Binary Polynomial Optimization: Theory, Algorithms, and Applications

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Problem definition

- Let $V = \{1, \ldots, n\}$, let E be a set of subsets of cardinality at least two of V, and let V_1, V_2 be a partition of V. Consider the optimization problem:

$$\max\left\{\sum_{e\in E} c_e \prod_{v\in e} z_v : \; z_v \in [0,1] \; \forall v \in V_1, \; z_v \in \{0,1\} \; \forall v \in V_2\right\}.$$

  - $V_1 = \emptyset$: Pseudo-Boolean optimization, unconstrained binary polynomial optimization, unconstrained binary nonlinear optimization
  - $V_2 = \emptyset$: maximizing a multilinear function over a box

- Define $z_e := \prod_{v\in e} z_v$ for all $e \in E$:

$$\begin{aligned} \max \quad & \sum_{e\in E} c_e z_e, \\ \text{s.t.} \quad & z_e = \prod_{v\in e} z_v, \; \forall e \in E \\ & z_v \in \{0,1\}, \; \forall v \in V. \end{aligned}$$

The multilinear polytope

- We define the multilinear set as:

$$\mathcal{S} = \Big\{ z \in \{0,1\}^{|V|+|E|} : z_e = \prod_{v \in e} z_v, \forall e \in E \Big\}.$$

- Example:

$$\mathcal{S} = \Big\{ z \in \{0,1\}^8 : z_{12} = z_1 z_2, z_{24} = z_2 z_4, z_{123} = z_1 z_2 z_3, z_{134} = z_1 z_3 z_4 \Big\}.$$

- We define the multilinear polytope as the convex hull of the multilinear set:

$$\mathsf{MP} = \operatorname{conv}(\mathcal{S})$$

- If $|e| = 2$ for all $e \in E$, then MP is the Boolean quadric polytope QP (Padberg, 89) and hence the cut polytope under a bijective linear transformation.

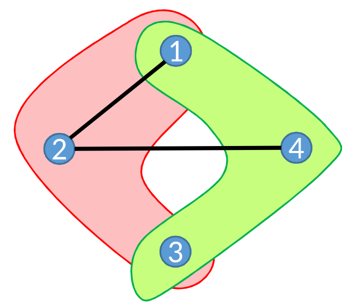
The hypergraph representation of multilinear sets

- With any hypergraph $G = (V, E)$, we associate a multilinear set $\mathcal{S}_G$ defined as:

$$\mathcal{S}_G = \{z \in \{0,1\}^d : z_e = \prod_{v \in e} z_v,\ e \in E\},$$

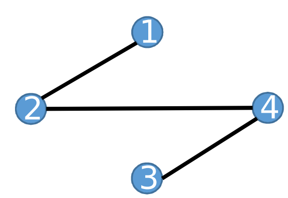
where $d = |V| + |E|$. We define $\text{MP}_G = \text{conv}(\mathcal{S}_G)$.

$$z_{12} = z_1 z_2$$
$$z_{24} = z_2 z_4$$
$$z_{123} = z_1 z_2 z_3$$
$$z_{134} = z_1 z_3 z_4$$



- For quadratic sets, we obtain the graph representation of QP_G (Padberg, 89)

$$z_{12} = z_1 z_2$$
$$z_{24} = z_2 z_4$$
$$z_{34} = z_3 z_4$$



- The rank of G is the maximum cardinality of any edge in E.

Standard linearization of multilinear sets

- Replace each multilinear term $z_e = \prod_{v \in e} z_v$, by its convex hull over the unit hypercube and use $\bigcap_i \mathrm{conv}(\mathcal{S}_i) \supseteq \mathrm{conv}(\bigcap_i \mathcal{S}_i)$ to obtain the standard linearization $\mathsf{MP}_G^{\mathsf{LP}}$ of $\mathcal{S}_G$:

$$\mathsf{MP}_G^{\mathsf{LP}} = \Big\{ z : \ z_v \leq 1, \ \forall v \in V, z_e \geq 0, \ z_e \geq \sum_{v \in e} z_v - |e| + 1, \ \forall e \in E, \\ z_e \leq z_v, \forall v \in e, \ \forall e \in E \Big\}.$$

- Existing results for the Boolean quadric polytope:
  - $\mathsf{QP}_G = \mathsf{QP}_G^{\mathsf{LP}}$ iff G is an acyclic graph (Padberg 89).
  - Let QP_G^C be polytope obtained by adding all odd cycle inequalities to $\mathsf{QP}_G^{\mathsf{LP}}$; $\mathsf{QP}_G = \mathsf{QP}_G^C$ iff G is a series-parallel graph (Barahona 86, Padberg 89).
  - Optimizing over $\mathsf{QP}_G^{\mathsf{LP}}$ and QP_G^C can be done in polynomial-time.

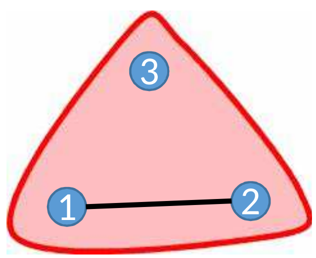
- Goal: obtaining similar results for higher degree multilinear sets in terms of easily verifiable conditions on the structure of underlying hypergraphs.

Cycles in hypergraphs

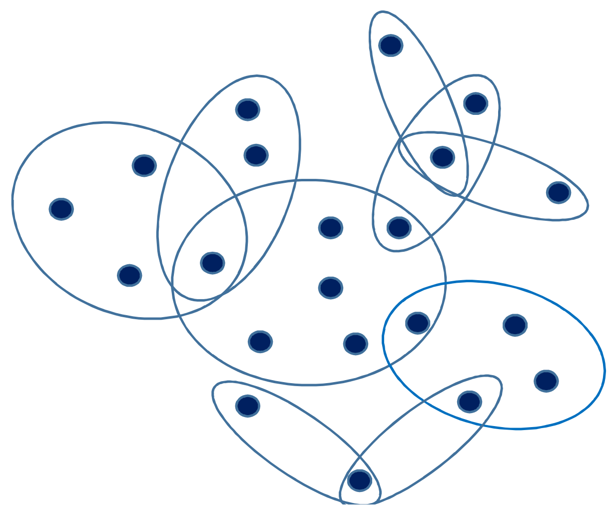
- Hypergraph acyclicity in increasing degree of generality: Berge-acyclicity, γ-acyclicity, β-acyclicity, and α-acyclicity.

- A Berge-cycle in G of length t for some $t \geq 2$, is a sequence $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ with the following properties:
  - $v_1, v_2, \ldots, v_t$ are distinct nodes of G,
  - $e_1, e_2, \ldots, e_t$ are distinct edges of G,
  - $v_i, v_{i+1} \in e_i$ for $i = 1, \ldots, t-1$, and $v_t, v_1 \in e_t$.

- A hypergraph is Berge-acyclic when it contains no Berge-cycles.



Berge-cycle:
$C = v_1, e_{12}, v_2, e_{123}, v_1$



A Berge-acyclic Hypergraph

Decomposability of multilinear sets

- Given $V' \subset V$, the section hypergraph of G induced by V' is $G' = (V', E')$, where $E' = \{e \in E : e \subseteq V'\}$.

- Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we denote by $G_1 \cup G_2$, the hypergraph $(V_1 \cup V_2, E_1 \cup E_2)$, and by $G_1 \cap G_2$, the hypergraph $(V_1 \cap V_2, E_1 \cap E_2)$.

- Let G_1, G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$. We say that $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$, if

$$\text{conv}\mathcal{S}_G = \text{conv}\mathcal{S}_{G_1} \cap \text{conv}\mathcal{S}_{G_2}.$$

- A hypergraph $G = (V, E)$ is complete if all subsets of V of cardinality at least two are in E.

- Theorem: Let G_1, G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a complete hypergraph. Then the set $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$.

The standard linearization vs. the convex hull relaxation

- Theorem: $\text{MP}_G^{\text{LP}} = \text{MP}_G$ if and only if G is a Berge-acyclic hypergraph.

- Proof sketch:

  – If G has a Berge-cycle of length two; i.e., $E(C) = \{e_1, e_2\}$ with $|e_1 \cap e_2| \geq 2$, the following is valid for $\mathcal{S}_G$:

$$\sum_{v \in e_2 \setminus e_1} z_v + z_{e_1} - z_{e_2} \leq |e_2 \setminus e_1|$$

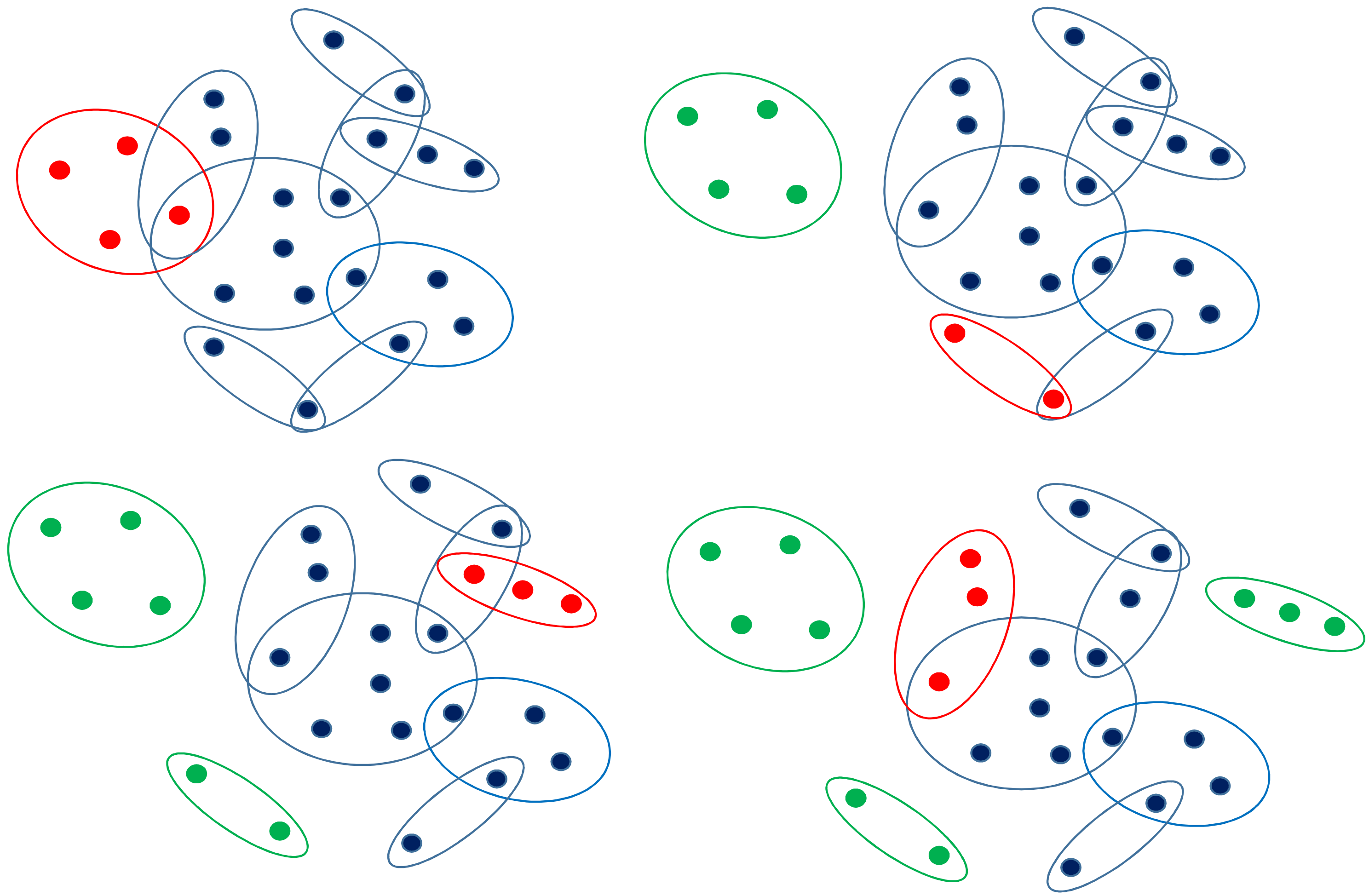
  Consider $\tilde{z}_v = 1$ for all $v \in e_2 \setminus e_1$, $\tilde{z}_v = 1/2$ for all $v \in e_1$, $\tilde{z}_v = 0$ for the remaining nodes in G, $\tilde{z}_{e_1} = 1/2$, $\tilde{z}_{e_2} = 0$, $\tilde{z}_e = 1$ for all $e \subseteq e_2 \setminus e_1$, $\tilde{z}_e = 0$ for all $e \not\subseteq e_1 \cup e_2$ and $\tilde{z}_e = 1/2$ for all remaining edges in G. $\tilde{z} \in \text{MP}_G^{\text{LP}}$. Substituting $\tilde{z}$ in the above inequality yields $|e_2 \setminus e_1| + 1/2 - 0 \not\leq |e_2 \setminus e_1|$.

  – Let C be a Berge-cycle of minimum length t, where $t \geq 3$. Since $|e_i \cap e_j| \leq 1$ for all $e_i, e_j \in E$, the subhypergraph $G_{V(C)}$ is a graph consisting of a chordless cycle. To show $\text{MP}_G \subset \text{MP}_G^{LP}$ is suffices to show that $\text{MP}_{G_{V(C)}} \subset \text{MP}_{G_{V(C)}}^{\text{LP}}$. The polytope $\text{MP}_{G_{V(C)}}$ is integral while $\text{MP}_{G_{V(C)}}^{\text{LP}}$ is not integral.

$\Rightarrow$ if G contains a Berge-cycle, we have $\text{MP}_G \subset \text{MP}_G^{\text{LP}}$

The standard linearization vs. the convex hull relaxation

- Suppose that G is a Berge-acyclic hypergraph. Then there exists an edge $\tilde{e}$ of G such that $\tilde{e} \cap \{v : \exists e \in E(G) \setminus \tilde{e}, v \in e\} = \{\tilde{v}\}$, for some $\tilde{v} \in V(G)$.

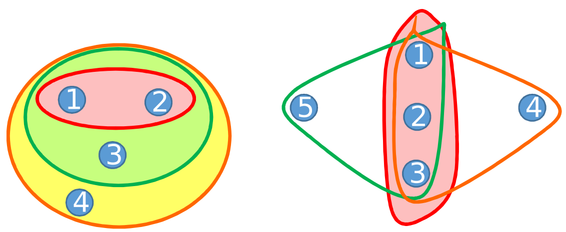


$\Rightarrow$ if G is Berge-acyclic, we have $\text{MP}_G = \text{MP}_G^{\text{LP}}$

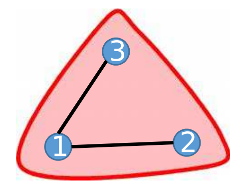
γ-acyclic hypergraphs

- A γ-cycle in G is a Berge-cycle $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ such that $t \geq 3$, and the node v_i belongs to e_{i-1}, e_i and no other e_j, for all $i = 2, \ldots, t$.

- A hypergraph is called γ-acyclic if it contains no γ-cycles.



No γ-cycles; Berge-cycles of length two and three.

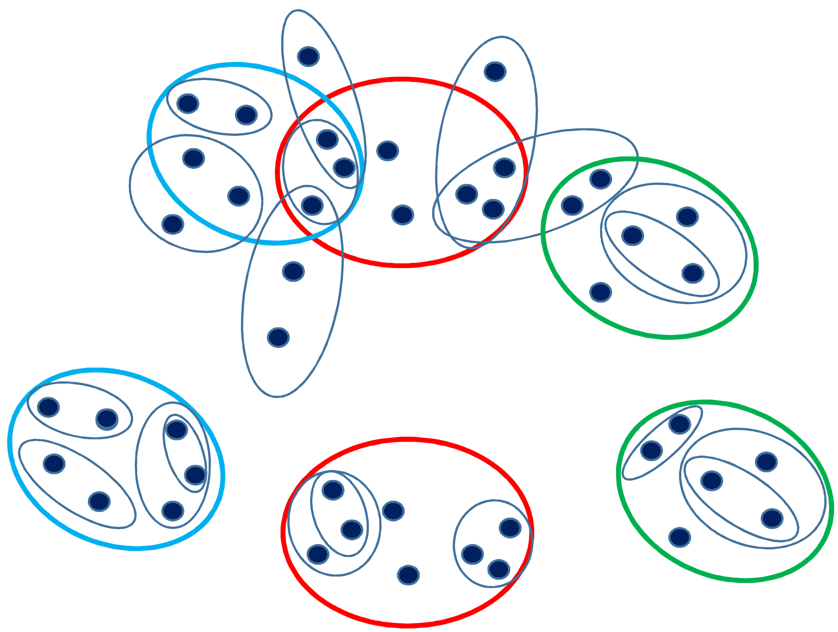


γ-cycle: $C = v_1, e_{12}, v_2, e_{123}, v_3, e_{13}, v_1$

A γ-acyclic hypergraph

γ-acyclicity and laminarity

- Given $G = (V, E)$ and $\bar{V} \subseteq V$, the subhypergraph of G induced by $\bar{V}$ is $G_{\bar{V}}$ with node set $\bar{V}$ and with edge set $\{e \cap \bar{V} : e \in E,\ |e \cap \bar{V}| \geq 2\}$.
- A hypergraph G is laminar, if for any $e_1, e_2 \in E$, one of the following is satisfied: (i) $e_1 \cap e_2 = \emptyset$, (ii) $e_1 \subset e_2$, (iii) $e_2 \subset e_1$.



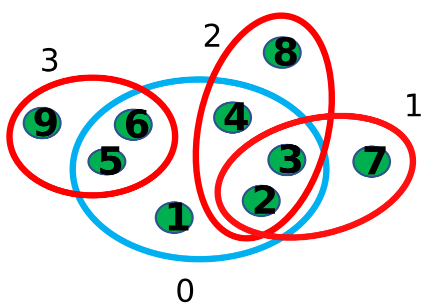
- Let $G = (V, E)$ be a γ-acyclic hypergraph, and let $e \in E$. Then the subhypergraph G_e is laminar.

Flower inequalities

- Let $e_0 \in E$ and let e_k, $k \in K$, be the set of all edges adjacent to e_0. Let $T \subseteq K$ such that
$$\left|(e_0 \cap e_i) \setminus \bigcup_{j \in T \setminus \{i\}} (e_0 \cap e_j)\right| \geq 2, \quad \forall i \in T. \tag{1}$$

- The flower inequality centered at e_0 with neighbors e_k, $k \in T$ is:
$$\sum_{v \in e_0 \setminus \cup_{k \in T} e_k} z_v + \sum_{k \in T} z_{e_k} - z_{e_0} \leq |e_0 \setminus \cup_{k \in T} e_k| + |T| - 1.$$

- We refer to the flower inequalities for all nonempty $T \subseteq K$ satisfying (1), as the system of flower inequalities centered at e_0. The flower relaxation MP_G^F is the polytope obtained by adding the system of flower inequalities centered at each edge of G to $\mathrm{MP}_G^{\mathrm{LP}}$.



$$z_1 + z_4 + z_5 + z_6 + z_{e_1} - z_{e_0} \leq 4$$
$$z_1 + z_5 + z_6 + z_{e_2} - z_{e_0} \leq 3,$$
$$z_1 + z_2 + z_3 + z_4 + z_{e_3} - z_{e_0} \leq 4,$$
$$z_1 + z_4 + z_{e_1} + z_{e_3} - z_{e_0} \leq 3, \; z_1 + z_{e_2} + z_{e_3} - z_{e_0} \leq 2$$

A sufficient condition for decomposability of multilinear sets

- Given $V' \subset V$, the section hypergraph of G induced by V' is $G' = (V', E')$, where $E' = \{e \in E : e \subseteq V'\}$. Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we denote by $G_1 \cup G_2$, the hypergraph $(V_1 \cup V_2, E_1 \cup E_2)$.

- Let G_1, G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$. We say that $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$, if

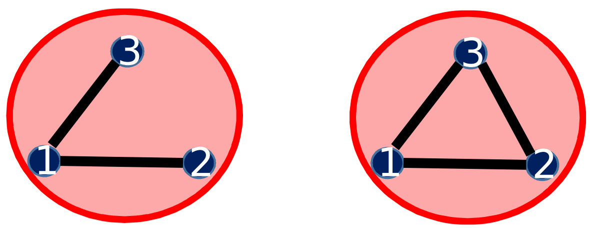
$$\text{conv}\mathcal{S}_G = \text{conv}\mathcal{S}_{G_1} \cap \text{conv}\mathcal{S}_{G_2}.$$

- Theorem: Let G_1, G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$. Suppose that $\bar{p} := V(G_1) \cap V(G_2) \in V(G) \cup E(G)$, and that for every edge e of G containing nodes in $V(G_1) \setminus V(G_2)$ either $e \supset \bar{p}$, or $e \cap \bar{p} = \emptyset$. Then $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$.

The flower relaxation vs. the convex hull relaxation

- Theorem: $\text{MP}_G^F = \text{MP}_G$ if and only if G is a γ-acyclic hypergraph.

- Proof sketch: If G has a γ-cycle, then after fixing $z_v = 1$ for all v in certain subset $V' \subset V(G)$, we either obtain a chordless cycle or one of the following structures:

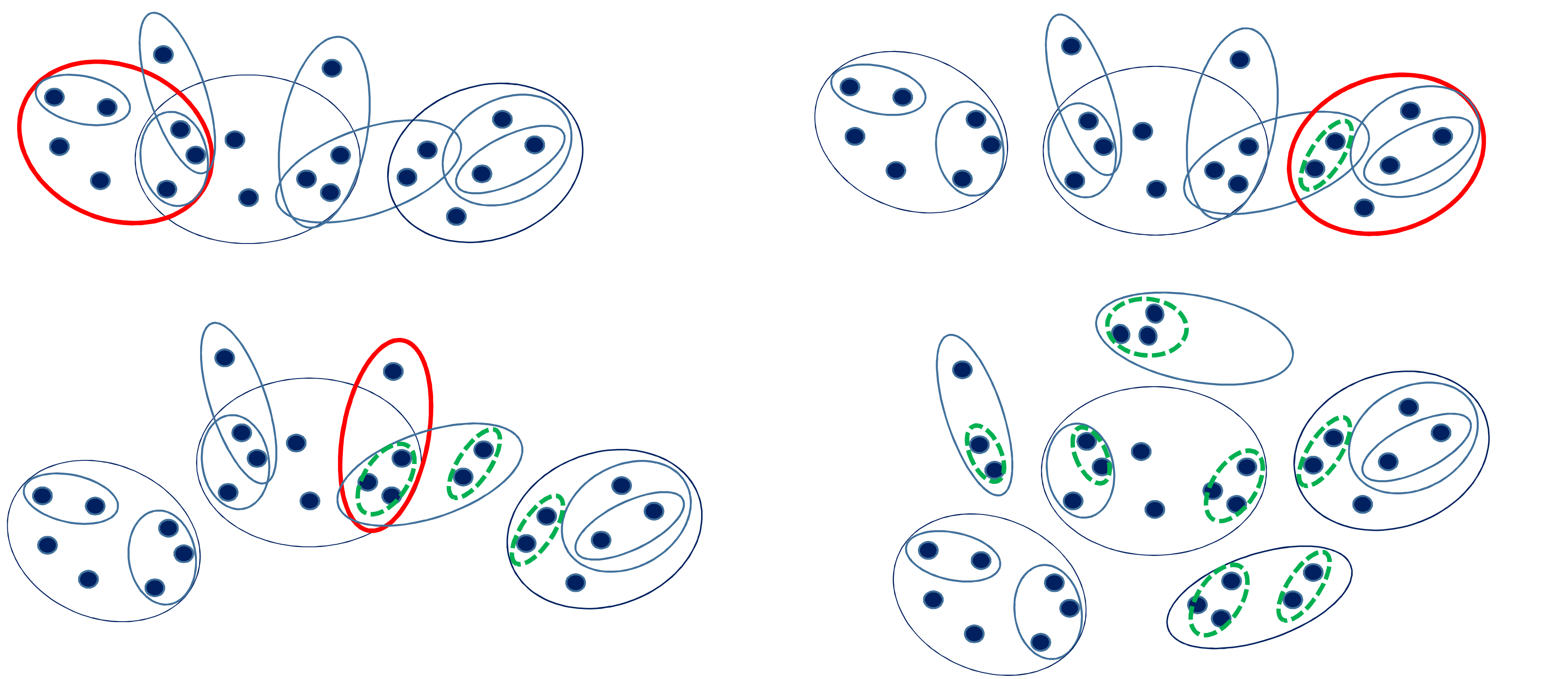


$$-z_1 + z_{12} + z_{13} - z_{123} \leq 0$$

$\Rightarrow$ if G contains a γ-cycle, we have $\text{MP}_G \subset \text{MP}_G^F$

The flower relaxation vs. the convex hull relaxation

- Suppose that G is γ-acyclic and that G has at least two maximal edges. Consider a maximal edge e' of G, and define E' to be the set of edges contained in e', and $\bar{V} := e' \cap (\cup_{e \in E \setminus E'} e)$. Then e' is a leaf of G, if $\bar{V} \subset \tilde{e}$ for some $\tilde{e} \in E \setminus E'$. We claim that G contains a leaf.



$\Rightarrow$ G decomposes into a collection of laminar hypergraphs

The flower relaxation vs. the convex hull relaxation

- Theorem: Let $G = (V, E)$ be a laminar hypergraph. Given an edge $e \in E$, we define $I(e) := \{p \in V \cup E : p \subset e,\ p \not\subset e',\ \text{for } e' \in E,\ e' \subset e\}$. Then MP_G is described by the following system:

$$\begin{aligned} z_v &\leq 1 && \forall v \in V \\ -z_e &\leq 0 && \forall e \in E \text{ such that } e \not\subset f, \text{ for } f \in E \\ -z_p + z_e &\leq 0 && \forall e \in E,\ \forall p \in I(e) \\ \sum_{p \in I(e)} z_p - z_e &\leq |I(e)| - 1 && \forall e \in E. \end{aligned}$$

- Our proof relies on a fundamental result due to Conforti and Cornuéjols regarding the connection between integral polyhedra and balanced matrices.

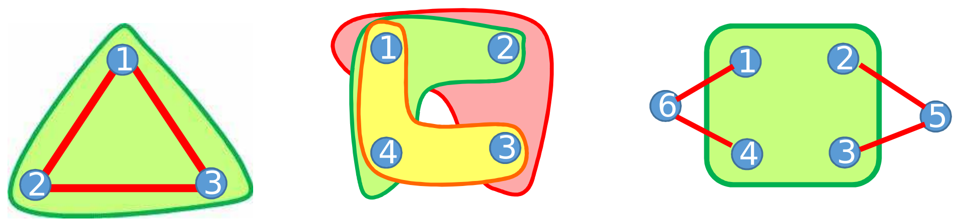
- For a γ-acyclic hypergraph G, the multilinear polytope has a polynomial-size extended formulation.

- Applying Fourier-Motzkin elimination to project out auxiliary edges, it follows that for a γ-acyclic G, we have $\text{MP}_G = \text{MP}_G^F$.

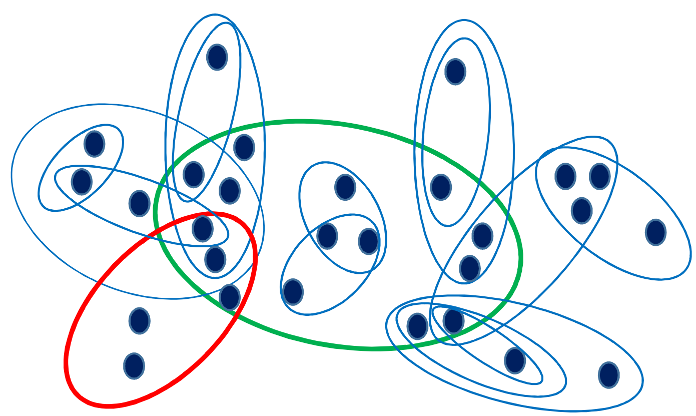
β-acyclic hypergraphs

- A β-cycle in G is a γ-cycle $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ such that the node v_1 belongs to e_1, e_t and no other e_j.

- A hypergraph is called β-acyclic if it contains no β-cycles.



Examples of β-cycles



A β-acyclic hypergraph

Running intersection inequalities

- A multiset F of subsets of a finite set V has the running intersection property if there exists an ordering $p_1, p_2, \ldots, p_m$ of the sets in F such that

$$\forall k \in \{2, \ldots, m\}, \exists j < k : \; N(p_k) := p_k \cap \Big(\bigcup_{i<k} p_i \Big) \subseteq p_j.$$

  We refer to such an ordering as a running intersection ordering of F.

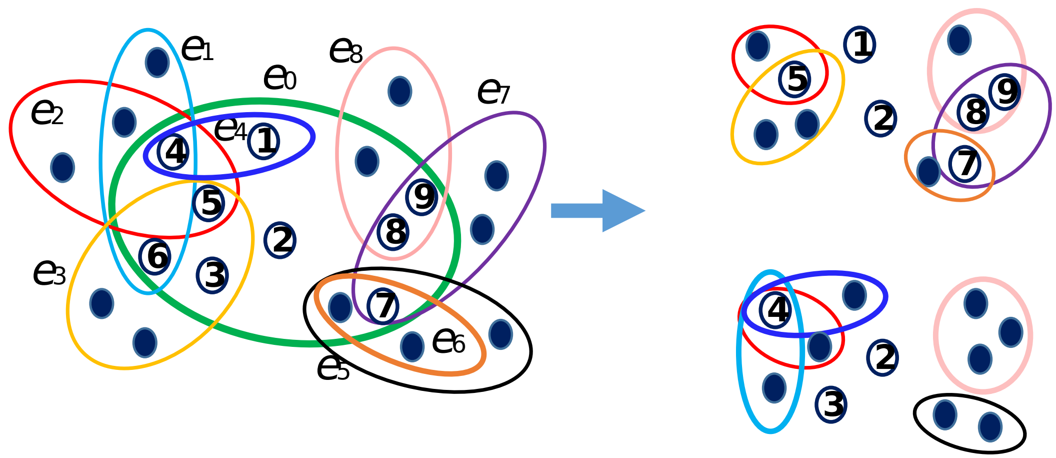
- Let e_0 and e_k, $k \in K$, be a collection of edges adjacent to e_0 such that $\tilde{E} := \{e_0 \cap e_k : k \in K\}$ has the running intersection property. Consider a running intersection ordering of $\tilde{E}$. For each $k \in K$, let $w_k \subseteq N(e_0 \cap e_k)$ such that $w_k \in \emptyset \cup V \cup E$. We define a running intersection inequality centered at e_0 with neighbours e_k, $k \in K$ as

$$-\sum_{k \in K} z_{w_k} + \sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} z_v + \sum_{k \in K} z_{e_k} - z_{e_0} \leq \omega - 1,$$

  where we define $z_{\emptyset} = 0$, and $\omega = \Big|\{k \in K : w_k = \emptyset\}\Big| + \Big|\{e_0 \setminus \bigcup_{k \in K} e_k\}\Big|$.

Running intersection inequalities

- Letting $w_k = \emptyset$ for all $k \in K$, running intersection inequalities simplify to flower inequalities.



$$-z_{v_5} - z_{v_7} - z_{v_8} + z_{v_1} + z_{v_2} + z_{e_2} + z_{e_3} + z_{e_6} + z_{e_7} + z_{e_8} - z_{e_0} \leq 3,$$

$$-z_{v_5} - z_{v_7} - z_{v_9} + z_{v_1} + z_{v_2} + z_{e_2} + z_{e_3} + z_{e_6} + z_{e_7} + z_{e_8} - z_{e_0} \leq 3,$$

$$-2z_{v_4} + z_{v_2} + z_{v_3} + z_{e_1} + z_{e_2} + z_{e_4} + z_{e_5} + z_{e_8} - z_{e_0} \leq 4$$

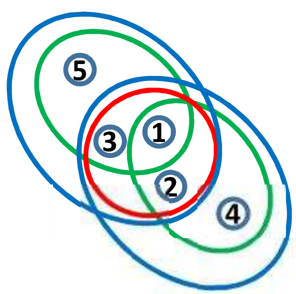
- Any running intersection ordering of $\tilde{E}$ leads to the same system of running intersection inequalities centered at e_0 with neighbors e_k, $k \in K$.

The running intersection relaxation

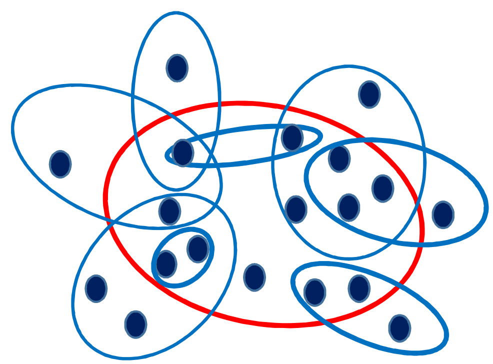
- The running intersection relaxation $\text{MP}_G^{\mathrm{RI}}$ is the polytope obtained by adding to $\text{MP}_G^{\mathrm{LP}}$ all possible running intersection inequalities for $\mathcal{S}_G$.

- If MP_G is not β-acyclic, then $\text{MP}_G \subset \text{MP}_G^{\mathrm{RI}}$.

- Let G be a β-acyclic hypergraph. Suppose that there exist no three edges $e_0, e_1, e_2 \in E$ such that $|e_0 \cap e_1 \cap e_2| \geq 2$, $(e_0 \cap e_1) \setminus e_2 \neq \emptyset$, and $(e_0 \cap e_2) \setminus e_1 \neq \emptyset$. Then $\text{MP}_G = \text{MP}_G^{\mathrm{RI}}$.



$$-z_1 + z_{123} + z_{124} + z_{135} - z_{1234} - z_{1235} \leq 0$$



Separation of flower and running intersection inequalities

- Even for a γ-acyclic hypergraph $G = (V, E)$, the number of facets of MP_G^F may not be bounded by a polynomial in $|V|, |E|$.

- Given a rank-r γ-acyclic hypergraph $G = (V, E)$, the separation problem over all flower inequalities can be solved in $O(r|E|^2(|V| + |E|))$ operations.

- The separation problem for flower inequalities over general hypergraphs is NP-hard (reduction from 3D matching).

- The separation problem for running intersection inequalities for fixed-rank hypergraphs can be solved in polynomial-time. More precisely, in $O(|E|(r2^r|E| + 2^{r^2}r^3))$ operations.

Numerical Experiments

- We characterize each problem by its degree (d), number of variables (n), number of constraints (q), and density (ν).
- Polynomial problems of degree 3 with

$$(n, \nu) \in \{(10, 0.75), (15, 0.25), (15, 0.15), (20, 0.1), (20, 0.05)\},$$

and multilinear problems of degree 3 with

$$(n, \nu) \in \{(10, 1.0), (15, 0.5), (20, 0.15), (20, 0.1), (25, 0.05), (30, 0.02)\}.$$

- Polynomial problems of degree 4 with

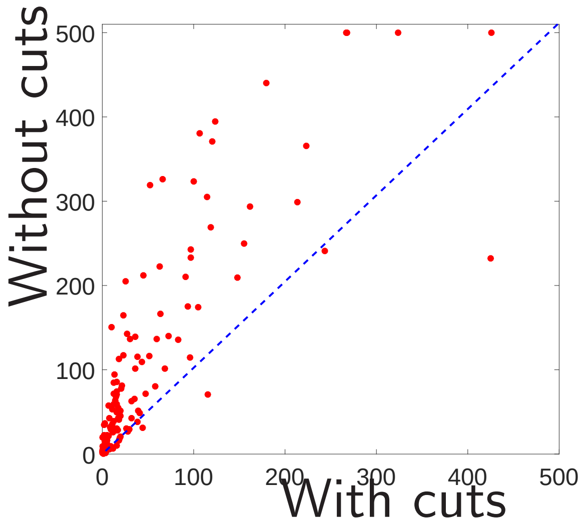
$$(n, \nu) \in \{(10, 0.25), (10, 0.15), (15, 0.05), (15, 0.02), (20, 0.01)\},$$

and multilinear problems of degree 4 with

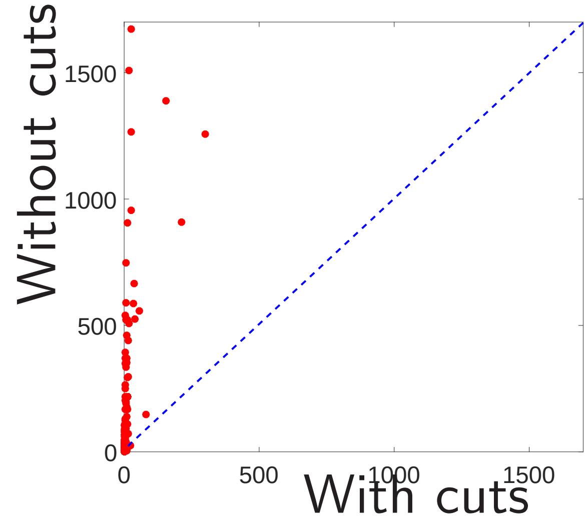
$$(n, \nu) \in \{(10, 1.0), (15, 0.15), (20, 0.02), (20, 0.01), , (25, 0.01), (25, 0.005)\}.$$

- In both sets, we let $q \in \{0, n/5, n/2, n\}$. For each combination, 5 random instance are generated.
- Relative/absolute optimality tolerance $= 10^{-6}$ and time limit $= 500s$.

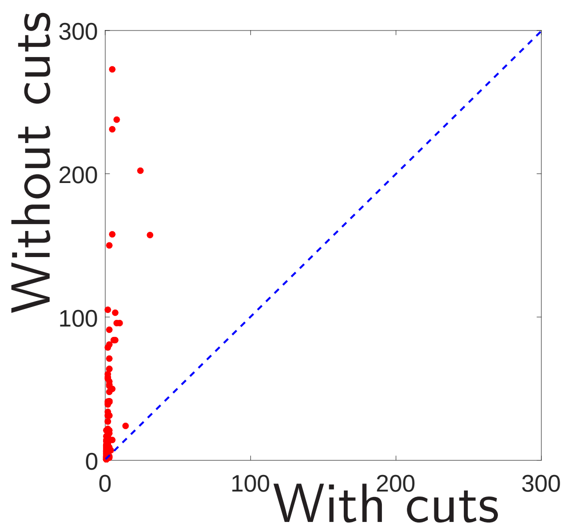
220 polynomial optimization problems of degree three



(a) CPU time (s)



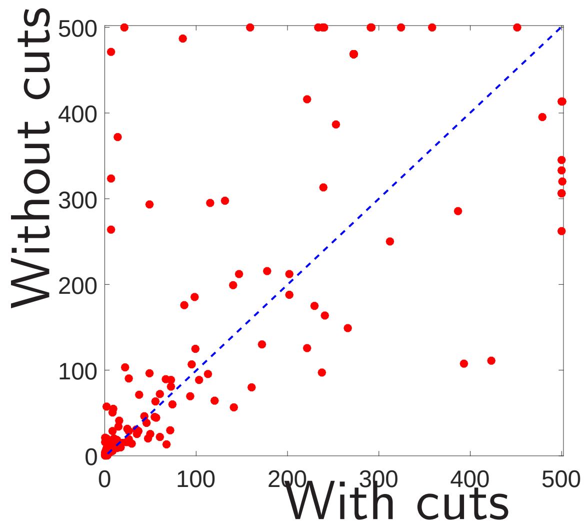
(b) Iterations



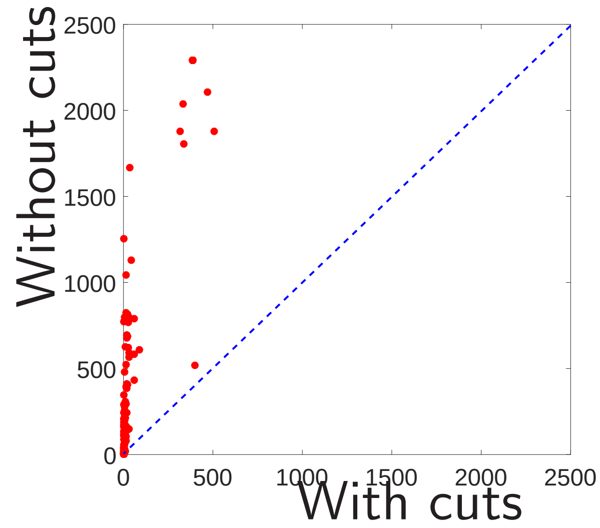
(c) Memory

- **Average reductions of 60% in CPU time, 78% in number of nodes, and 70% in maximum number of nodes in memory.**

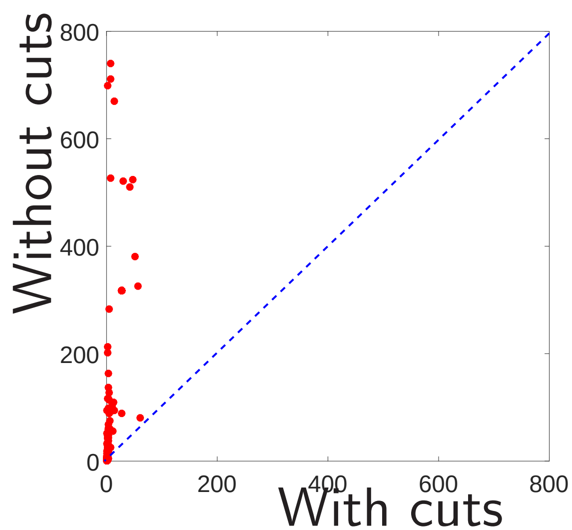
220 polynomial optimization problems of degree four



(d) CPU time (s)



(e) Iterations



(f) Memory

- **Average reductions of 43% in CPU time, 76% in number of nodes, and 72% in maximum number of nodes in memory.**

Numerical Experiments – computer vision instances

- The purpose of image restoration is to estimate the original image from the degraded data. An image is modeled as a $l \times h$ matrix where each binary element x_{ij} represents a pixel.
- The image restoration problem is defined as the objective function $f(x) = H(x) + L(x)$ to be minimized, where $H(x)$ is linear and models similarity between the input blurred image and the output, $L(x)$ is a multilinear function of degree four and models smoothness.
- Test set taken from [CramaRodrigez16] with images sizes $\{10 \times 10\}$, $\{10 \times 15\}$, $\{15 \times 15\}$.

Effect of adding cuts	CPU time	Iterations	Nodes
Better by a factor at least 2	17 (38%)	10 (23%)	10 (23%)
Between 30% and 100% better	13 (30%)	0 (0%)	0 (0%)
Difference smaller than 30%	14 (32%)	34 (77%)	34 (77%)
Between 30% and 100% worse	0 (0%)	0 (0%)	0 (0%)
Worse by a factor of at least 2	0 (0%)	0 (0%)	0 (0%)

- **Average reductions of 63% in CPU time, 42% in number of iterations, and 30% in maximum number of nodes in memory.**

What about the multilinear polytope of β-acyclic hypergraphs?

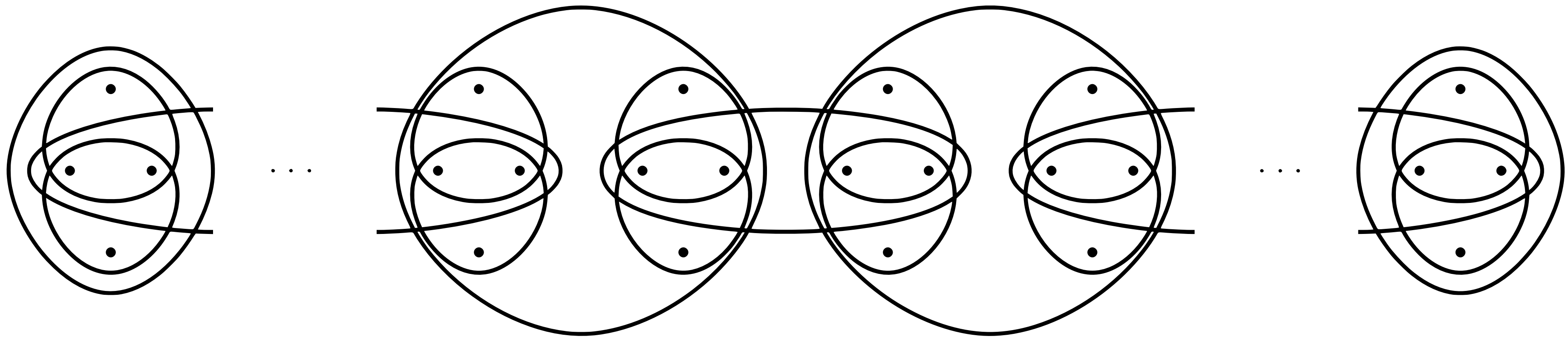
- From a computational perspective, sparsity is key to the effectiveness of cutting planes in a branch-and-cut framework.

- For a rank r hypergraph, flower inequalities contain at most $\frac{r}{2}$ nonzero coefficients, and running intersection inequalities contain at most $2(r-1)$ nonzero coefficients.

- For β-acyclic hypergraphs, MP_G may contain dense facet-defining inequalities with $\theta(|E|)$ nonzero coefficients.

- In practice, we almost always have $r \ll |E|$.

Example



- Let $n \geq 2$ and consider the β-acyclic hypergraph $G = (V, E)$ with $V = \bigcup_{i\in[n]} V^i$, $E = H \cup \bigcup_{i\in[n]} E^i$, where $V^1 = \{v_3^1, v_4^1, v_7^1, v_8^1\}$, $V^i = \{v_1^i, \cdots, v_8^i\}$ for all $i \in [n-1] \setminus \{1\}$, $V^n = \{v_1^n, v_2^n, v_5^n, v_6^n\}$,

$$H = \Big\{\{v_3^i, v_4^i, v_1^{i+1}, v_2^{i+1}\},\ i \in [n-1]\Big\}$$

$$E^1 = \Big\{\{v_3^1, v_4^1, v_7^1\}, \{v_3^1, v_4^1, v_8^1\}, V^1\Big\}$$

$$E^i = \Big\{\{v_1^i, v_2^i, v_5^i\}, \{v_1^i, v_2^i, v_6^i\}, \{v_3^i, v_4^i, v_7^i\}, \{v_3^i, v_4^i, v_8^i\}, V^i\Big\}, \quad \forall i \in [n-1] \setminus \{1\}$$

$$E^n = \Big\{\{v_1^n, v_2^n, v_5^n\}, \{v_1^n, v_2^n, v_6^n\}, V^n\Big\}.$$

- Then the following inequality containing $|E|$ nonzero coefficients defines a facet of MP_G:

$$-\sum_{i\in[n]} z_{V^i} - \sum_{e\in H} z_e + \sum_{i\in[n]} \sum_{e\in E^i\setminus\{V^i\}} z_e \leq 2n-3.$$

The multilinear polytope of acyclic hypergraphs

- The multilinear polytope of Berge-acyclic hypergraphs is the standard linearization; polynomial-size description: $|V|+|E|$ variables and $|V|+(r+2)|E|$ inequalities.

- The multilinear polytope of γ-acyclic hypergraphs is the flower relaxation; polynomial-size extended formulation: at most $|V|+2|E|$ variables ($|E|$ additional variables) and $|V|+(r+2)|E|$ inequalities.

- The multilinear polytope of kite-free β-acyclic hypergraphs is the running intersection relaxation; polynomial-size extended formulation: at most $|V|+2|E|$ variables ($|E|$ additional variables) and $|V|+(r+2)|E|$ inequalities.

- The multilinear polytope of α-acyclic hypergraphs does not admit a polynomial-size extended formulation unless $P = NP$

- Does the multilinear polytope of β-acyclic admit a polynomial-size extended formulation?

The multilinear polytope of β-acyclic hypergraphs

- Theorem: Let $G = (V, E)$ be a β-acyclic hypergraph of rank r. Then there exists an extended formulation of MP_G comprising of at most $(3r-4)|V|+4|E|$ inequalities, with at most $(r-2)|V|$ extended variables.

- Fewer inequalities than the standard linearization for β-acyclic hypergraphs with $|E| \geq 3|V|$

- The inequalities defining the extended formulation are very sparse: they contain at most four variables with non-zero coefficients.

β-acyclicity and nest points

- A node $v \in V$ is a nest point of G if the set of the edges of G containing v can be ordered so that $e_1 \subset e_2 \subset \cdots \subset e_k$.

- We define the hypergraph obtained from $G = (V, E)$ by removing a node $v \in V$ as $G - v := (V', E')$, where $V' := V \setminus \{v\}$ and $E' := \{e \setminus \{v\} : e \in E, |e \setminus \{v\}| \geq 2\}$.

- A nest point sequence of length s for some $s \leq |V|$ of G is an ordering $v_1, \cdots, v_s$ of s distinct nodes of G, such that v_1 is a nest point of G, v_2 is a nest point of $G - v_1$, and so on

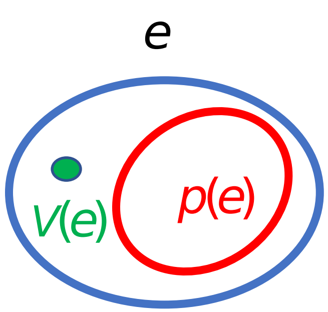
- Theorem: A hypergraph $G = (V, E)$ is β-acyclic if and only if it has a nest point sequence of length $|V|$.

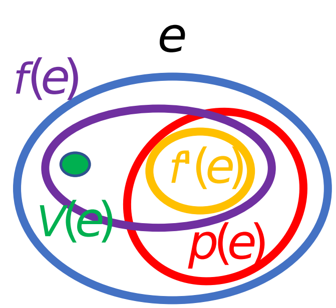
- Let $v_1, \cdots, v_s$ be a nest point sequence of G. The expansion of G (w.r.t. $v_1, \cdots, v_s$) is the hypergraph $G' = (V, E')$, where E' is obtained from E by adding, for each $e \in E$, the sets of cardinality at least two among $e \setminus \{v_1\}$, $e \setminus \{v_1, v_2\}, \cdots, e \setminus \{v_1, \cdots, v_s\}$.

The Extended formulation

- Theorem: Let $G = (V, E)$ be a β-acyclic hypergraph expanded w.r.t. $v_1, \ldots, v_n$. For every $e \in E$, denote by $v(e)$ the first node in the sequence $v_1, \ldots, v_n$ contained in e, and define $p(e) := e \setminus \{v(e)\}$. Define $M := \{e \in E : \exists g \in E, g \subset e, v(e) \in g\}$. For every $e \in M$, let $f(e) \subset e$ be the edge of maximum cardinality with $v(e) \in f(e)$, and let $f'(e) := f(e) \setminus \{v(e)\}$. Denote by $\bar{E}$ the set of maximal edges of G. Then, MP_G is defined by:

$$
\begin{aligned}
&0 \leq z_u \leq 1, \quad \forall u \in V, \quad z_e \geq 0, \quad \forall e \in \bar{E}\\
&z_e - z_{p(e)} \leq 0, \quad \forall e \in E\\
&z_e - z_{f(e)} \leq 0, \ -z_{f'(e)} + z_{p(e)} + z_{f(e)} - z_e \leq 0, \quad \forall e \in M\\
&z_e - z_{v(e)} \leq 0, \ z_{v(e)} + z_{p(e)} - z_e \leq 1, \quad \forall e \in E \setminus M.
\end{aligned}
$$





Recap

- Goal: constructing strong and cheap polyhedral relaxations for multilinear sets
- The standard linearization coincides with the multilinear polytope of Berge-acyclic hypergraphs, very weak relaxations in general
- Flower relaxation gives the multilinear polytope of γ-acyclic hypergraphs
- Running intersection inequalities dominate flower inequalities when the neighbours intersect and satisfy the running intersection property
- A very simple compact extended formulation for the multilinear polytope of β-acyclic hypergraphs, but no characterization in the original space

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- A. Del Pia and A. Khajavirad. The multilinear polytope for acyclic hypergraphs, *SIAM Journal on Optimization*, 2018.
- A. Del Pia and A. Khajavirad. The running intersection relaxation of the multilinear polytope, *Mathematics of Operations Research*, 2021.
- A. Del Pia, A. Khajavirad, and N. V. Sahinidis. On the impact of running-intersection inequalities for globally solving polynomial optimization problems, *Mathematical Programming Computation*, 2020.
- A. Del Pia and A. Khajavirad. The multilinear polytope of beta-acyclic hypergraphs has polynomial extension complexity, *arXiv:2212.11239*, 2022.

Other results on the facial structure of the multilinear polytope

- A. Del Pia and S. Di Gregorio, Chvatal rank in binary polynomial optimization, INFORMS Journal on Optimization 3(4) 315-349 (2021)
  - Running intersection inequalities are CG cuts for the standard linearization MP_G^{LP}
  - For kite-free β-acyclic hypergraphs, the polytope MP_G^{LP} has Chvatal rank 1.

- A. Del Pia and M. Walter, Simple odd β-cycle inequalities for binary polynomial optimization, Proceedings of IPCO 2022 LNCS 13265 181-194 (2022)
  - Odd β-cycle inequalities: when added to the flower relaxation, they give the multilinear polytope of a cycle hypergraph
  - $G = (V, E)$, with $E = \{e_1, \cdots, e_m\}$, where $m \geq 3$ is a cycle hypergraph, if every edge e_i has nonempty intersection only with e_{i-1} and e_{i+1} for all $i \in \{1, \cdots, m\}$. When $m = 3$, it is also required that $e_1 \cap e_2 \cap e_3 = \emptyset$.
  - Odd β-cycle inequalities have Chvatal rank 2.
  - Odd β-cycle inequalities can be separated in strongly polynomial time

Some open questions

- What is the first CG closure of $\mathrm{MP}_G^{\mathrm{LP}}$?
  - For BQP, adding the triangle inequalities to the standard linearization gives the first CG closure.

- What "the RLT level" of flower inequalities or running intersection inequalities?

- What is "the SoS level" of flower inequalities or running intersection inequalities?

- Other classes of acyclic hypergraphs between β-acyclic and α-acyclic with polytime complexity.

Comparison with recursive McCormick relaxations

- A. Khajavirad, On the strength of recursive McCormick relaxations for binary polynomial optimization, Operations Research Letters, 2023.