Exercise Sheet

For notation not defined in here, refer to the slides.

1. Stable matchings in the marriage model

Exercise 1

Consider the following class of marriage instances $\{I^k\}_{k\in\mathbb{N}}$ defined recursively as follows. I^1 has students $1, 2$, schools $1, 2$, and preference lists of students and schools defined as:

$$\left|\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right. \qquad\qquad \left|\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array}\right. .$$

For $k \geq 2$, I^k has students $1, \ldots, 2^k$, schools $1, \ldots, 2^k$, and preference lists defined as:

$$\left|\begin{array}{cc} I_a^{k-1} & I_a^{k-1} \oplus 2^{k-1} \\ I_a^{k-1} \oplus 2^{k-1} & I_a^{k-1} \end{array}\right. \qquad\qquad \left|\begin{array}{cc} I_b^{k-1} \oplus 2^{k-1} & I_b^{k-1} \\ I_b^{k-1} & I_b^{k-1} \oplus 2^{k-1}. \end{array}\right. ,$$

where I_a^{k-1} (resp., I_b^{k-1}) denotes the preference lists of students (resp., schools) in I^{k-1}, and $\oplus 2^{k-1}$ shifts all entries of a matrix by 2^{k-1}. Show that, for $k \in \mathbb{N}$, I^k has no less than 2^{2^k-1} stable matchings.

Exercise 2.0

Let M, M' be stable matchings. Define $M^\uparrow$ to be the set of pairs where each student is assigned to their favorite partner between M, M'. Show that $M^\uparrow$ is a stable matching.

Exercise 2.1

Let $\mathcal{S}$ be the set of stable matchings of a marriage instance. Show that $(\mathcal{S}, \succeq)$ is a distributive lattice.

Exercise 3

Give a polynomial-time algorithm for the `Red-Blue Unstable Matching Problem` defined below:

Given: An instance I of the marriage problem with weights w on the edges E ("blue"), plus an additional disjoint set F of edges ("red") with weights w.

Find: Among those that are stable in I, a matching M maximizing $w(M) - w(\text{edges from } F \text{ that block } M)$.

(Auxiliary facts that may help:

- *ij is in some stable matching (i.e., it is a stable pair) iff it is contained in the student-optimal stable matching, or in ρ^+ for some rotation ρ;*
- *For a student-school pair (i, j) that is not in any stable matching, there exists at most one rotation ρ and schools j', j'' so that: $(i, j') \in \rho^-, (i, j'') \in \rho^+$, and $j' >_i j >_i j''$.*
- *In each sequence of matchings obtained starting from the student-optimal stable matching and iteratively eliminating rotations until the school-optimal, we rotate all rotations.)*

2. Pareto-optimal matchings

Exercise 4.0

Show that the TTC algorithm is strategy-proof for students and outputs a matching that is Pareto-optimal for students.

Exercise 4.1

For an infinite set of values $n \in \mathbb{N}$, give a family of marriage instances with n agents such that there is a matching M that Pareto-dominates the student-optimal stable matching M_0 and moreover, for $\Theta(n)$ students i, the rank of $M(i)$ (i.e., the position of $M(i)$ in i's list) is $\Theta(n)$ positions better than the rank of $M_0(i)$.

(Hint: start from an instance I with exponentially many stable matchings, and show how to add one student and one school as to obtain an instance I', so that the school-optimal stable matching of I can be extended to the unique stable matching in I', and the student-optimal stable matching in I can be extended to a Pareto-optimal matching in I'.)

3. Popular matchings

Exercise 5

Prove that a matching M is 1-popular if and only if

- Every f-school is matched in M;
- For each student i, $M(i) \in \{f(i), s(i)\}$.

Exercise 6

Show how to modify the algorithm seen in the talk that outputs a 1-popular matching (or decide none exists) to an algorithm that outputs a 1-popular matching of maximum size.

4. Choice functions and stable matchings

Exercise 7

Show that, if choice functions of both sides of the markets are substitutable and consistent, stable matchings form a lattice.

Exercise 8

Show that, under the assumptions that choice functions of both sides of the markets are substitutable and consistent, Roth's algorithm outputs a stable matching.