New optimized Robin-Robin domain decomposition methods using Krylov solvers for the Stokes-Darcy system

Xiaoming He
Department of Mathematics & Statistics
Missouri University of Science & Technology, Rolla, MO

Joint work with
Yassine Boubendir, New Jersey Institute of Technology
Yingzhi Liu, University of Macau
Yinnian He, Xi'an Jiaotong University

Partially supported by NSF grants DMS-1722647, DMS-1720014, and DMS-2011843

Outline

Outline

Introduction: Stokes-Darcy model and multi-physics DDM

Modal analysis for the Robin-Robin DDM

Optimized parameters for the Robin-Robin DDM

Orthodir algorithm for the Robin-Robin DDM

Numerical examples

Introduction: applications of Stokes-Darcy model

- Subsurface flow in Karst aquifers
- Interaction between surface water flows and subsurfaces flows
- Oil reservoir in vuggy porous medium
- Industrial filtrations, field-flow fractionation
- Blood motion in lungs, solid tumors and vessels
- Meshy zone in alloy solidification
- Remediation of soils by means of bacterial colonies
- Topology optimization
- Heat transfer in walls with fibrous insulation
- Spontaneous combustion of coal stockpiles
- ....

Introduction: Stokes-Darcy model

We consider a coupled Stokes-Darcy system on a bounded domain $\Omega = \Omega_D \bigcup \Omega_S \subset \mathbb{R}^{\mathbf{d}}$, $(\mathbf{d} = 2, 3)$.

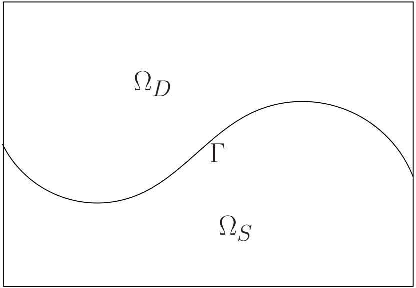


Figure: A sketch of the porous medium domain Ω_D, fluid domain Ω_S, and the interface Γ.

Introduction: Stokes-Darcy model

In Ω_D, the porous media flow is assumed to satisfy the following saturated flow model and Darcy's law.

$$\begin{aligned} \nabla \cdot \vec{u}_D &= f_D, \\ \vec{u}_D &= -\mathbb{K}\nabla\phi_D, \end{aligned}$$

where

- $\vec{u}_D$: fluid discharge rate in the porous medium
- ϕ_D: hydraulic head
- $\mathbb{K}$: hydraulic conductivity tensor
- f_D: sink/source term

We consider the second-order form of the Darcy system

$$-\nabla \cdot (\mathbb{K}\nabla\phi_D) = f_D.$$

Introduction: Stokes-Darcy model

In Ω_S, the fluid flow is assumed to satisfy the Stokes equations

$$\begin{aligned} -\nabla \cdot \mathbb{T}(\overrightarrow{u}_S, p_S) &= \overrightarrow{f}_S, \\ \nabla \cdot \overrightarrow{u}_S &= 0. \end{aligned}$$

where

- $\overrightarrow{u}_S$: fluid velocity
- p_S: kinematic pressure
- $\overrightarrow{f}_S$: external body force
- μ: kinematic viscosity of the fluid
- $\mathbb{T}(\overrightarrow{u}_S, p_S) = 2\mu\mathbb{D}(\overrightarrow{u}_S) - p_S\mathbb{I}$: stress tensor
- $\mathbb{D}(\overrightarrow{u}_S) = 1/2(\nabla\overrightarrow{u}_S + \nabla^T\overrightarrow{u}_S)$: rate of deformation tensor

Introduction: Stokes-Darcy model

Two interface conditions in the normal direction:

- Continuity of the normal velocity across the interface (conservation of mass):

$$\overrightarrow{u}_S \cdot \overrightarrow{n}_S \quad = \quad -\overrightarrow{u}_D \cdot \overrightarrow{n}_D.$$

- Balance of force normal to the interface:

$$-\overrightarrow{n}_S \cdot (\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) \quad = \quad g(\phi_D - z).$$

  where z is the height and g is the gravity constant.

One interface condition in the tangential direction:

- Beavers-Joseph-Saffman-Jones (BJSJ):

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) \quad = \quad \alpha \boldsymbol{\tau}_j \cdot \overrightarrow{u}_S.$$

  where $\boldsymbol{\tau}_j$ $(j = 1, \cdots, d-1)$ denote mutually orthogonal unit tangential vectors to the interface Γ.

Introduction: Survey on DDM for Stokes-Darcy

- M. Discacciati, E. Miglio and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math., 43(1-2):57-74, 2002.
- M. Discacciati and A. Quarteroni. Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations. Comput. Vis. Sci., 6: 93-103, 2004.
- R. Hoppe, P. Porta and Y. Vassilevski. Computational issues related to iterative coupling of subsurface and channel flows, CALCOLO, 44(1); 1-20, 2007.
- M. Discacciati, A. Quarteroni and A. Valli. Robin-Robin domain decomposition methods for the Stokes-Darcy coupling, SIAM J. Numer. Anal., 45(3):1246-1268, 2007.
- B. Jiang. A parallel domain decomposition method for coupling of surface and groundwater flows, Comput. Methods Appl. Mech. Engrg., 198(9-12): 947-957, 2009.
- W. Chen, M. Gunzburger, F. Hua and X. Wang. A parallel Robin-Robin domain decomposition method for the Stokes-Darcy system, SIAM. J. Numer. Anal., 49: 1064-1084, 2011.

Introduction: Survey on DDM for Stokes-Darcy

- Y. Cao, M. Gunzburger, X.-M. He, and X. Wang. Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beaver-Joseph interface condition, Numer. Math., 117: 601-629, 2011.
- Y. Boubendir and S. Tlupova. Domain decomposition methods for solving Stokes-Darcy problems with boundary integrals, SIAM J. Sci. Comput., 35: B82-B106, 2013.
- D. Vassilev, C. Wang and I. Yotov. Domain decomposition for coupled Stokes and Darcy flows. Comput. Methods Appl. Mech. Engrg., 268: 264-283, 2014.
- X.-M. He, J. Li, Y. Lin, and J. Ming. A domain decomposition method for the steady-state Navier-Stokes-Darcy model with Beavers-Joseph interface condition, SIAM J. Sci. Comput., 37(5): S264-S290, 2015.
- M. Discacciati, and L. Gerardo-Giorda. Optimized Schwarz methods for the Stokes-Darcy coupling, IMA J. Numer. Anal., 38: 1959-1983, 2018.
- Y. Liu, Y. Boubendir, X.-M. He, and Y. He. New optimized Robin-Robin domain decomposition methods using Krylov solvers for the Stokes-Darcy system, SIAM J. Sci. Comput., 44(4): B1068-B1095, 2022.

Introduction: notations

- Spaces:

$$\begin{aligned} X_S &= \{\overrightarrow{v} \in [H^1(\Omega_S)]^d \mid \overrightarrow{v} = 0 \text{ on } \partial\Omega_S \backslash \Gamma\}, \\ Q_S &= L^2(\Omega_S), \\ X_D &= \{\psi \in H^1(\Omega_D) \mid \psi = 0 \text{ on } \partial\Omega_D \backslash \Gamma\}. \end{aligned}$$

- Bilinear forms:

$$\begin{aligned} a_D(\phi_D, \psi) &= (\mathbb{K}\nabla\phi_D, \nabla\psi)_{\Omega_D}, \\ a_S(\overrightarrow{u}_S, \overrightarrow{v}) &= 2\mu(\mathbb{D}(\overrightarrow{u}_S), \mathbb{D}(\overrightarrow{v}))_{\Omega_S}, \\ b_S(\overrightarrow{v}, q) &= -(\nabla \cdot \overrightarrow{v}, q)_{\Omega_S}. \end{aligned}$$

- P_τ denotes the projection onto the tangent space on Γ, i.e.,

$$P_\tau \overrightarrow{u} = \sum_{j=1}^{d-1} (\overrightarrow{u} \cdot \boldsymbol{\tau}_j)\boldsymbol{\tau}_j.$$

Introduction: Multi-physics DDM with Robin conditions

- For the Darcy system, we impose the Robin boundary condition: given a constant $\gamma_p > 0$ and given a function η_p defined on Γ,

$$\gamma_p \mathbb{K} \nabla \widehat{\phi}_D \cdot \vec{n}_D + g \widehat{\phi}_D = \eta_p, \text{ on } \Gamma.$$

- The corresponding weak formulation for the Darcy system is given by: for $\eta_p \in L^2(\Gamma)$, find $\widehat{\phi}_D \in X_D$ such that

$$a_D(\widehat{\phi}_D, \psi) + \langle \frac{g \widehat{\phi}_D}{\gamma_p}, \psi \rangle = (f_D, \psi)_{\Omega_D} + \langle \frac{\eta_p}{\gamma_p}, \psi \rangle, \ \forall \psi \in X_D.$$

Introduction: Multi-physics DDM with Robin conditions

- For the Stokes system, we impose the Robin boundary conditions: given a constant $\gamma_f > 0$ and given functions η_f defined on Γ,

$$\vec{n}_S \cdot (\mathbb{T}(\widehat{\vec{u}}_S, \widehat{p}_S) \cdot \vec{n}_S) + \gamma_f \widehat{\vec{u}}_S \cdot \vec{n}_S = \eta_f, \text{ on } \Gamma,$$

- The corresponding weak formulation for the Navier-Stokes system is given by: for $\eta_f \in L^2(\Gamma)$, find $\widehat{\vec{u}}_S \in X_S$ and $\widehat{p}_S \in Q_S$ such that

$$\begin{aligned} & a_S(\widehat{\vec{u}}_S, \vec{v}) + b_S(\vec{v}, \widehat{p}_S) - b_S(\widehat{\vec{u}}_S, q) \\ & + \gamma_f \langle \widehat{\vec{u}}_S \cdot \vec{n}_S, \vec{v} \cdot \vec{n}_S \rangle + \alpha \langle P_\tau \widehat{\vec{u}}_S, P_\tau \vec{v} \rangle \\ = \ & (\vec{f}_S, \vec{v})_{\Omega_S} + \langle \eta_f, \vec{v} \cdot \vec{n}_S \rangle, \ \forall \ (\vec{v}, q) \in X_S \times Q_S. \end{aligned}$$

Introduction: Multi-physics DDM with Robin conditions

- Compatibility conditions:

$$\begin{aligned} \eta_f &= \gamma_f \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S - g\widehat{\phi}_D + gz, \\ \eta_p &= \gamma_p \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S + g\widehat{\phi}_D. \end{aligned}$$

or equivalent conditions:

$$\begin{aligned} \eta_f &= a\eta_p + bg\widehat{\phi}_D + gz, \\ \eta_p &= c\eta_f + d\widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S + gz, \end{aligned}$$

where

$$a = \frac{\gamma_f}{\gamma_p}, \quad b = -\left(1 + \frac{\gamma_f}{\gamma_p}\right), \quad c = -1, \quad d = \gamma_f + \gamma_p.$$

Introduction: Multi-physics DDM with Robin conditions

1. Initial values η_p^0 and η_f^0 are guessed. They may be taken to be zero.
2. For $k = 0, 1, 2, \ldots$, independently solve the Stokes and Darcy systems with Robin boundary conditions. More precisely, $\phi_D^k \in X_D$ is computed from

$$a_D(\phi_D^k, \psi) + \langle \frac{g\phi_D^k}{\gamma_p}, \psi \rangle = \langle \frac{\eta_p^k}{\gamma_p}, \psi \rangle + (f_D, \psi)_{\Omega_D}, \ \forall \psi \in X_D, \quad (1.1)$$

and $(\vec{u}_S^k, p_S^k) \in X_S \times Q_S$ are computed from:

$$\begin{aligned} & a_S(\vec{u}_S^k, \vec{v}) + b_S(\vec{v}, p_S^k) - b_S(\vec{u}_S^k, q) \\ & + \gamma_f \langle \vec{u}_S^k \cdot \vec{n}_S, \vec{v} \cdot \vec{n}_S \rangle + \alpha \langle P_\tau \vec{u}_S^k, P_\tau \vec{v} \rangle \\ & = \langle \eta_f^k, \vec{v} \cdot \vec{n}_S \rangle + (\vec{f}_S, \vec{v})_{\Omega_S}, \ \forall (\vec{v}, q) \in X_S \times Q_S. \end{aligned} \quad (1.2)$$

3. η_p^{k+1} and η_f^{k+1} are updated in the following manner:

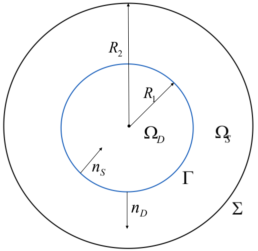
$$\begin{aligned} \eta_f^{k+1} &= a\eta_p^k + bg\phi_D^k + gz, \\ \eta_p^{k+1} &= c\eta_f^k + d\vec{u}_S^k \cdot \vec{n}_S + gz. \end{aligned}$$

Outline

Modal analysis

For modal analysis, we consider:

- The iteration algorithm in continuous context.
- A circular geometric domain



- Simplify the analysis by setting $\mathbb{K} = KI$, $g = 1$ and $z = 0$.

Modal analysis

In continuous context, the Robin-Robin DDM can be described as

- Give initial guess $\eta^0 = (\eta_f^0, \eta_p^0)^T$.
- For $k = 0, 1, 2, \ldots$, solve Darcy equations

$$\begin{cases} \vec{u}_D^k + \mathbb{K}\nabla\phi_D^k = 0, \ \nabla\cdot\vec{u}_D^k = 0 & \text{in } \Omega_D, \\ \gamma_p \mathbb{K}\nabla\phi_D^k\cdot\vec{n}_D + \phi_D^k = \eta_p^k & \text{on } \Gamma, \end{cases} \tag{2.3}$$

  and Stokes equations

$$\begin{cases} \mu\Delta\vec{u}_S^k - \nabla p_S^k = 0, \ \nabla\cdot\vec{u}_S^k = 0 & \text{in } \Omega_S, \\ \vec{u}_S^k = 0 & \text{on } \Sigma, \\ \vec{u}_S^k\cdot\vec{\tau}_S = 0, & \text{on } \Gamma, \\ \vec{n}_S\cdot(\mathbb{T}(\vec{u}_S^k, p_S^k)\cdot\vec{n}_S) + \gamma_f\vec{u}_S^k\cdot\vec{n}_S = \eta_f^k & \text{on } \Gamma, \end{cases} \tag{2.4}$$

  where $\Gamma = \partial\Omega_D \cap \partial\Omega_S$ and $\Sigma = \partial\Omega_S \setminus \Gamma$.
- Update iteration by

$$\eta^{k+1} = \begin{pmatrix} a\eta_p^k + b\phi_D^k \\ c\eta_f^k + d\,\vec{u}_S^k\cdot\vec{n}_S \end{pmatrix}. \tag{2.5}$$

- Define the Darcy operator $\mathcal{D} : \eta_p \in L^2(\Gamma) \mapsto \mathcal{D}\eta_p \in L^2(\Gamma)$ as

$$\mathcal{D}\eta_p = a\eta_p + b\phi_D, \tag{2.6}$$

- Define the Stokes operator $\mathcal{S} : \eta_f \in L^2(\Gamma) \mapsto \mathcal{S}\eta_f \in L^2(\Gamma)$ as

$$\mathcal{S}\eta_f = c\eta_f + d\overrightarrow{u}_S \cdot \overrightarrow{n}_S, \tag{2.7}$$

- The iteration operator $\mathcal{A} : \eta \in (L^2(\Gamma))^2 \mapsto \mathcal{A}\eta \in (L^2(\Gamma))^2$ can be written as

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{S} & 0 \end{pmatrix}. \tag{2.8}$$

Modal analysis

Proposition

The operator $\mathcal{D}$ *has the decomposition* $\mathcal{D}\eta_p = \sum_{m\in\mathbf{Z}} \mathcal{D}_m \eta_{p,m} H_m(\theta)$ *with*

$$\mathcal{D}_0 = -1, \quad \mathcal{D}_m = \frac{\gamma_f K|m|/R_1 - 1}{\gamma_p K|m|/R_1 + 1} \quad (m \neq 0), \tag{2.9}$$

where $\eta_p = \sum_{m\in\mathbf{Z}} \eta_{p,m} H_m(\theta)$.

Proposition

The operator $\mathcal{S}$ *has the decomposition* $\mathcal{S}\eta_f = \sum_{m\in\mathbf{Z}} \mathcal{S}_m \eta_{f,m} H_m(\theta)$ *with*

$$\mathcal{S}_0 = -1, \quad \mathcal{S}_m = \frac{\gamma_p M_m/\mu - N_m}{\gamma_f M_m/\mu + N_m} \quad (m \neq 0), \tag{2.10}$$

where $\eta_f = \sum_{m\in\mathbf{Z}} \eta_{f,m} H_m(\theta)$.

Modal analysis

Here

$$H_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta},\ \theta \in [0, 2\pi],\ m \in \mathbf{Z},$$

is the basis functions in $L^2(\Gamma)$,

$$M_m = \begin{cases} -\dfrac{R_1^2}{2}(\lambda^2 - 1) + h_1 \ln \lambda, & |m| = 1, \\ -\dfrac{R_1^{|m|+1}}{2}(\lambda^2 - 1) + \dfrac{h_m}{2(|m|-1)R_1^{|m|-1}}(1 - \lambda^{-2(|m|-1)}), & |m| > 1, \end{cases} \tag{2.11}$$

$$N_m = R_1^{|m|} + \frac{h_m}{R_1^{|m|}} + \frac{2}{R_1} M_m, \quad |m| \geq 1, \tag{2.12}$$

with

$$h_m = \begin{cases} \dfrac{R_1^2}{2} \dfrac{(\lambda^4 - 1)/2 + (\lambda^2 - 1)}{\ln \lambda + (\lambda^2 - 1)/2}, & |m| = 1, \\ R_1^{2|m|} \lambda^{2(|m|+1)} \dfrac{|m| - 1}{|m| + 1} \dfrac{1 + \lambda^{-2(|m|+1)}((\lambda^2 - 1)(|m| + 1) - 1)}{(\lambda^2 - 1)(|m| - 1) + 1 - \lambda^{-2(|m|-1)}}, & |m| > 1, \end{cases}$$

and $\lambda = R_2/R_1 > 1$. Here R_1 is the radius of the Darcy domain Ω_D, R_2 is the radius of the Stokes-Darcy domain Ω.

Modal analysis

Lemma

Let M_m and N_m be defined as in (2.11) and (2.12), respectively, and $C_m = \frac{M_m}{N_m}$. Then, for $|m| > 1$, we have

$$C_m^{-1} = \frac{N_m}{M_m} = \frac{2}{R_1}|m| \left(1 + O\left(\frac{|m|^2}{\lambda^{2|m|-2} - |m|^2}\right)\right). \quad (2.13)$$

Remark

When $m \to \infty$, we have $O\left(\frac{|m|^2}{\lambda^{2|m|-2} - |m|^2}\right) \to 0$ with $\lambda > 1$, hence $\frac{N_m}{M_m} \to \frac{2}{R_1}|m|$. This almost linear dependence is also observed geometrically in the next plot.

Modal analysis

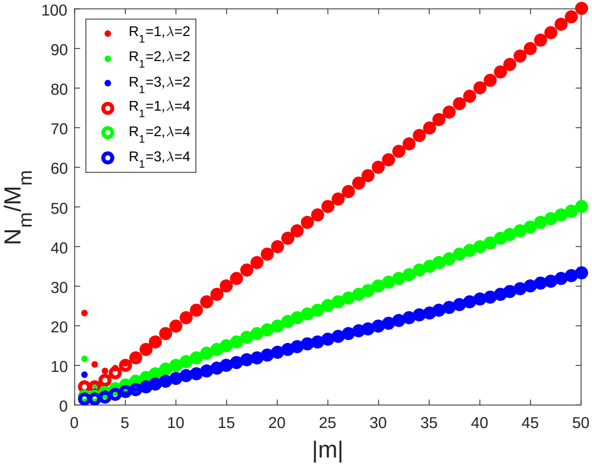


Figure: Change of $\frac{N_m}{M_m}$ with respect to m: almost linear performance after a big enough m.

Modal analysis

- The iterative operator can be written as $\mathcal{A} = \sum_{m\in\mathbf{Z}} \mathcal{A}_m H_m(\theta)$, where

$$\mathcal{A}_m := \begin{pmatrix} 0 & \mathcal{D}_m \\ \mathcal{S}_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\gamma_f K|m|/R_1 - 1}{\gamma_p K|m|/R_1 + 1} \\ \frac{\gamma_p M_m/\mu - N_m}{\gamma_f M_m/\mu + N_m} & 0 \end{pmatrix}. \qquad (2.14)$$

Theorem

Let $\rho(\mathcal{A}_m)$ be the spectral radius of $\mathcal{A}_m$ defined in (2.14). When $\gamma_f = \gamma_p$, we have

$$\rho(\mathcal{A}_0) = 1 \quad \textit{and} \quad \rho(\mathcal{A}_m) < 1 \quad \textit{for } m \neq 0.$$

When $\gamma_f \neq \gamma_p$, by choosing γ_f and γ_p satisfying

$$\gamma_f \gamma_p K/(2\mu) = 1, \;\; \gamma_p K|m|/R_1 + C_m \gamma_f/\mu \gg \gamma_f K|m|/R_1 + C_m \gamma_p/\mu,$$

we have $|\rho(\mathcal{A}_m)| < 1$.

Modal analysis

- Spectral radius: $\rho(\gamma_f, \gamma_p, m) = \left| \left(\frac{\gamma_f Km/R_1 - 1}{\gamma_p Km/R_1 + 1} \right) \left(\frac{\gamma_p C_m/\mu - 1}{\gamma_f C_m/\mu + 1} \right) \right|$.

- Using $C_m^{-1} \approx 2/R_1|m|$, spectral radius $\rho(\gamma_f, \gamma_p, m)$ can be approximately reduced to

$$\rho(\gamma_f, \gamma_p, m) \approx \left| \left(\frac{2\widetilde{\mu} m - \gamma_p}{2\widetilde{\mu} m + \gamma_f} \right) \left(\frac{1 - \gamma_f \widetilde{K} m}{1 + \gamma_p \widetilde{K} m} \right) \right|,$$

  where $\widetilde{\mu} = \mu/R_1$ and $\widetilde{K} = K/R_1$.

- In fact, the above conclusion is consistent with the corresponding conclusion in the following reference which uses Fourier analysis based on a geometric assumption of a straight line:
  M. Discacciati, and L. Gerardo-Giorda. Optimized Schwarz methods for the Stokes-Darcy coupling, IMA J. Numer. Anal., 38: 1959-1983, 2018.

Outline

Min-Max approach: hyperbolic relation (review)

- M. Discacciati, and L. Gerardo-Giorda. Optimized Schwarz methods for the Stokes-Darcy coupling, IMA J. Numer. Anal., 38: 1959-1983, 2018.
- Hyperbolic relation between Robin parameters γ_f and γ_p:

$$\gamma_f \gamma_p = \frac{2\widetilde{\mu}}{\widetilde{K}}.$$

- Min-max problem with hyperbolic relation (M-H):

$$\min_{\gamma_f \gamma_p = \frac{2\widetilde{\mu}}{\widetilde{K}}} \max_{m \in [m_{\min}, m_{\max}]} \rho(\gamma_f, \gamma_p, m)$$
$$= \min_{\gamma_f \gamma_p = \frac{2\widetilde{\mu}}{\widetilde{K}}} \max \{\rho(\gamma_f, \gamma_p, m_{\min}), \rho(\gamma_f, \gamma_p, m_{\max})\}. \qquad (3.15)$$

Min-Max approach: linear relation

- When $m = m_{\min}$ and $m = m_{\max}$, the parameter pairs (γ_f, γ_p) reach optima at

$$\left(\frac{1}{\widetilde{K} m_{\min}}, 2\widetilde{\mu} m_{\min} \right) \text{ and } \left(\frac{1}{\widetilde{K} m_{\max}}, 2\widetilde{\mu} m_{\max} \right).$$

- Linear relation between Robin parameters γ_f and γ_p:

$$\begin{aligned} \gamma_p &= \left(-2\widetilde{\mu} \widetilde{K} m_{\min} m_{\max} \right) \gamma_f + 2\widetilde{\mu} (m_{\min} + m_{\max}) \\ &:= p\gamma_f + q \end{aligned} \tag{3.16}$$

for any $\gamma_f \in \mathcal{I}_f$, where $\mathcal{I}_f = \left[\frac{1}{\widetilde{K} m_{\max}}, \frac{1}{\widetilde{K} m_{\min}} \right]$.

- Min-max problem with linear relation (M-L):

$$\min_{\gamma_p = p\gamma_f + q} \max_{m \in [m_{\min}, m_{\max}]} \rho(\gamma_f, \gamma_p, m). \tag{3.17}$$

Min-Max approach: linear relation

Theorem

Let (γ_f^*, γ_p^*) and $(\gamma_f^\star, \gamma_p^\star)$ be the solution of (3.15) and (3.17), respectively. If $\widetilde{K}$ tends to zero and $m_{\max} > m_{\min}$, then

$$\max_{m \in [m_{\min}, m_{\max}]} \rho(\gamma_f^*, \gamma_p^*, m) > \max_{m \in [m_{\min}, m_{\max}]} \rho(\gamma_f^\star, \gamma_p^\star, m). \tag{3.18}$$

Min-Max approach: spectral comparison

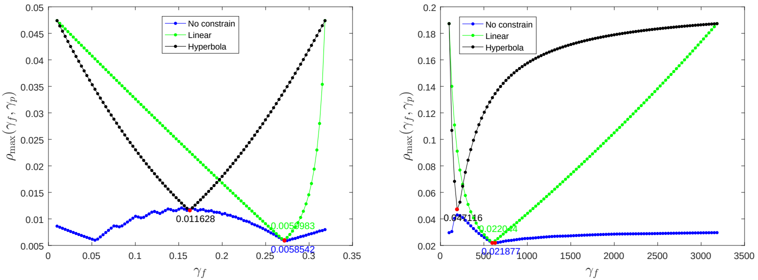


Figure: Comparison of the maximum spectral radius with respect to γ_f and the corresponding optimal γ_p: $\mu = 1, K = 1$ (left) or $\mu = 1e-1, K = 1e-4$ (right).

Expectation approach: hyperbolic relation (review)

- Spectral cluster can improve the convergence of Krylov methods.
- Define $\mathcal{A}_f = \left\{ \gamma_f > 0 : \rho(\gamma_f, \gamma_p, m)\big|_{\gamma_f \gamma_p = \frac{2\widetilde{\mu}}{\widetilde{K}}} \leq 1, \quad \forall m \in [m_{\min}, m_{\max}] \right\}$.
- Expectation minimization problem with hyperbolic relation (E-H):

$$\min_{\substack{\gamma_f \in \mathcal{A}_f \\ \gamma_f \gamma_p = \frac{2\widetilde{\mu}}{\widetilde{K}}}} E(\gamma_f, \gamma_p) := \min_{\gamma_f \in \mathcal{A}_f} E(\gamma_f), \tag{3.19}$$

where

$$\begin{aligned} E(\gamma_f) &= \frac{1}{m_{\max} - m_{\min}} \int_{m_{\min}}^{m_{\max}} \rho(\gamma_f, \gamma_p, m) \mathrm{d}m \\ &= \frac{\gamma_f^2 \widetilde{K}}{2\widetilde{\mu}} + \frac{(\gamma_f^2 \widetilde{K} + 2\widetilde{\mu})^2}{2\widetilde{\mu}\widetilde{K}(2\widetilde{\mu} m_{\max} + \gamma_f)(2\widetilde{\mu} m_{\min} + \gamma_f)} \\ &\quad - \frac{\gamma_f(\gamma_f^2 \widetilde{K} + 2\widetilde{\mu})}{2\widetilde{\mu}^2(m_{\max} - m_{\min})} \ln\left(\frac{2\widetilde{\mu} m_{\max} + \gamma_f}{2\widetilde{\mu} m_{\min} + \gamma_f} \right). \end{aligned}$$

Expectation approach: linear relation

- Expectation minimization problem with linear relation (E-L):

$$\min_{\substack{\gamma_f \in \mathcal{I}_f \\ \gamma_p = p\gamma_f + q}} E(\gamma_f, \gamma_p), \tag{3.20}$$

where

$$\begin{aligned}
E(\gamma_f, \gamma_p) &:= \frac{1}{m_{\max} - m_{\min}} \int_{m_{\min}}^{m_{\max}} \rho(\gamma_f, \gamma_p, m) \mathrm{d}m \\
&= \frac{1}{m_{\max} - m_{\min}} \left(\int_{m_{\min}}^{m_{1c}} -g(\gamma_f, \gamma_p, m)\mathrm{d}m + \int_{m_{1c}}^{m_{2c}} g(\gamma_f, \gamma_p, m)\mathrm{d}m + \int_{m_{2c}}^{m_{\max}} -g(\gamma_f, \gamma_p, m)\mathrm{d}m \right) \\
&= \frac{(\gamma_f + \gamma_p)\left(2\widetilde{\mu} + \widetilde{K}\gamma_p^2\right)}{\widetilde{K}\gamma_p^2(m_{\max} - m_{\min})\left(\gamma_f\gamma_p\widetilde{K} - 2\widetilde{\mu}\right)} \ln\left(\left(\frac{\widetilde{K}\gamma_p m_{\max} + 1}{\widetilde{K}\gamma_p m_{\min} + 1} \right) \left(\frac{\widetilde{K}\gamma_p m_{1c} + 1}{\widetilde{K}\gamma_p m_{2c} + 1} \right)^2 \right) \\
&- \frac{(\gamma_f + \gamma_p)\left(2\widetilde{\mu} + \widetilde{K}\gamma_f^2\right)}{2\widetilde{\mu}(m_{\max} - m_{\min})\left(\gamma_f\gamma_p\widetilde{K} - 2\widetilde{\mu}\right)} \ln\left(\left(\frac{\gamma_f + 2\widetilde{\mu} m_{\max}}{\gamma_f + 2\widetilde{\mu} m_{\min}} \right) \left(\frac{\gamma_f + 2\widetilde{\mu} m_{1c}}{\gamma_f + 2\widetilde{\mu} m_{2c}} \right)^2 \right) \\
&+ \frac{\gamma_f}{\gamma_p} \left(1 - \frac{2(m_{2c} - m_{1c})}{m_{\max} - m_{\min}} \right).
\end{aligned}$$

Expectation approach: spectral comparison

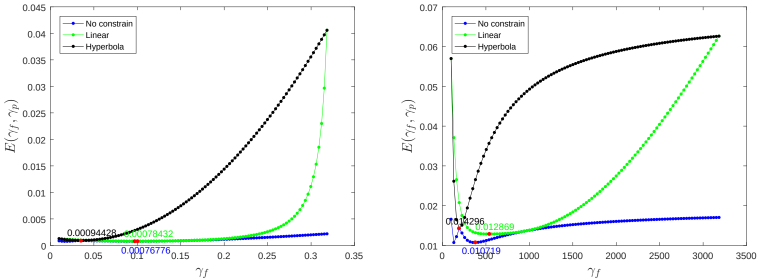


Figure: Comparison of the maximum spectral radius with respect to γ_f and the corresponding optimal γ_p: $\mu = 1, K = 1$ (left) or $\mu = 1e-1, K = 1e-4$ (right).

Outline

Robin-Robin Orthodir DDM

- Algebraic system of Stokes equation (1.2): $A_1 u_1 = b_1 + l_1$.
- Algebraic system of Darcy equation (1.1): $A_2 u_2 = b_2 + l_2$.
- The Robin-Robin DDM is a Jacobian iteration of the problem

$$\mathcal{A}\eta = \eta,$$

or

$$\widetilde{\mathcal{A}}\eta := (I - \widehat{\mathcal{A}})\eta = g_0,$$

where $\widehat{\mathcal{A}}\eta := \{\mathcal{A}\eta | l_1 = 0, l_2 = 0\}$, $g_0 := \{\mathcal{A}\eta | b_1 = 0, b_2 = 0\}$.

Robin-Robin Orthodir DDM

Algorithm 1 Robin-Robin Orthodir DDM for Stoke-Darcy problem

Initialize $\eta^0 = 0$.
Solve $g_0 = \mathcal{A}\eta^0$ with $b_1 = 0$ and $b_2 = 0$.
Set $r^0 = p^0 = g_0$.
for $j = 0, 1, \ldots$ **do**
    Compute $\widehat{\mathcal{A}}p^j$ by solving $\mathcal{A}p^j$ with $l_1 = 0$ and $l_2 = 0$, and then set $\widetilde{\mathcal{A}}p^j = p^j - \widehat{\mathcal{A}}p^j$. Compute $\widetilde{\mathcal{A}}^2 p^j$ using the same routine but with $\widetilde{\mathcal{A}}p^j$ instead of p^j.
    $\alpha_j = \dfrac{\langle r^j, \widetilde{\mathcal{A}}p^j \rangle}{\langle \widetilde{\mathcal{A}}p^j, \widetilde{\mathcal{A}}p^j \rangle}$.
    $X^{j+1} = X^j + \alpha_j p^j$.
    $r^{j+1} = r^j - \alpha_j \widetilde{\mathcal{A}}p^j$.
    for $i = 0, \ldots, j$ **do**
        $\beta_{ij} = -\dfrac{\langle \widetilde{\mathcal{A}}^2 p^j, \widetilde{\mathcal{A}}p^i \rangle}{\langle \widetilde{\mathcal{A}}p^i, \widetilde{\mathcal{A}}p^i \rangle}$.
    end for
    $p^{j+1} = \widetilde{\mathcal{A}}p^j + \sum_{i=0}^{j} \beta_{ij} p^i$.
end for

Outline

Numerical example 1: straight interface

- Domain: $\Omega = (0,1) \times (0,2)$ where $\Omega_D = (0,1) \times (0,1)$, $\Omega_S = (0,1) \times (1,2)$ and interface $\Gamma = (0,1) \times \{1\}$
- Parameters: $\alpha = \alpha_0 \sqrt{\mu/K}$, $\alpha_0 = 1$, $g = 1$ and $z = 0$.
- Exact solution:

$$\begin{cases} \phi_D = (-\alpha_0 x(y-1) + y^3/3 - y^2 + y)/K + 2\mu x, \\ \overrightarrow{u}_S = (\sqrt{\mu K}, \alpha_0 x), \\ p_S = 2\mu(x+y-1) + 1/(3K). \end{cases}$$

Numerical example 1: straight interface

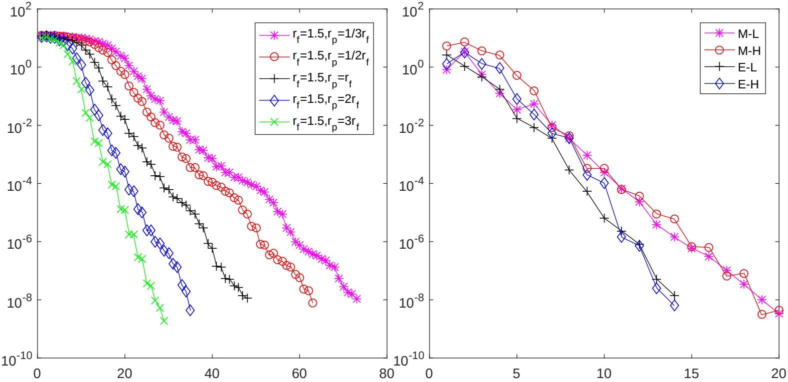


Figure: Orthodir DDM with different Robin parameters for $\mu = 1, K = 10^{-2}$ and $h = 1/32$. Left: nonoptimized Robin parameters; Right: optimal Robin parameters obtained from the four different optimal approaches.

Numerical example 1: straight interface

Table: The optimal parameter pairs (γ_f, γ_p) and the number of iterations with four optimal approaches: M-L, M-H, E-L, and E-H.

μ	K	γ_f	γ_p	$\rho_{\max}$	$E(\gamma_f, \gamma_p)$	Iter	
1	1	0.2703	36.6256	0.0060	0.0041	7	(M–L)
		0.1618	12.3606	0.0116	0.0089	8	(M–H)
		0.1014	143.3135	0.0324	0.0008	6	(E–L)
		0.0363	55.1120	0.0393	0.0009	7	(E–H)
1	1e-6	5.6434e+04	171.6983	0.0024	0.0014	18	(M–L)
		1.9245e+04	103.9255	0.0048	0.0016	21	(M–H)
		5.6434e+04	171.6983	0.0024	0.0014	18	(E–L)
		1.9245e+04	103.9255	0.0048	0.0016	21	(E–H)
1e-1	1e-4	595.3315	16.9741	0.0222	0.0129	31	(M–L)
		207.9411	9.6181	0.0457	0.0145	33	(M–H)
		533.3490	17.3656	0.0260	0.0129	31	(E–L)
		192.4455	10.3926	0.0474	0.0143	33	(E–H)

Numerical example 1: straight interface

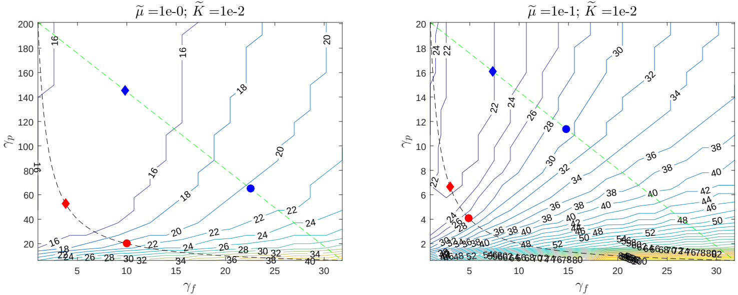


Figure: Contour distribution of the number of iterations and the optimal (γ_f, γ_p) pairs: blue circle (M-L), blue diamond (E-L), red circle (M-H) and red diamond (E-L), respectively.

Numerical example 2: curved interface

- Domain: $\Omega = (-1.5, 1.5) \times (-1.5, 1.5)$ where interface $\Gamma = \{y = -0.5\sin(\pi(x + 1.5)), -1.5 \le x \le 1.5\}$, Stokes and Darcy regions are the top and bottom parts, respectively.
- Parameters: $\alpha = \sqrt{\mu/K}$, $g = 1$ and $z = 0$.
- Stokes boundary condition: $\overrightarrow{u}_S = (0, x^2 - 4)$.
- Darcy boundary condition: $\mathbb{K}\nabla\phi_D \cdot \overrightarrow{n}_D = 0$ on the left and right boundaries; $\phi_D = 0$ on the bottom boundary.
- Source terms: $\overrightarrow{f}_S = 0$, $f_D = 0$.

Numerical example2: curved interface

Table: The optimal parameter pairs (γ_f, γ_p) and the numbers of iterations with four optimal approaches: M-L, M-H, E-L, and E-H.

(μ, K)	$h = 1/8$			$h = 1/128$			
	γ_f	γ_p	Iter	γ_f	γ_p	Iter	
(1,1)	2.44e-01	1.80e+01	12	2.66e-01	1.39e+02	10	(M-L)
	1.77e-01	1.13e+01	12	1.58e-01	1.27e+01	10	(M-H)
	1.57e-01	3.17e+01	12	5.01e-02	6.84e+02	10	(E-L)
	9.44e-02	2.12e+01	12	1.20e-02	1.67e+02	10	(E-H)
(1,1e-2)	1.91e+01	2.64e+01	24	2.28e+01	2.35e+02	24	(M-L)
	1.22e+01	1.63e+01	25	9.29e+00	2.15e+01	22	(M-H)
	1.42e+01	3.41e+01	22	5.96e+00	6.60e+02	24	(E-L)
	8.74e+00	2.29e+01	23	1.52e+00	1.32e+02	30	(E-H)
(1e-2,1e-2)	1.15e+01	3.83e-01	35	8.34e+00	6.00e+00	35	(M-L)
	7.20e+00	2.78e-01	35	1.68e+00	1.19e+00	31	(M-H)
	1.14e+01	3.86e-01	35	2.31e+00	7.52e+00	40	(E-L)
	7.20e+00	2.78e-01	35	8.84e-01	2.26e+00	38	(E-H)
(1e-6,1e-6)	1.13e+05	3.88e-05	9	5.49e+04	6.72e-04	11	(M-L)
	7.06e+04	2.83e-05	9	5.66e+03	3.53e-04	11	(M-H)
	1.13e+05	3.88e-05	9	8.66e+04	5.92e-04	11	(E-L)
	7.06e+04	2.83e-05	9	5.66e+03	3.53e-04	11	(E-H)

Numerical example2: curved interface

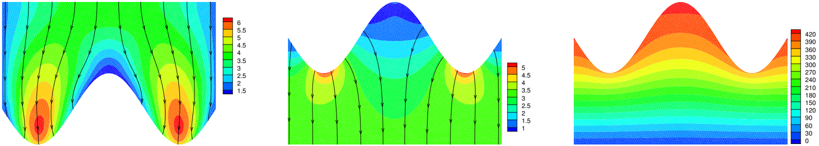


Figure: Stokes velocity field (left), Darcy velocity field (middle) and pressure (right) with $(\mu, K) = (1, 10^{-2})$.

Ongoing & future work

Modal analysis and optimized DDM with Orthodir algorithm for

- Stokes-Darcy model with Beavers-Joseph interface condition
- Navier-Stokes-Darcy model
- Dual-Porosity-Navier-Stokes model
- Helmholtz equation
- Phase field models
- ....