t, x_1, \ldots, x_N) =$ joint law of the system $(X_1, \ldots, X_N)$ at time t,

together with its various marginals

$F_{N,k}(t, x_1, \ldots, x_k) =$ law of the partial system $(X_1, \ldots, X_k)$ at time t.

Of course, we have the simple relation

$$F_{N,k}(t, x_1, \ldots, x_k) = \int_{\Omega^{N-k}} F_N(t, x_1, \ldots, x_N)\, dx_{k+1} \ldots dx_N.$$

Our new result

Theorem

- *Assume that* $K \in L^2_{loc}(\Omega;\mathbb{R}^d)$, *and for convenience* $K \in L^\infty_{loc}(|x| > 1)$.
- *Assume that* $f \in L^\infty(\mathbb{R}^+;\mathcal{P}(\Omega)\cap L^\infty(\Omega))$ *is a weak solution to the Vlasov equation with initial data* f°, *and with bounded Fisher information*

$$\int_0^T \left(\int_\Omega |\nabla \log f|^2 f\right)^{\frac{1}{2}} < \infty.$$

- *Assume that* $F_N \in L^\infty_{loc}(\mathbb{R}^+;L^1(\Omega^N))$ *is the joint law of a solution to the many-particle system with initial data* $(f^\circ)^{\otimes N}$.

Then propagation of chaos holds: for all $k \geq 0$, *the* k*th marginal* $F_{N,k}$ *converges to* $f^{\otimes k}$ *as* $N \uparrow \infty$ *in the sense of distributions on* $[0,T]\times\Omega^k$.

A duality approach

The joint law solves the Liouville or forward Kolmogorov equation,

$$\partial_t F_N + \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{x_i} F_N = \frac{\varepsilon_N^2}{2} \sum_{i=1}^{N} \Delta_{x_i} F_N,$$

$$F_N(t=0) = (f^\circ)^{\otimes N}.$$

Our approach is based on the analysis of the dual backward Kolmogorov equation,

$$\partial_t \Phi_N + \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{x_i} \Phi_N = -\frac{\varepsilon_N^2}{2} \sum_{i=1}^{N} \Delta_{x_i} \Phi_N,$$

$$\Phi_N(t=T) = \bar{\Phi}_N.$$

We have the relation

$$\int F_N(t=T)\, \bar{\Phi}_N \, dx_1 \dots dx_N = \int (f^\circ)^{\otimes N} \, \Phi_N(t=0)\, dx_1 \dots dx_N.$$

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But how are we supposed to investigate the limit of the backward Kolmogorov equation? It is posed in dimension dN with $N \to \infty$...

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Our goal is to show that as $N \to \infty$, for the limiting solution f

$$\int (f(t = T))^{\otimes N} \, \bar{\Phi}_N \, dx_1 \dots dx_N - \int (f^\circ)^{\otimes N} \, \Phi_N(t = 0) \, dx_1 \dots dx_N \longrightarrow 0.$$

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For example with $\bar{\Phi}_N = \frac{1}{N}\sum_i \bar{\varphi}(x_i)$, this yields the weak convergence of the 1-marginal, with similar formula for other marginals

$$\int F_N(t=T)\,\bar{\Phi}_N\,dx_1\dots dx_N - \int (f(t=T))^{\otimes N}\,\bar{\Phi}_N\,dx_1\dots dx_N$$

$$= \int F_{N,1}(t=T,x)\,\bar{\varphi}(x)\,dx - \int f(t=T,x)\,\bar{\varphi}(x)\,dx \longrightarrow 0.$$

A duality approach

We have the relation

$$\int F_N(t = T)\,\bar{\Phi}_N\,dx_1 \ldots dx_N = \int (f^\circ)^{\otimes N}\,\Phi_N(t = 0)\,dx_1 \ldots dx_N.$$

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$$\int (f(t = T))^{\otimes N}\,\bar{\Phi}_N\,dx_1 \ldots dx_N - \int (f^\circ)^{\otimes N}\,\Phi_N(t = 0)\,dx_1 \ldots dx_N \longrightarrow 0.$$

We want to prove that

$$\frac{d}{dt}\int (f(t))^{\otimes N}\,\Phi_N\,dx_1 \ldots dx_N \longrightarrow 0.$$

But we do not know how to do this directly...

A new notion of dual cumulant

The key idea is to introduce a clustering expansion around Φ_N. For example if we take $\bar{\Phi}_N = \frac{1}{N}\sum_{i=1}^N \bar{\varphi}(x_i)$, then we would like to have this particular form preserved asymptotically with for $t < T$,

$$\Phi_N(t) = \frac{1}{N}\sum_{i=1}^N \varphi(t, x_i) + \text{vanishing terms},$$

at least when integrated around $(f(t))^{\otimes N}$.
This leads us to define the dual cumulants $C_{N,n}$ for all n s.t.

$$\Phi_N(t) = C_{N,0} + \sum_{i=1}^N C_{N,1}(x_i) + \sum_{i \neq j} C_{N,2}(x_i, x_j) + \dots$$

A new notion of dual cumulant

To be precise and define the $C_{N,n}$ uniquely, we pose

$$\Phi_N(t) = \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} C_{N,n}(x_\sigma),$$

where P_n^N denotes the set of all subsets of $[N] := \{1, \dots, N\}$ with n elements, and where for an index subset $\sigma = \{i_1, \dots, i_k\}$ we write $x_\sigma := (x_{i_1}, \dots, x_{i_k})$.
Moreover, we impose that $C_{N,n}$ is a symmetric function in its n variables and satisfies

$$\int_\Omega C_{N,n}(x_1, \dots, x_n)\, f(x_j)\, dx_j = 0, \qquad \text{for all } 1 \le j \le n.$$

This is exactly the dual of the definition of classical cumulants that are useful around equilibria.

Alternative definitions of dual cumulants

Define the dual marginals of Φ_N through

$$M_{N,n}(t, x_1, \ldots, x_n) = \int_{\Omega^{N-n}} \Phi_N \, f^{\otimes(N-n)} \, dx_{n+1} \, dx_N.$$

The $M_{N,n}$ allows to invert the definition of the dual cumulants to

$$C_{N,n}(x_1, \ldots, x_n) = \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma \in P_k^n} M_{N,k}(x_\sigma).$$

The dual cumulant $C_{N,n}$ is also the orthogonal projection of $M_{N,n}$ over the set of symmetric ψ s.t. $\int_\Omega \psi(x_1, \ldots, x_n) \, f(x_j) \, dx_j = 0$ for the inner product of $L^2_{f^{\otimes n}}$

$$< \psi, \phi >= \int_{\Omega^n} \psi \, \phi \, f^{\otimes n} \, dx_1 \ldots dx_n.$$

Both definitions allow to easily derive equations on the $C_{N,n}$.

Scaling and bounds on the dual cumulants

Since $\bar{\Phi}_N$ is uniformly bounded in L^∞, so is $\Phi_N(t)$ for all t. This yields the following bound, which is exactly the scaling of the law of large numbers.

Lemma

The dual cumulants as defined satisfy that for all $t \leq T$,

$$\left(\int_{\Omega^n} |C_{N,n}|^2 f^{\otimes n} \right)^{\frac{1}{2}} \leq \binom{N}{n}^{-\frac{1}{2}} \|\bar{\phi}_N\|_{L^\infty} \sim N^{-n/2}.$$

Proof.

From the formula on the $C_{N,n}$, we simply have that

$$\int_{\Omega^N} |\Phi_N|^2 f^{\otimes N} = \sum_{n=0}^{N} \binom{N}{n} \int_{\Omega^n} |C_{N,n}|^2 f^{\otimes n}.$$

The system for the dual marginals

We define the effective kernel through

$$V_f(x,y) = (K(x-y) - K\star f(x_i))\cdot\nabla_{x_i}\log f(x_i).$$

We then have that

$$\begin{aligned}
\partial_t M_{N,n} \;=\; & -\frac{1}{N-1}\sum_{i\neq j}^{n} K(x_i-x_j)\cdot\nabla_{x_i} M_{N,n} \\
& + \frac{N-n}{N-1}\sum_{j=1}^{n}\int_\Omega V_f(x_*,x_j)\, M_{N,n+1}(x_{[n]},x_*)\, f(x_*)\, dx_* \\
& - \frac{N-n}{N-1}\sum_{i=1}^{n}\int_\Omega K(x_i-x_*)\cdot\nabla_{x_i} M_{N,n+1}(x_{[n]},x_*)\, f(x_*)\, dz_* \\
& + \frac{(N-n)(N-n-1)}{N-1}\int_{\Omega^2} V_f(x_*,x_*')\, M_{N,n+2}(x_{[n]},x_*,x_*')\, f(x_*)f(x_*')\, dx_* dx_*'.
\end{aligned}$$

The system for the dual cumulants

$$
\begin{aligned}
&\partial_t C_{N,n} - \frac{N-n}{N-1}\sum_{j=1}^{n}\int_{\Omega} V_f(x_*,x_j)\, C_{N,n}(x_{[n]\setminus\{j\}},x_*)\, f(x_*)\, dx_* \\
&+ \frac{N-n}{N-1}\sum_{i=1}^{n}(K*f)(x_i)\cdot\nabla_{x_i} C_{N,n} + \frac{1}{N-1}\sum_{i\neq j} K(x_i-x_j)\cdot\nabla_{x_i} C_{N,n} \\
&- \frac{(N-n)(N-n-1)}{N-1}\int_{\Omega^2} V_f(x_*,x_*')\, C_{N,n+2}(x_{[n]},x_*,x_*')\, f(x_*) f(x_*')\, dx_*\, dx_*' \\
&+ \frac{N-n}{N-1}\sum_{i=1}^{n}\nabla_{x_i}\cdot\int_{\Omega}\big(K(x_i-x_*) - K*f(x_i)\big)\, C_{n+1}(x_{[n]},x_*)\, f(x_*)\, dx_* \\
&- \frac{N-n}{N-1}\sum_{j=1}^{n}\int_{\Omega} V_f(x_*,x_j)\, C_{N,n+1}(x_{[n]},x_*)\, f(x_*)\, dx_* \\
&+ \frac{1}{N-1}\sum_{i\neq j} K(x_i-x_j)\cdot\nabla_{x_i} C_{N,n-1}(x_{[n]\setminus\{j\}}) \quad = \quad R_{N,n}.
\end{aligned}
$$

The remainder term

The remainder term $R_{N,n} \in W^{-2,1}_{loc}([0,T] \times \Omega^n)$ is orthogonal to the subset of cumulants in the following weak sense,

$$\int_0^T \int_{\Omega^n} h_n R_{N,n} = 0 \quad \text{for all } h_n \in C_c^\infty([0,T] \times \Omega^n)$$

$$\text{such that } \int_\Omega h_n(t, x_{[n]})\, dx_j = 0 \text{ a.e. for all } 1 \leq j \leq n.$$

The full system on the dual cumulants

$$\partial_t C_{N,n} = \frac{1}{N-1} S_N^{n,+} C_{N,n-1} + S_N^{n,\circ} C_{N,n} + S_N^{n,-} C_{N,n+1} + N S_N^{n,=} C_{N,n+2},$$

where we have set $C_{N,-1}, C_{N,N+1}, C_{N,N+2} \equiv 0$, and

$$\begin{aligned} S_N^{n,+} C_{N,n-1} &:= \sum_{i\neq j}^{n} (K * f)(x_i) \cdot \nabla_{x_i} C_{N,n-1}(x_{[n]\setminus\{j\}}) \\ &- \sum_{i\neq j}^{n} \int_\Omega V_f(x_*, x_j)\, C_{N,n-1}(x_{[n]\setminus\{i,j\}}, x_*)\, f(x_*)\, dx_* \\ &- \sum_{i\neq j}^{n} K(x_i - x_j) \cdot \nabla_{x_i} C_{N,n-1}(x_{[n]\setminus\{j\}}), \end{aligned}$$

The full system on the dual cumulants

$$\partial_t C_{N,n} = \frac{1}{N-1} S_N^{n,+} C_{N,n-1} + S_N^{n,\circ} C_{N,n} + S_N^{n,-} C_{N,n+1} + N S_N^{n,=} C_{N,n+2},$$

where we have set $C_{N,-1}, C_{N,N+1}, C_{N,N+2} \equiv 0$, and

$$\begin{aligned} S_N^{n,\circ} C_{N,n} :=& -\frac{N-n}{N-1} \sum_{i=1}^{n} (K * f)(x_i) \cdot \nabla_{x_i} C_{N,n} - \frac{1}{N-1} \sum_{i \neq j}^{n} K(x_i - x_j) \cdot \nabla_{x_i} C_{N,n} \\ &+ \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\Omega} V_f(x_*, x_j)\, C_{N,n}(x_{[n]\setminus\{j\}}, x_*)\, f(x_*)\, dx_* \\ &+ \frac{1}{N-1} \sum_{i \neq j}^{n} \int_{\Omega} K(x_i - x_*) \cdot \nabla_{x_i} C_{N,n}(x_{[n]\setminus\{j\}}, x_*)\, f(x_*)\, dx_* \\ &- \frac{1}{N-1} \sum_{i \neq j}^{n} \int_{\Omega} V_f(x_*, x_j)\, C_{N,n}(x_{[n]\setminus\{i\}}, x_*)\, f(x_*)\, dx_* \\ &+ \frac{1}{N-1} \sum_{i \neq j}^{n} \int_{\Omega^2} V_f(x_*, x_*')\, C_{N,n}(x_{[n]\setminus\{i,j\}}, x_*, x_*')\, f(x_*) f(x_*')\, dx_*\, dx_*', \end{aligned}$$

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where we have set $C_{N,-1}, C_{N,N+1}, C_{N,N+2} \equiv 0$, and

$$S_N^{n,-} C_{N,n+1} \;:=\; \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\Omega} V_f(x_*, x_j)\, C_{N,n+1}(x_{[n]}, x_*)\, f(x_*)\, dx_*$$

$$-\;\; \frac{N-n}{N-1} \sum_{i=1}^{n} \int_{\Omega} K(x_i - x_*) \cdot \nabla_{x_i} C_{N,n+1}(x_{[n]}, x_*)\, f(x_*)\, dx_*$$

$$-\;\; 2\frac{N-n}{N-1} \sum_{i=1}^{n} \int_{\Omega^2} V_f(x_*, x_*')\, C_{N,n+1}(x_{[n]\setminus\{i\}}, x_*, x_*')\, f(x_*) f(x_*')\, dx_*\, dx_*',$$

$$S_N^{n,=} C_{N,n+2} \;:=\; \frac{(N-n)(N-n-1)}{N(N-1)}$$

$$\times \int_{\Omega^2} V_f(x_*, x_*')\, C_{N,n+2}(x_{[n]}, x_*, x_*')\, f(x_*) f(x_*')\, dx_*\, dx_*'.$$

A very simple limit

While those equations look horrible, almost every term vanishes asymptotically when taking the right scaling.

Consider the rescaled correlation, and their weak limit in $L^2_{f^{\otimes n}}$

$$\bar{C}_{N,n} := \binom{N}{n}^{\frac{1}{2}} C_{N,n}, \qquad \bar{C}_n = \lim_{N\to\infty} \bar{C}_{N,n}.$$

Then, if $K \in L^2$, the $\bar{C}_n$ satisfy the limiting hierarchy

$$\partial_t \bar{C}_n + \sum_{i=1}^{n} (K \star f)(x_i) \cdot \nabla_{x_i} \bar{C}_n$$

$$= \sum_{j=1}^{n} \int_{\Omega} V_f(x_*, x_j)\, \bar{C}_n(x_{[n]\setminus\{j\}}, x_*)\, f(x_*)\, dx_*$$

$$+\sqrt{n+1}\,\sqrt{n+2}\, \int_{\Omega^2} V_f(x_{n+1}, x_{n+2})\, \bar{C}_{n+2}\, f(x_{n+1}) f(x_{n+2})\, dx_{n+1}\, dx_{n+2}.$$

Concluding the proof

- With $\bar{\Phi}_N$ such as $\frac{1}{N}\sum_i \varphi(x_i)$ (or any smooth function of the empirical measures), we have that $C_{N,n}(t=T) = O(N^{-n})$ so $\bar{C}_n(t=T) = 0$ for all $n \geq 1$.
- It is straightforward to prove the uniqueness of the limiting hierarchy among solutions $\bar{C}_n$ with $\sup_n \|\bar{C}_n\|_{L^2_{f^{\otimes n}}} < \infty$.This implies that $\lim \bar{C}_{N,n}(t) = \bar{C}_n(t) = 0$ for all $t \leq T$.
- Finally, we just notice that

$$\frac{d}{dt}\int_{\Omega^N} \Phi_N\, f^{\otimes N} = \sqrt{2}\int_{\Omega^2} V_f(x_1,x_2)\,\bar{C}_{N,2}\, f(x_1) f(x_2)\, dx_1\, dx_2 \longrightarrow 0.$$

Conclusions

- We can also provide a quantitative argument with explicit rates of convergence by propagating on the whole system the norm

$$\|\bar{C}_{N,n}\|_n = \inf_{\bar{C}_{N,n}=\alpha_n+\beta_n} \|\alpha_n\|_{L^2_{f^{\otimes n}}} + N^\theta \|\beta_n\|_{H^{-1}_{f^{\otimes n}}}$$

- We expect to be able to derive the Vlasov-Poisson system in dimension 2 by combining this method with averaging lemmas on the $\bar{C}_{N,n}$.
- The derivation of Vlasov-Poisson in dimension 3 and above is still open though.
- This approach is likely helpful for more than mean-field limits: The detailed structure of the dual cumulants also provide insights on the dynamics at various scales.

Some additional remarks on the result

- The key assumption $K \in L^2_{loc}$ corresponds exactly to a scaling of the law of large numbers.
- We only need F_N to be a solution to the many-particle system in a very weak sense, a weak duality solution.
- The result can easily be extended to initial data that are not exactly tensorized but we need $f_N^\circ - (f^\circ)^{\otimes N} = o(N^{-1/2})$ in some strong distance (relative entropy for example).
- In the case of 1st order dynamics, we can improve the assumption to $|x|\,K \in L^2$.
- In dimension 3, the previous theorem allows to push up to $K(q) \lesssim |q|^{-3/2}$ for kinetic systems vs. the previous $|q|^{-1}$.

The definition of weak duality solution

Instead of defining weak solutions to the Liouville or forward Kolmogorov equation, we say that $F_N \in L^\infty([0,\ T], L^1(\Omega^N))$ is a weak duality solution iff for any $\bar{\phi}_N \in L^\infty(\Omega)$ with decay at infinity, there exists at least one weak solution Φ_N to the backward Kolmogorov

$$\partial_t \Phi_N + \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{x_i} \Phi_N = -\frac{\varepsilon_N^2}{2} \sum_{i=1}^{N} \Delta_{x_i} \Phi_N,$$

$$\Phi_N(t = T) = \bar{\Phi}_N,$$

and if the following relation holds

$$\int F_N(t = T)\, \bar{\Phi}_N \, dx_1 \dots dx_N = \int (f^\circ)^{\otimes N}\, \Phi_N(t = 0)\, dx_1 \dots dx_N.$$

This is much weaker than the classical definition of duality solutions which imposes

The dual of classical cumulants

Our definition of dual cumulants is the dual of the classical notion of cumulants. Classical cumulants are useful to study the behavior of solutions F_N close to some equilibrium μ_N and are defined through

$$F_N(t) = \mu_N \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} c_{N,n}(x_\sigma),$$

and the cancellation rule

$$\int_\Omega c_{N,n}(x_1, \dots, x_n)\, \mu_N \, dx_j = 0, \qquad \text{for all } 1 \leq j \leq n.$$

Unfortunately this typically requires that $\int \frac{|F_N|^2}{\mu_N} = O(1)$ to derive useful bounds on the c_n. This forces F_N to be very close to μ_N, which is not useful for us here, but is very helpful in the right context, see for example Duerinckx Saint-Raymond 21.

The scaling of the law of large numbers

We also have an alternative proof that directly connects the scaling of $C_{N,n}$ to the law of large number.

Consider $n = 1$ for simplicity and for any test function $\psi(x)$ with $\int_\Omega \psi\, f\, dx$, write

$$\int_\Omega \psi(x)\, C_{N,1}(x)\, f(x)\, dx = \int_\Omega \psi(x)\, M_{N,1}(x)\, f(x)\, dx$$

$$= \int_{\Omega^N} \psi(x_1)\, \Phi_N\, f^{\otimes N}\, dx_1 \dots dx_N$$

$$= \int_{\Omega^N} \frac{1}{N} \sum_i \psi(x_i)\, \Phi_N\, f^{\otimes N}\, dx_1 \dots dx_N$$

$$\leq \|\Phi_N\|_{L^\infty} \left(\int_{\Omega^N} \left| \frac{1}{N} \sum_i \psi(x_i) \right|^2 f^{\otimes N}\, dx_1 \dots dx_N \right)^{1/2}$$

$$\leq N^{-1/2}\, \|\Phi_N\|_{L^\infty}\, \|\psi\|_{L^2_f}.$$