Matching Theory and School Choice

Yuri Faenza

IEOR Department, Columbia University

RECAP: MARRIAGE MODEL & STABILITY

The marriage model:

1	A	B	C	D		A	4	3	2	1
2	B	D	A	C		B	3	4	1	2
3	C	B	A	D		C	2	1	4	3
4	C	D	A	B		D	1	2	3	4

A student-school pair (i, j) is called blocking for a matching M if:

$$j >_i M(i) \quad \text{and} \quad i >_j M(j).$$

RECAP: MARRIAGE MODEL & STABILITY

The marriage model:

1	A	B	C	D
2	B	D	A	C
3	C	B	A	D
4	C	D	A	B

A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

A student-school pair (i, j) is called blocking for a matching M if:

$$j >_i M(i) \quad \text{and} \quad i >_j M(j).$$

A matching without blocking pairs is called stable.

Recap: Marriage model & stability

The marriage model:

1	A	B	C	D	A	4	3	2	1
2	B	D	A	C	B	3	4	1	2
3	C	B	A	D	C	2	1	4	3
4	C	D	A	B	D	1	2	3	4

A student-school pair (i, j) is called blocking for a matching M if:

$$j >_i M(i) \quad \text{and} \quad i >_j M(j).$$

A matching without blocking pairs is called stable.

Part I: A stable matching (of maximum weight) can be computed in time polynomial in the number n of agents by exploiting the distributive lattice structure of stable matchings.

RECAP: DRAWBACKS OF THE CLASSICAL APPROACH

Stability is a very stringent condition:

- It disqualifies many “good” and “fair” solutions. (Abdulkadiroğlu, Pathak, and Roth 09) showed empirically that by forgoing stability, we can obtain a matching that is much “better” for students.
  - Part II (a): Pareto-optimality (and vNM stability) allows us to output matchings that are more favorable to students.
- It may leave many empty seats: the cardinality of a stable matching can be half that of a maximum-size matching.

1 | A B
2 | A

A | 1 2
3 | 1

Part II:

Changing the output

(b) Popularity

POPULAR MATCHINGS

Stability gives each student-school pair a veto power over matchings.

Popularity replaces the veto power with majority vote, with the goal of enlarging the set of feasible matchings.

1	A	B
2	A	

A	1	2
B	1	

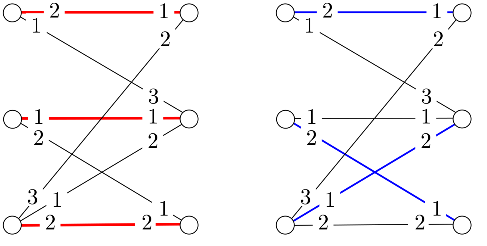
A, 1

2, B

THE $\trianglerighteq$ AND $\triangleright$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq M'$ if

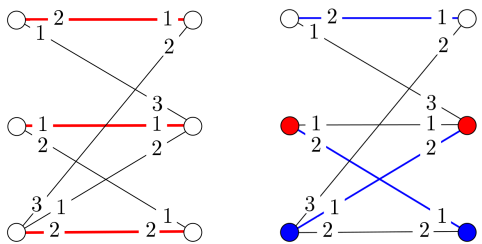
$$\underbrace{|\{x : M(x) >_x M'(x)\}|}_{\#\text{ of votes for } M} \geq \underbrace{|\{x : M'(x) >_x M(x)\}|}_{\#\text{ of votes for } M'},$$



THE $\trianglerighteq$ AND $\triangleright$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq M'$ if

$$\underbrace{|\{x : M(x) >_x M'(x)\}|}_{\#\text{ of votes for } M} \geq \underbrace{|\{x : M'(x) >_x M(x)\}|}_{\#\text{ of votes for } M'},$$



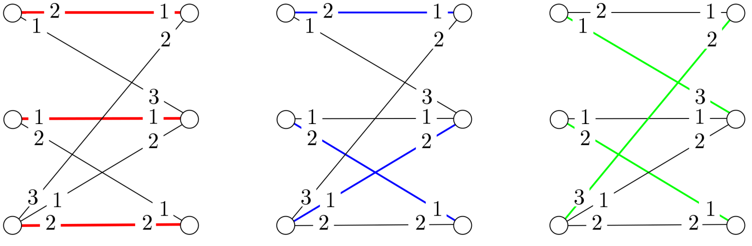
M   M'

$M \trianglerighteq M' \trianglerighteq M$

THE $\trianglerighteq$ AND $\triangleright$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq M'$ if

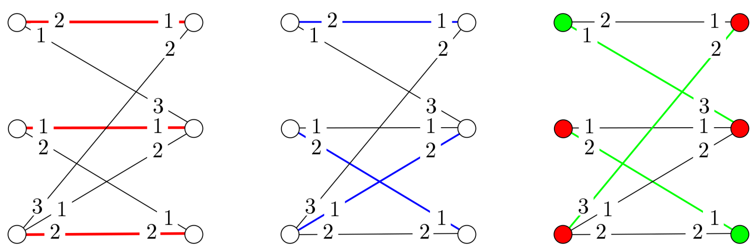
$$\underbrace{|\{x : M(x) >_x M'(x)\}|}_{\#\text{ of votes for } M} \geq \underbrace{|\{x : M'(x) >_x M(x)\}|}_{\#\text{ of votes for } M'},$$



THE $\trianglerighteq$ AND $\triangleright$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq M'$ if

$$\underbrace{|\{x : M(x) >_x M'(x)\}|}_{\#\text{ of votes for } M} \geq \underbrace{|\{x : M'(x) >_x M(x)\}|}_{\#\text{ of votes for } M'},$$



M   M''   M ▷ M''

and $M \triangleright M'$ if the inequality is strict (M beats M').

Popular matchings

A matching that is never beaten is called popular.
Equivalently, popular matchings are weak Condorcet winners.

Popular matchings:

- Shift the focus from "veto power" to "collective decision"; ✓
- contain stable matching as a special case; ✓
- allow for matchings of size larger than stable matchings. ✓

Lemma. Every stable matching is a popular matching of minimum size.

EVERY STABLE MATCHING IS A POPULAR MATCHING OF MINIMUM SIZE

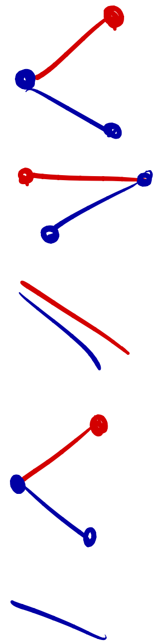
M stable matching

M' popular matching

If M was to match any node not matched by M'

then $M \rhd M'$,

contradicting popularity of M'



∀ edge of M', at least one of the endpoints prefer M to M' (or they are both indifferent between M, M')

⇓

$M \unrhd M' \unrhd M$

Structural and Algorithmic results

Bad structural news:

- The $\triangleright$ operator is not transitive.
- So there can be cycles $M \triangleright M' \triangleright M'' \triangleright M$.

Good algorithmic news! Efficient algorithms for:

- testing if a matching is popular;
- finding a popular matching of maximum size;
- finding a popular matching with / without an edge or deduce that such a matching does not exist.

Testing popularity

$\Phi(M', M) = \#\text{votes for } M' - \#\text{votes for } M \overset{?}{\leq} 0 \quad \forall M' \iff M \text{ popular}$

$$\Delta_M(i,j) = \begin{cases} -2 & \text{if } M(i) >_i j \text{ and } M(j) \geq_j i \\ +2 & \text{if } j >_i M(i) \text{ and } i >_j M(j) \\ 0 & \text{oth.} \end{cases} \quad \forall\, ij \text{ edge}$$

$$\Delta_M(i,i) = \begin{cases} -1 & \text{if } i \text{ is matched in } M \\ 0 & \text{otherwise} \end{cases} \quad \forall\, i \text{ agent}$$

$$\Phi(M', M) = \sum_{ij \in M'} \Delta_M(i,j) + \sum_{i \text{ unmatched in } M'} \Delta_M(i,i) \overset{?}{\leq} 0 \quad \forall \text{ MATCHINGS } M'$$

$G \to \tilde{G}$

$E \to \tilde{E}$

THE INCIDENCE MATRIX OF $\tilde{G}$ IS TU

$$\boxed{\begin{aligned} \max \sum_{ij \in \tilde{E}} \Delta_M(i,j)\, x_{ij} &\overset{?}{=} 0 \\ x(\tilde{\delta}(i)) &= 1 \quad \forall \text{ agent } i \\ 0 \leq x &\leq 1 \end{aligned}}$$

Finding a popular matching of maximum cardinality

1	A	B		
2	A	B	C	
3	A	B		
4	A	B	D	

A	2	4	1	3
B	4	2	3	1
C	2			
D	4			

Hardness Results

It is NP-Hard to:

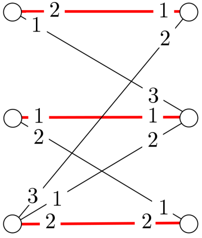
- find a popular matching with / without two edges;
- find a popular matching of max weight (assuming $w \geq 0$)
  – also, hard to approximate to a factor better than 2 (this is tight);
- find a popular matching of min weight (assuming $w \geq 0$)
  – also, inapproximable up to any factor.

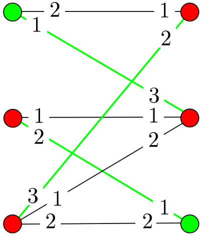
A Relaxation. How about relaxing popularity to slight unpopularity for the sake of tractability of the above problems?

Unpopularity factor and quasi-popular matchings

The unpopularity factor of a matching M is:

$$u(M) := \max_{M'} \frac{\#\text{ of votes for } M'}{\#\text{ of votes for } M}.$$





Observation. *M popular* $\Leftrightarrow u(M) \leq 1$.

Call M quasi-popular if $u(M) \leq 2$.

Quasi-popular matchings and Optimality

Optimizing over the set of matchings M with $u(M) \leq 1$ (popular) is hard, maybe optimize over the set of matchings M with $u(M) \leq 2$ (quasi-popular)?

QUASI-POPULAR MATCHINGS AND OPTIMALITY

Optimizing over the set of matchings M with $u(M) \leq 1$ (popular) is hard, maybe optimize over the set of matchings M with $u(M) \leq 2$ (quasi-popular)?

THEOREM. *The min-weight quasi-popular matching problem is NP-hard to approximate up to any factor.*

THEOREM. *The popular and the quasi-popular matching polytopes have at least near-exponential extension complexity.*

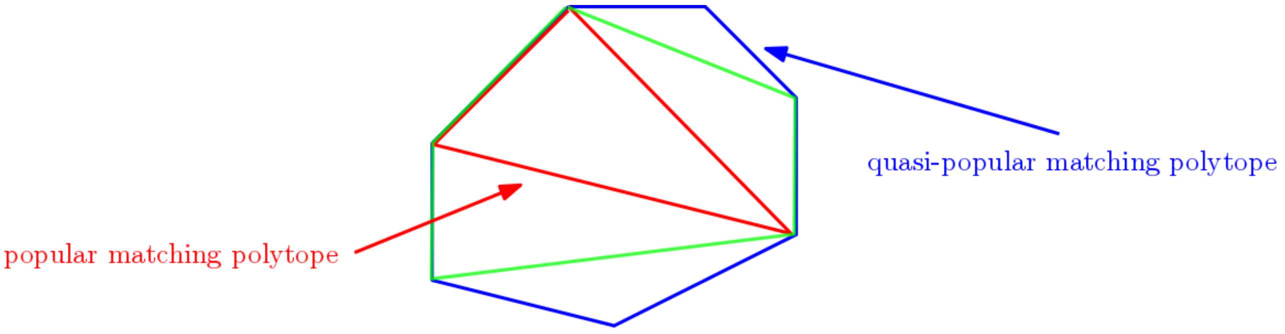
A Bicriteria Theorem

We know the min-weight popular matching problem and the min-weight quasi-popular matching problem are NP-hard to approximate up to any factor.

Bicriteria Theorem.

Given weights on the edges, we can find efficiently a quasi-popular matching with weight at most that of a min-weight popular matching.

The techniques

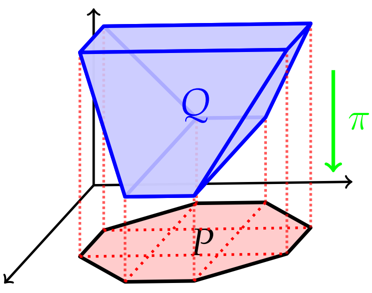


- Let $\mathcal{P}_G$ be the popular matching polytope.
- Let $\mathcal{Q}_G$ be the quasi-popular matching polytope.
- We show an integral polytope $\mathcal{C}$ sandwiched between $\mathcal{P}_G$ and $\mathcal{Q}_G$ such that $\mathcal{C}$ has a compact extended formulation.

Extended formulations

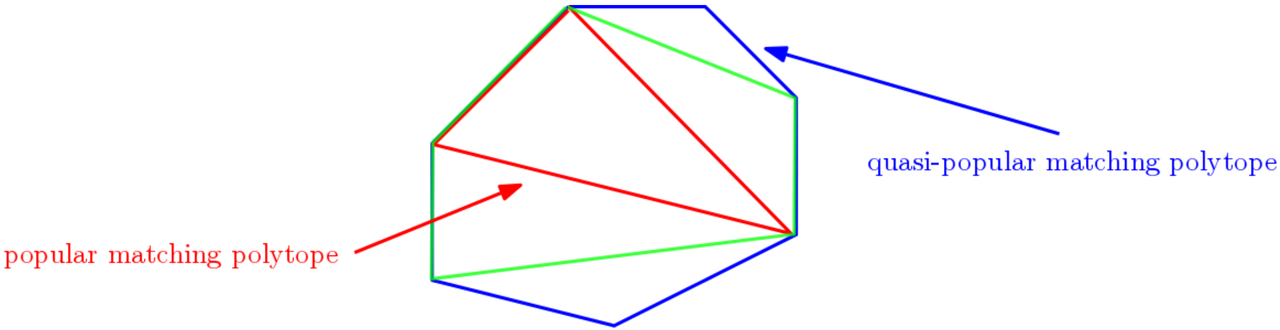
Let P, Q be polytopes.

- If there is an affine map π such that $\pi(Q) = P$, then Q is an extension of P.
- $Ax \leq b$ such that $Q = \{x : Ax \leq b\}$ is an extended formulation for P;
- It is compact if A has poly(d) rows, with $d =$ dimension of P.



$$\max_{x \in P} c^T x = \max_{(x,y) \in Q} c^T x + 0^T y$$

The techniques



- Let $\mathcal{P}_G$ be the popular matching polytope.
- Let $\mathcal{Q}_G$ be the quasi-popular matching polytope.
- We show an integral polytope $\mathcal{C}$ sandwiched between $\mathcal{P}_G$ and $\mathcal{Q}_G$ such that $\mathcal{C}$ has a compact extended formulation.
- Optimizing over $\mathcal{C}$ (using an LP algorithm) leads to the main result.

Open problems

- Can we obtain a polynomial-time bicriteria algorithm if we allow only matchings M with $u(M) < 2$?
  - In particular, can we get arbitrarily close to popularity?
- Are there efficient bicriteria approximation algorithms for other hard matching problems under preferences?

One-sided popularity

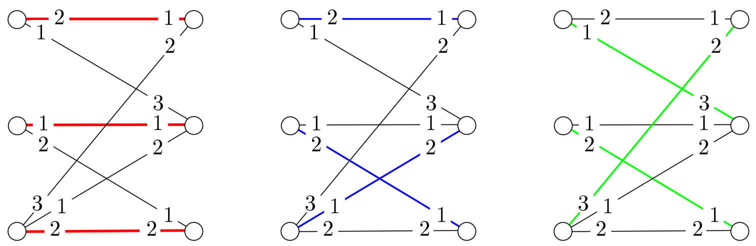
Stability is a symmetric concept; (student-) Pareto-optimality favours students over schools;

Popularity is also a symmetric concept; can we “skew it” towards students?

THE $\trianglerighteq^1$ AND $\triangleright^1$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq^1 M'$ if

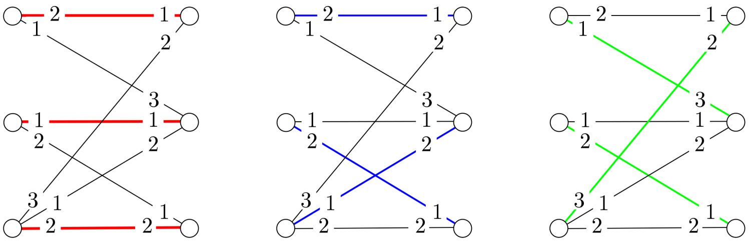
$$\underbrace{|\{x \text{ student} : M(x) >_x M'(x)\}|}_{\#\text{ of students votes for } M} \geq \underbrace{|\{x \text{ student} : M'(x) >_x M(x)\}|}_{\#\text{ of students votes for } M'},$$



THE $\trianglerighteq^1$ AND $\triangleright^1$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq^1 M'$ if

$$\underbrace{|\{x \text{ student} : M(x) >_x M'(x)\}|}_{\#\text{ of students votes for } M} \geq \underbrace{|\{x \text{ student} : M'(x) >_x M(x)\}|}_{\#\text{ of students votes for } M'},$$

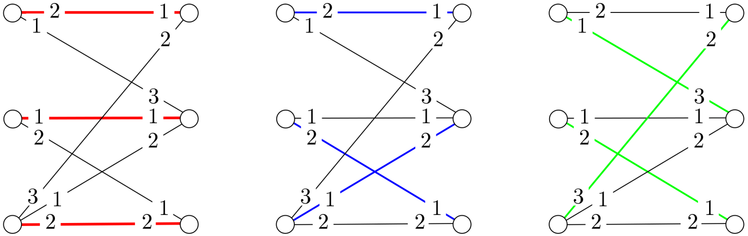


and $M \triangleright M'$ if the inequality is strict (M student-beats M').

THE $\trianglerighteq^1$ AND $\triangleright^1$ OPERATORS

Given two matchings M, M' in a marriage instance, we write $M \trianglerighteq^1 M'$ if

$$\underbrace{|\{x \text{ student} : M(x) >_x M'(x)\}|}_{\#\text{ of students votes for } M} \geq \underbrace{|\{x \text{ student} : M'(x) >_x M(x)\}|}_{\#\text{ of students votes for } M'},$$



and $M \triangleright M'$ if the inequality is strict (M student-beats M').

A matching that is never student-beaten is called 1-popular.

About 1-popular matchings

1 C   1 A
2 A   D   2 B
3 B   3 C

A 1-popular matching may not exist:

1	A	B	C
2	A	B	C
3	A	B	C

Some facts

For each student i, add a school $\ell(i)$ ("last resort") that:

- forms a feasible pair with i only;
- appears last in i's preference list.

$\Rightarrow$ we can restrict to perfect matchings.

For a student i, let $f(i)$ be the school that ranks first in i's list.
A school j is an f-school if $j = f(i)$ for some i.

1	A	C	B	$\ell(1)$
2	A	D	B	$\ell(2)$
3	A	B		$\ell(3)$
4	B	C	D	$\ell(4)$

f-schools = A, B

SOME FACTS

For each student i, add a school $\ell(i)$ ("last resort") that:

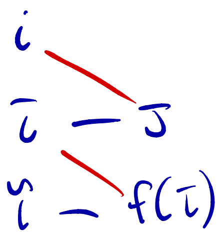
- forms a feasible pair with i only;
- appears last in i's preference list.

$\Rightarrow$ we can restrict to perfect matchings.

For a student i, let $f(i)$ be the school that ranks first in i's list.

A school j is an f-school if $j = f(i)$ for some i.

Lemma. Let M be a popular matching and j an f-school. Then j is matched in M and $f(M(j)) = j$.



$j \neq f(\bar{i})$

$j = f(i)$

A characterization of 1-popular matchings

Lemma. Let M be a 1-popular matching and j an f-school. Then j is matched in M and $f(M(j)) = j$.

For a student i, let $s(i)$ be the first school in i's list that is not an f-school.

1	A	C	B	$\ell(1)$
2	A	D	B	$\ell(2)$
3	A	B		$\ell(3)$
4	B	C	D	$\ell(4)$

f-schools = A, B

A characterization of 1-popular matchings

Lemma. Let M be a 1-popular matching and j an f-school. Then j is matched in M and $f(M(j)) = j$.

For a student i, let $s(i)$ be the first school in i's list that is not an f-school.

1	A	C	B	$\ell(1)$
2	A	D	B	$\ell(2)$
3	A	B		$\ell(3)$
4	B	C	D	$\ell(4)$

Lemma. Let M be a 1-popular matching. Then $M(i) \in \{f(i), s(i)\}$ for each student i.

Theorem. A matching M is 1-popular if and only if

- Every f-school is matched in M;
- For each student i, $M(i) \in \{f(i), s(i)\}$.

Exercise 7. Prove the theorem above.

Finding a 1-popular matching

To find a 1-popular matching, one needs to find a matching M such that:

- Every f-school is matched in M;
- For each student i, $M(i) \in \{f(i), s(i)\}$;

or conclude that no such matching exists.

This can be done with standard algorithms. With some care, one deduces:

Theorem. Let I be a marriage instance with n agents and E pairs. In time $O(n + |E|)$ one can decide if 1-popular matchings exist in I and, if they do, find one.

Exercise 8. Give an efficient algorithm to find a 1-popular matching of maximum size.

Highlights from Part II (b)

- Every stable matching is popular, but popular matchings may have larger size than stable matchings;
- Some popular matchings can be found efficiently; however, weighted popular matching problems are hard.
- Quasi-popularity can be used to obtain bicriteria approximations efficiently.
- 1-sided popularity allows to skew popularity to favor students.

Bibliography for part II (b) (Popularity)

Properties of popular matchings

- Gärdenfors, Peter. Match making: assignments based on bilateral preferences. Behavioral Science 20.3 (1975): 166-173.
- Biró, Péter, Robert W. Irving, and David F. Manlove. Popular matchings in the marriage and roommates problems. Proceedings of CIAC 2010.

Maximum-size popular matching, popular edge problem and related results

- Huang, Chien-Chung, and Telikepalli Kavitha. Popular matchings in the stable marriage problem. Information and Computation 222 (2013): 180-194.
- Kavitha, Telikepalli. A size-popularity tradeoff in the stable marriage problem. SIAM Journal on Computing 43.1 (2014): 52-71.
- Cseh, Ágnes, and Telikepalli Kavitha. Popular edges and dominant matchings. Mathematical Programming 172 (2018): 209-229.
- Huang, Chien-Chung, and Telikepalli Kavitha. Popularity, mixed matchings, and self-duality. Mathematics of Operations Research 46.2 (2021): 405-427.

Bibliography for part II (Popularity), continued

Hardness results

- Faenza, Y., Kavitha, T., Powers, V., & Zhang, X. (2019). Popular matchings and limits to tractability. Proceedings of SODA'19.
- Gupta, S., Misra, P., Saurabh, S., & Zehavi, M. (2021). Popular matching in roommates setting is NP-hard. ACM Transactions on Computation Theory (TOCT), 13(2), 1-20.
- Cseh, Á., Faenza, Y., Kavitha, T., & Powers, V. (2022). Understanding popular matchings via stable matchings. SIAM Journal on Discrete Mathematics, 36(1), 188-213.

Approximate popularity

- Faenza, Yuri, and Telikepalli Kavitha. Quasi-popular matchings, optimality, and extended formulations. Mathematics of Operations Research 47.1 (2022): 427-457.

One-sided popularity

- Abraham, D. J., Irving, R. W., Kavitha, T., & Mehlhorn, K. (2007). Popular matchings. SIAM Journal on Computing, 37(4), 1030-1045.

Content

Part I: The classical model (Gale and Shapley, 62):

- Structural and algorithmic properties of stable matchings in the marriage model;
- Impact for school choice.
- Drawbacks.

Part II: Beyond (Gale and Shapley, 62), changing the output:

(a) Pareto-optimality & von Neumann-Morgenstern Stability;

(b) Popularity;

Part III: Beyond (Gale and Shapley, 62), changing the input:

- Polytopes ~~Ties~~, choice functions, and applications to school choice.

Part III:
Changing the input

$C_a(S)$

Choice functions

S

,

S

- For each agent a and sets S of agents from the opposite set,

$$C_a(S) \subseteq S$$

$C_a(S)$

denotes the set of agents that a would choose from S.

THE SC-MODEL

Finite sets of firms F and workers W.

For each $f \in F$, $S \subseteq W(f)$, choice function $\mathcal{C}_f : 2^{W(f)} \to 2^{W(f)}$ satisfies:

- $\mathcal{C}_f(S) \subseteq S$;
- Substitutability: $b \in \mathcal{C}_f(S), T \subseteq S \;\Rightarrow\; b \in \mathcal{C}_f(T \cup \{b\})$;
- Consistency: $\mathcal{C}_f(S) \subseteq T \subseteq S \Rightarrow \mathcal{C}_f(S) = \mathcal{C}_f(T)$.

Similar properties apply to $\mathcal{C}_w$ for every $w \in W$.

$M \subseteq W \times F$ is a stable matching if:

- $M(w) \subseteq F(w)\ \forall w \in W$;
- $M(f) \subseteq W(f)\ \forall f \in F$;
- individual rationality: $\mathcal{C}_a(M(a)) = M(a)\ \forall a \in F \cup W$;
- no blocking pairs: $w \in \mathcal{C}_f(M(f) \cup \{w\}), f \in \mathcal{C}_w(M(w) \cup \{f\}) \;\Rightarrow\; wf \in M$.

Matchings and choice functions

Models with substitutable and consistent choice functions arise in:

- Staffing problems
- United Kingdom residency matching market
- Course allocations
- Achieving diversity in school cohorts (see later in the talk).

Goal

Can we extend the algorithmic results on stable matching in the marriage model to this more general setting?

X_f^s = workers that have not rejected f in rounds $1, \dots, s-1$

ROTH'S ALGORITHM

Algorithm 1 Firm-proposing DA algorithm for an instance $(F, W, \mathcal{C}_F, \mathcal{C}_W)$.

initialize the step count $s \leftarrow 0$
for each firm f **do** initialize $X_f^{(s)} \leftarrow W(f)$ **end for**
repeat
    for each worker w **do**
        $\boxed{X_w^{(s)}} \leftarrow \{f \in F : w \in \mathcal{C}_f(X_f^{(s)})\}$
        $\boxed{Y_w^{(s)}} \leftarrow \mathcal{C}_w(X_w^{(s)})$
    end for
    for each firm f **do**
        update $X_f^{(s+1)} \leftarrow X_f^{(s)} \setminus \{w \in W : f \in X_w^{(s)} \setminus Y_w^{(s)}\}$
    end for
    update the step count $s \leftarrow s + 1$
until $X_f^{(s)} = X_f^{(s-1)}$ for every firm f
Output: matching $\overline{\mu}$ with $\overline{\mu}(w) = Y_w^{(s-1)}$ for every worker w

ALL FIRMS PROPOSING TO WORKER W IN ROUND S

ALL FIRMS THAT WORKER W TEMPORARILY ACCEPT IN ROUND S

Roth's algorithm

Algorithm 1 Firm-proposing DA algorithm for an instance $(F, W, \mathcal{C}_F, \mathcal{C}_W)$.

initialize the step count $s \leftarrow 0$
for each firm f **do** initialize $X_f^{(s)} \leftarrow W(f)$ **end for**
repeat
    for each worker w **do**
        $X_w^{(s)} \leftarrow \{f \in F : w \in \mathcal{C}_f(X_f^{(s)})\}$
        $Y_w^{(s)} \leftarrow \mathcal{C}_w(X_w^{(s)})$
    end for
    for each firm f **do**
        update $X_f^{(s+1)} \leftarrow X_f^{(s)} \setminus \{w \in W : f \in X_w^{(s)} \setminus Y_w^{(s)}\}$
    end for
    update the step count $s \leftarrow s + 1$
until $X_f^{(s)} = X_f^{(s-1)}$ for every firm f
Output: matching $\overline{\mu}$ with $\overline{\mu}(w) = Y_w^{(s-1)}$ for every worker w

Define $M \succeq M'$ if, for every firm $f \in F$, $\mathcal{C}_f(M(f) \cup M'(f)) = M(f)$.

Theorem. Roth's algorithm outputs a stable matching that is firm-optimal.

How about a stable matching of minimum weight?

We don't know...maybe not. Reason for being skeptical:

- The stable matchings in the `SC`-model form a lattice which however is not, in general, distributive.

Restricting to the QF-model

Finite sets of firms F and workers W.

For each $f \in F$, $S \subseteq W(f)$, choice function $\mathcal{C}_f : 2^{W(f)} \to 2^{W(f)}$ satisfies:

- $\mathcal{C}_f(S) \subseteq S$;
- Substitutability: $b \in \mathcal{C}_f(S), T \subseteq S \;\Rightarrow\; b \in \mathcal{C}_f(T \cup \{b\})$;
- Consistency: $\mathcal{C}_f(S) \subseteq T \subseteq S \Rightarrow \mathcal{C}_f(S) = \mathcal{C}_f(T)$.
- Quota-filling: $|C_f(S)| = \min\{|S|, q_f\}$.

Similar properties apply to $\mathcal{C}_w$ for every $w \in W$.

Theorem. The set of stable matchings in the QF-model forms a distributive lattice.

ALGORITHMIC AND LP RESULTS

Theorem. There is an algorithm that finds a stable matching of maximum weight in the `QF-model` in $O(|F|^3|W|^3\texttt{oracle-call})$ time.

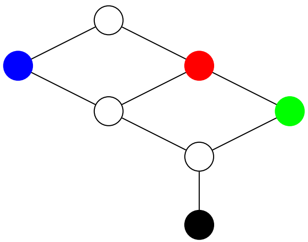
- where `oracle-call`= time to compute $C_a(S)$ $\forall a, S$.

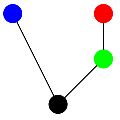
Theorem. There exists an LP formulation for the convex hull of stable matchings in the `QF-model` with $O(|F|^2|W|^2)$ constraints.

Birkhoff's representation theorem

Birkhoff's representation theorem. Let $(L, \succeq)$ be a distributive lattice. There exists $R \subseteq L$ and a bijection ψ between elements of L and upper closed sets of $(R, \succeq)$.

- $S \subseteq R$ is a upper closed set of $(R, \succeq)$ if $e \in R, e' \succeq e \Rightarrow e' \in S$;
- $(R, \succeq)$ is the representation poset;
- Often, $|R| \ll |L|$.





Back to the roadmap for [OPT-SM]

(a) Transform [OPT-SM] to a linear optimization problem over the UCS of the representation poset $(R, \succeq)$ of $(\mathcal{S}, \succeq)$.

$$\begin{aligned}\min_{M \in \mathcal{S}} w^T M &= \min_{\{\rho_1,\ldots,\rho_k\} \text{ UCS of } (R,\succeq)} w(M^0/\rho_1/\ldots/\rho_k) \\ &= w(M^0) + \min_{\{\rho_1,\ldots,\rho_k\} \text{ UCS of } (R,\succeq)} \textstyle\sum_{j=1}^k w(\rho_j) \\ &= w(M^0) + \min_{X \text{ UCS of } (R,\succeq)} \textstyle\sum_{\rho \in X} w(\rho)\end{aligned}$$

(b) Show that $|R| = poly(n)$.

Lemma. $(i, j) \in \rho^+$ (resp. ρ^-) for at most one rotation ρ
$\Rightarrow\ |R| = O(n^2)$.

(c) Find an UCS of minimum weight in $(R, \succeq)$ in time poly($|R|$).

$S_3 = \{1,2,4\}$ $\chi^{S_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ $\psi(S_3) = y_2$ $\chi^{\psi(S_3)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Affinely representable lattices

Let $(L, \succeq)$ be a lattice over a ground set E and $(R, \succeq)$ its representation poset. $(R, \succeq)$ affinely represents L if there exist x_0, A such that

$$\chi^S = x_0 + A\chi^{\psi(S)}$$

for each $S \in L$.

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{tikzcd}[column sep=tiny] & \boxed{S_1 = \{1,2\}} \arrow[dl, dash] \arrow[dr, dash] & \\ \boxed{S_2 = \{1,3\}} \arrow[dr, dash] & & \boxed{S_3 = \{1,2,4\}} \arrow[dl, dash] \\ & \boxed{S_4 = \{1,3,4\}} & \end{tikzcd}$$

y_1 y_2

$\psi : \psi(S_1) = \emptyset,\ \psi(S_2) = \{y_1\},\ \psi(S_3) = \{y_2\},\ \psi(S_4) = \{y_1, y_2\}$

Affine representation: $\chi^S = \chi^{S_1} + A\chi^{\psi(S)}$ where $A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

AFFINE REPRESENTABILITY $\Rightarrow$ ALGORITHMIC TRACTABILITY

Lemma. Let $(L, \succeq)$ be a lattice over a ground set E and $(R, \succeq)$ its representation poset. Assume R affinely represents L and we are given x_0, A such that

$$\chi^S = x_0 + A\chi^{\psi(S)}$$

for each $S \in L$. Let $w \in \mathbb{Z}^E$. Then

$$\min_{S \in L} w^\intercal \chi^S$$

can be solved in time polynomial in $|R|$.

Affine representability $\Rightarrow$ algorithmic tractability

Lemma. Let $(L, \succeq)$ be a lattice over a ground set E and $(R, \succeq)$ its representation poset. Assume R affinely represents L and we are given x_0, A such that

$$\chi^S = x_0 + A\chi^{\psi(S)}$$

for each $S \in L$. Let $w \in \mathbb{Z}^E$. Then

$$\min_{S \in L} w^\intercal \chi^S$$

can be solved in time polynomial in $|R|$.

Proof.

$$\min_{S \in L} w^\intercal \chi^S = \min_{U \in \mathcal{U}(\mathcal{B})} w^\intercal (x_0 + A\chi^U) = w^\intercal x_0 + \min_{U \in \mathcal{U}(\mathcal{B})} (w^\intercal A)\chi^U.$$

$\min_{U \in \mathcal{U}(\mathcal{B})} (w^\intercal A)\chi^U$ polynomially reduces to min (s,t)-cut. $\square$

Affine representability $\Rightarrow$ polyhedral description

Lemma. Let $(L, \succeq)$ be a lattice over a ground set E and $(R, \succeq)$ its representation poset. Assume R affinely represents L, i.e., there exists x_0, A such that

$$\chi^S = x_0 + A\chi^{\psi(S)}$$

for each $S \in L$. There is an LP formulation (possibly with additional variables) for $(\chi^S : S \in L)$ of size polynomial in $|R|$. If moreover, A has full column rank, then this formulation does not have any additional variable.

Affine representability $\Rightarrow$ polyhedral description

Lemma. Let $(L, \succeq)$ be a lattice over a ground set E and $(R, \succeq)$ its representation poset. Assume R affinely represents L, i.e., there exists x_0, A such that

$$\chi^S = x_0 + A\chi^{\psi(S)}$$

for each $S \in L$. There is an LP formulation (possibly with additional variables) for $(\chi^S : S \in L)$ of size polynomial in $|R|$. If moreover, A has full column rank, then this formulation does not have any additional variable.

Proof.

The *order polytope* associated with poset $(R, \succeq)$

$$\mathbb{O}(R, \succeq) := \{y \in [0,1]^R : y_i \geq y_j, \ \forall i, j \in R \text{ s.t. } i \succeq j\}.$$

is the convex hull of the closed sets of $(R, \succeq)$.

Then

$$\operatorname{conv}(\chi^S : S \in L) = \{x_0\} \oplus A \cdot \mathbb{O}(R, \succeq) \qquad \square$$

Back to stable matchings

Thm 1. There is an algorithm that finds a stable matching of maximum weight in the `QF-model` in $O(|F|^3|W|^3\texttt{oracle-call})$ time.

Proof. Same approach as in the marriage model.

Thm 2. There exists an LP formulation for the convex hull of stable matchings in the `QF-model` with $O(|F|^2|W|^2)$ constraints.

Proof.

Thm 1 above $\oplus$

Affine representability $\Rightarrow$ polyhedral description $\oplus$

A full column rank $\square$

A direct description of the stable matching polytope

Theorem. Let I be a marriage instance, $\mathbb{P}(I)$ the associated matching polytope. Then:

$$\begin{array}{rrcll} \mathbb{P}(I) = \{x : & x(\delta(v)) & \leq & 1 & \text{for each agent } v \\ & x_{i,j} + \sum_{\bar{j}:\bar{j}>_i j} x_{i,\bar{j}} + \sum_{\bar{i}:\bar{i}>_j i} x_{\bar{i},j} & \geq & 1 & \text{for feasible pairs } (i,j) \\ & x & \geq & 0\}. & \end{array}$$

Oracle model vs explicit preference representation

Computational advantage of Gale-Shapley's algorithm:

- Agents need to communicate with the central planner only once.

Conversely, in the oracle model:

- A central planner needs to ask agent a for $\mathcal{C}_a(S)$ multiple times.

Is there a compact representation of choice functions for our models?

Theorem. We cannot compactly represent choice functions in the `QF-model`.

- there are doubly-exponentially many such functions.

NYC specialized high schools

- The admission to 8 specialized public high schools in NYC is run independently from the rest.
- Admissions to the NYC specialized high school is uniquely determined by score in the SHSAT exam.
- But there is disparity in opportunity!
- Nowadays, roughly 20% of the seats in each school are reserved to disadvantaged students (this admission path requires 3 weeks of additional classes).

PAST TENSE

Segregation Has Been the Story of New York City's Schools for 50 Years

Low black and Hispanic enrollment at Stuyvesant High School has reignited a debate about how to finally integrate the city's schools.

The New York Times

The Test That Changed Their Lives

The SHSAT, a grueling one-day exam, is considered a golden ticket into one of New York City's eight prestigious schools, if you score high enough. But the test is not perfect and has been the subject of public debate over its role in school segregation.

The New York Times

De Blasio Proposes Changes to New York's Elite High Schools

26

When we looked at the data...

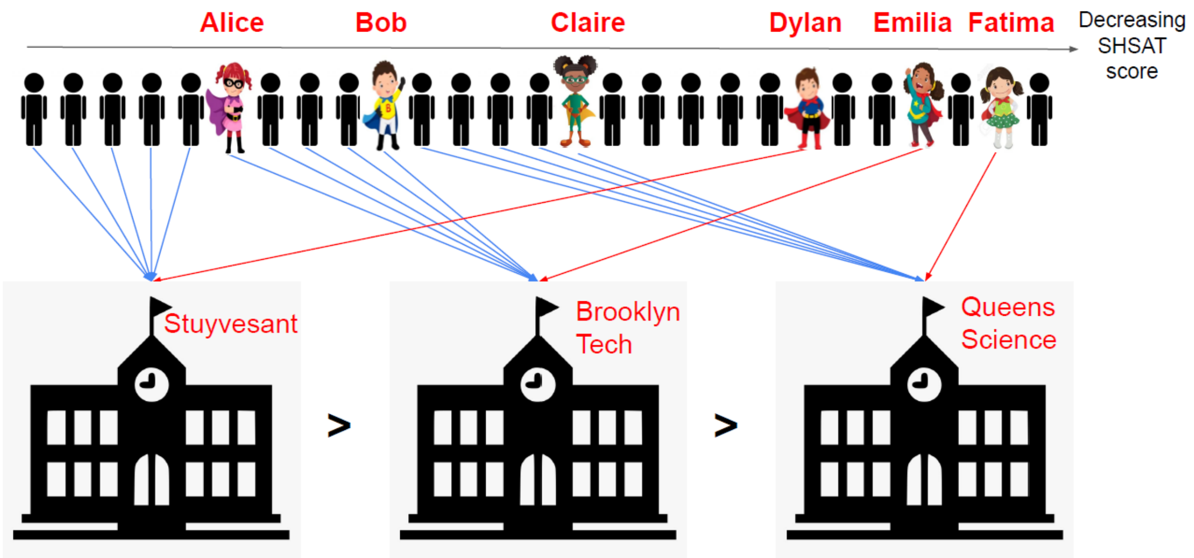
... we saw the following:

- Student A is disadvantaged, really wants to go to Stuyvesant, has a SHSAT score of 492, is admitted to Queens Science.
- Student B is disadvantaged, really wants to go to Stuyvesant, has a SHSAT score of 452, is admitted to Stuyvesant.

We say that (A, Stuyvesant) form an in-group blocking pair.

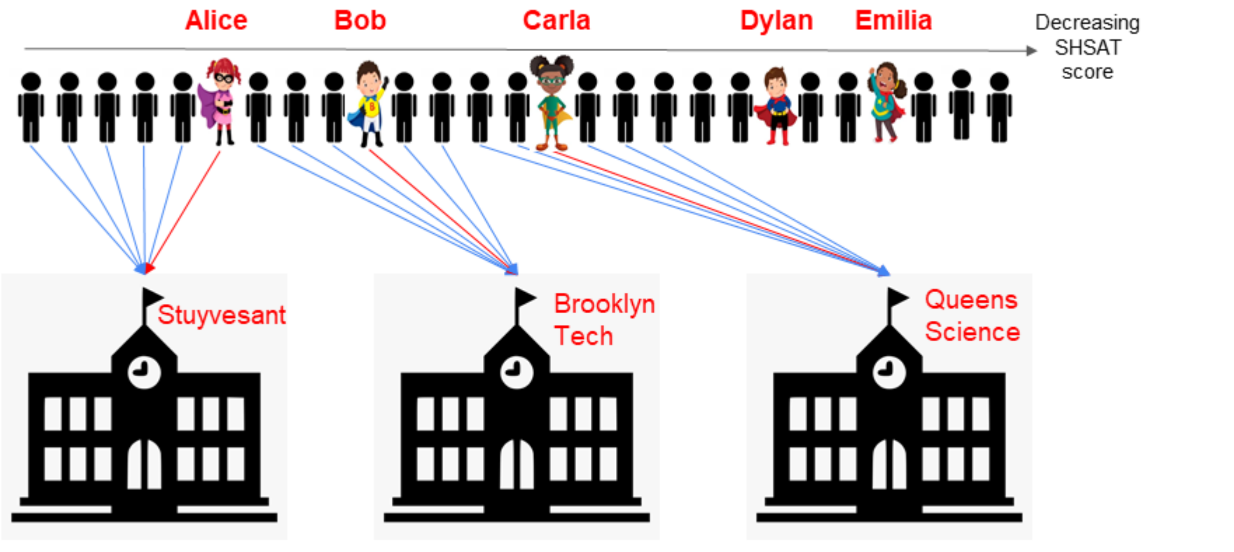
Around 30% of disadvantaged students that received an offer are part of an in-group blocking pair. Why is this the case?

The current mechanism



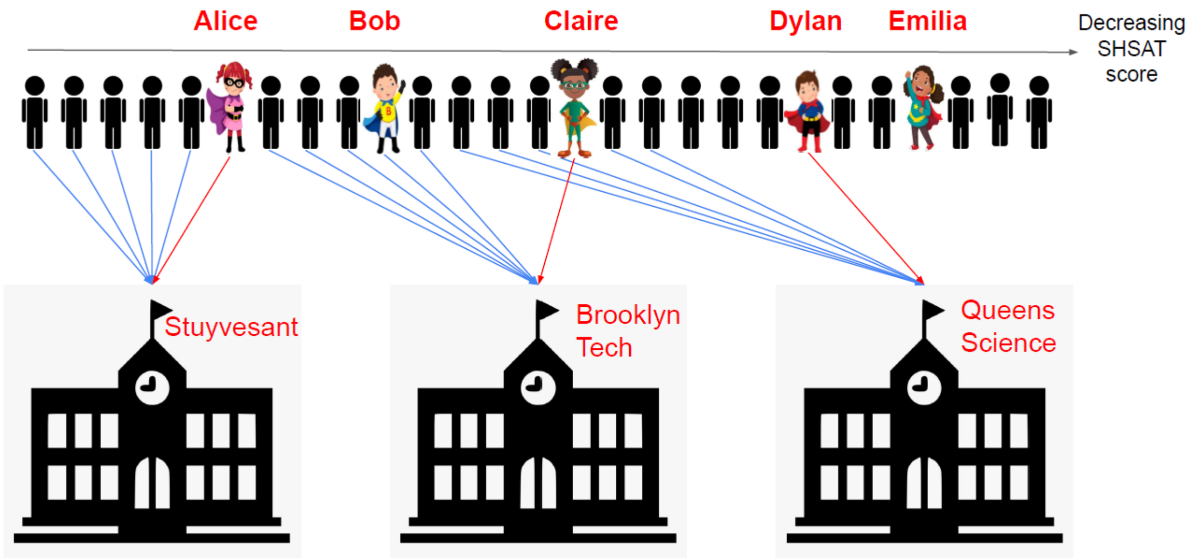
Assumption: All students list schools in the order 1,2,3; Each school has ***5 regular seats*** *and* ***1 DP seat***

The Minority Reserve mechanism



Assumption: All students list schools in the order 1,2,3; Each school has ***5 regular seats*** *and* ***1 DP seat****.*

The Joint Seat Allocation mechanism



Assumption: All students list schools in the order 1,2,3; Each school has ***5 regular seats*** *and* ***1 DP seat****.*

Properties of MR, JSA

- Can be reformulated in terms of `qf` choice functions, such as:

$$\mathcal{C}_c^{\mathrm{MR}}(S_1) = \underbrace{\min(S_1 \cap S^m, >_c, q_c^R)}_{\texttt{reserved seats}} \;\dot\cup\; \underbrace{\min\left(S_1 \setminus S_1^R, >_c, q_c - |S_1^R|\right)}_{\texttt{remaining seats}}.$$

- Strategy-proofness;
- No in-group blocking pairs.

A comparison of MR, JSA

- In theory, there is no domination between MR and JSA for disadvantaged students.
- On data, we were however seeing JSA being always weakly preferred by all disadvantaged students.
- High competitiveness hypothesis (hch) (informal)
  - `MR` assigns to each school no more disadvantaged students than the number of reserved seats the school has.
- **Theorem.** Under the hch, all disadvantaged students weakly prefer JSA to MR.

When does the hch hold?

Theorem. (informal) The hch holds whp if:

- Schools rank students following a master list and have the same quotas;
- The favorite school of each student is selected independently and uar;
- The $\approx (n \log n)$-th best advantaged student is ranked above the $\approx n$-th best disadvantaged student ($n = \#$ schools).

When does the hch hold?

Theorem. (informal) The hch holds whp if:

- Schools rank students following a master list and have the same quotas;
- The favorite school of each student is selected independently and uar;
- The $\approx (n \log n)$-th best advantaged student is ranked above the $\approx n$-th best disadvantaged student ($n = \#$ schools).

Theorem. (informal) The hch holds whp if:

- Schools rank students following a master list based on a students' perceived potential and have the same quotas;
- The favorite school of each student is selected independently and uar;
- Students have a perceived potential sampled from a Gaussian, with the mean for disadvantaged students being slightly less than the advantaged students.

The "slightly less" is largely satisfied, e.g., by the differences in test scores among the two groups of students in NYC.

Highlights for Part III

- Choice functions models allow us to address more complex preference settings.
- Tractability is guaranteed in oracle models, or if we drastically restrict the class of choice functions under consideration.
- These models can be applied to relevant real-world problems, and there is room for meaningful theory in applications!
- Affinely representable lattices may be impactful beyond stable matching problems.

Bibliography for Part III

Lattice structure of stable matchings in choice function models.

- Alkan, Ahmet. A class of multipartner matching markets with a strong lattice structure. Economic Theory 19 (2002): 737-746.
- Blair, Charles. The lattice structure of the set of stable matchings with multiple partners. Mathematics of operations research 13.4 (1988): 619-628.

Algorithms for stable matchings in choice function models.

- Faenza, Yuri, and Xuan Zhang. Affinely representable lattices, stable matchings, and choice functions. Mathematical Programming (2022): 1-40.
- Roth, Alvin E. Stability and polarization of interests in job matching."Econometrica: Journal of the Econometric Society (1984): 47-57.

Order polytope

- Stanley, Richard P. Two poset polytopes. Discrete & Computational Geometry 1.1 (1986): 9-23.

Bibliography for Part III, continued

The stable matching polytope

- Faenza, Yuri, and Xuan Zhang. Affinely representable lattices, stable matchings, and choice functions. Mathematical Programming (2022): 1-40.
- Roth, Alvin E., Uriel G. Rothblum, and John H. Vande Vate. Stable matchings, optimal assignments, and linear programming. Mathematics of Operations Research 18.4 (1993): 803-828.
- Vande Vate, John H. Linear programming brings marital bliss. Operations Research Letters 8.3 (1989): 147-153.

Counting choice functions

- Echenique, Federico. Counting combinatorial choice rules. Games and Economic Behavior 58.2 (2007): 231-245..
- Vuorinen, Aapeli. On quota-filling choice functions. Manuscript, 2023+.

Disadvantaged students in NYC high school and the high competitiveness hypothesis

- Faenza, Yuri, Swati Gupta, and Xuan Zhang. Discovering Opportunities in New York City's Discovery Program: Disadvantaged Students in Highly Competitive Markets. arXiv preprint arXiv:2203.00544 (2022).

Final comments

- Stable matchings are relevant in theory and practice.
- Nice theories have been developed, with impactful applications.
- The more we find out, the more directions open up.

Thank you for your attention.