Matching Theory and School Choice

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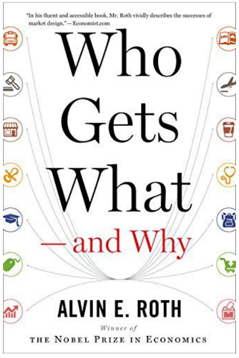
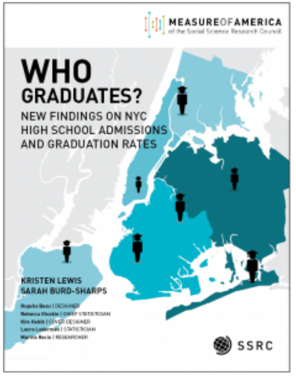
THE NEW YORK CITY SCHOOL SYSTEM

Every year NYC $\approx 80,000$ NYC students apply to one of the ≈ 800 public high school programs.

How should students be assigned to schools?





The pre-2003 method

Students were asked to list at most 5 schools in order of preference.

1) Schools see students' lists, then make offers to some students and place others in waiting lists.
2) Each student can accept one offer and one waiting list. If not all students are assigned to a school, go to 1).
3) After three iterations, students still unmatched are assigned a school decided by a central system.

Some issues with the pre-2003 system

- Students could apply only to 5 schools.
- Schools could see students' preference before taking decisions, and some schools would only admit students that ranked them first.

Hence, students had to be strategical on the schools they listed, rather than truthful.

Because of the need for multiple rounds, some students got many offers, some none.

It was tempting for schools to bypass the system, and give exploding offers to students before the process started:

- Some students would accept schools worse than what they could aim to.

ENTER (ABDULKADIROGLU, PATHAK, AND ROTH, 05)

REVISED ADMISSION FOR HIGH SCHOOLS

By David M. Herszenhorn

Oct. 3, 2003

New York City education officials have quietly instituted a new mandatory high school admissions policy that allows students this year to apply to as many as a dozen schools but will admit them to only one.

But admission to more than 200 other high schools will be decided by a computer using a complex formula intended to match student preferences with the schools they are qualified to attend. Schools will retain their admissions criteria, be it grades, an audition or simply a lottery, and will prepare a ranking of the students they want. Final assignments will be decided by computer.

Another major problem with the old system, officials said, was that each year tens of thousands of 13- and 14-year-olds were subjected to painful rejections, or an agonizing process of hoping to be lifted from waiting lists.

Enter (Abdulkadiroglu, Pathak, and Roth, 05)

Professor Roth of Harvard, who attended Martin Van Buren High School in Queens, said that as a parent he would prefer the new admissions system.

''I can now actually put down my first choice as my first choice,'' he said. The old system might force him to say to his child, ''You are unlikely to get into your first choice and it's too risky to try,'' he said.

In some cases, education officials said, the old system allowed for back-room deals and special favors because principals could easily understate the number of seats available for incoming students.

After most students had made their selection, officials said, the principals could then make the previously ''hidden'' seats available to children who might otherwise not have been accepted to that school.

Under the old rules, many students were assigned to their local zoned school, by default, which is an unpopular result in many neighborhoods.

Economic Sciences | Prize in Economic Sciences 2012 | Alvin E. Roth - Facts

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

Alvin E. Roth
Lloyd S. Shapley

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Alvin E. Roth
Facts

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Alvin E. Roth
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

Born: 18 December 1951, New York, NY, USA

Affiliation at the time of the award: Harvard University, Cambridge, MA, USA; Harvard Business School, Boston, MA, USA

Prize motivation: "for the theory of stable allocations and the practice of market design"

Prize share: 1/2

Stable matchings in the marriage model: impact

Stable matchings are employed to solve matching / assignment problems, including:

- School matches in NYC, Boston, Denver, ...;
- University placement, e.g., in India and Iran;
- the National Residency Matching Program (NRMP);
- Online dating;
- Firm / worker matchings;
- Jobs / server matchings;
- Ride sharing;
- ...

CONTENT

Part I: The classical model (Gale and Shapley, 62):

- Structural and algorithmic properties of stable matchings in the marriage model;
- Impact for school choice.
- Drawbacks.

Part II: Beyond (Gale and Shapley, 62), changing the output:

(a) Pareto-optimality & von Neumann-Morgenstern Stability;

(b) Popularity;

Part III: Beyond (Gale and Shapley, 62), changing the input:

- Ties, choice functions, and applications to school choice.

Part I:
The classical model

THE MARRIAGE MODEL

Each student strictly ranks schools.

- E.g. Bob ranks *Laguardia* first, *Midwood* second, while it does not consider *Staten Island Technical School* a suitable choice.

Each school has a quota and strictly ranks students.

- E.g. *Laguardia* ranks *Alexandra* first, *Bob* second, while it does not consider *Carl* suitable.
- Schools' rankings are based on many parameters: performance at tests, area of residence, etc.

$J >_i \bar{J}$

Ranking of an agent i: $>_i$
Partner of an agent i in a matching M: $M(i)$

Stability

1	A	B	C	D
2	B	D	A	C
3	C	B	A	D
4	C	D	A	B

A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

STABILITY

(A, 4) ? NOT A BLOCKING

(B, 3) ? YES!

1	A	B	C	D	A	4	3	2	1
2	B	D	A	C	B	3	4	1	2
3	C	B	A	D	C	2	1	4	3
4	C	D	A	B	D	1	2	3	4

A student-school pair (i, j) is called blocking for a matching M if:

$$j >_i M(i) \quad \text{and} \quad i >_j M(j).$$

Stability

1	~~A~~	B	C	D
2	~~B~~	D	A	C
3	~~C~~	B	A	D
4	C	D	A	B

A	4	3	2	1
B	3	4	1	~~2~~
C	2	1	4	~~3~~
D	1	2	3	4

A student-school pair (i, j) is called blocking for a matching M if:

$$j >_i M(i) \quad \text{and} \quad i >_j M(j).$$

A matching without blocking pairs is called stable.

Suppose $\exists\ (i,j)$: $j >_i M_0(i)$

STABILITY

i: | .... j .... $M_0(i)$

j rejected i at some iteration of the algorithm because j had an offer from $\bar{i}$

1	$\boxed{A}$	B	C	D
2	B	$\boxed{D}$	A	C
3	C	$\boxed{B}$	A	D
4	$\boxed{C}$	D	A	B

A	4	3	2	$\boxed{1}$
B	$\boxed{3}$	4	1	2
C	2	1	$\boxed{4}$	3
D	1	$\boxed{2}$	3	4

$M_0(j) \geq_j \bar{i} >_j i$

$\Rightarrow (i,j)$ NOT A BLOCKING PAIR

$\square$

A student-school pair (i,j) is called blocking for a matching M if:

$$j >_i M(i) \quad \text{and} \quad i >_j M(j).$$

A matching without blocking pairs is called stable.

Theorem [Gale-Shapley algorithm]. There is an algorithm that computes a stable matching M_0 in time $O(|E|)$, where E is the set of possible pairs.

Some interesting features of GS algorithm

- It outputs a stable matching; in particular, no student-school pair has incentive to find a deal "under the table".
- It requires only one round of communication and runs in time linear in the input size.

  $\approx$ 3 minutes for instances of the size of the NYC high school market.
- Among all stable matchings, it finds one that is that all students prefer (student-optimal).

Student-optimality

To show: $(*)$ $\left\{ \text{if } M \text{ is a stable matching, then } M_0(i) \geq M(i) \quad \forall \text{ student } i \right.$

(i,j) a STABLE PAIR if $\exists$ stable matching M: $(i,j) \in M$

$(*) \iff \nexists$ s.p. (i,j) : $j >_i M_0(i)$

$\iff \nexists$ s.p. (i,j) : j rejects i during some execution of GS $(**)$

Suppose by contradiction $(**)$ holds, take s.p. (i,j) rejected first by GS

Let M be a stable matching : $(i,j) \in M$.

Why did j reject i? $\boxed{\bar{\imath} >_j i = M(j)}$ $\Rightarrow$ $(\bar{\imath}, j)$ blocks M ↯

$\boxed{j \geq_{\bar{\imath}} M(\bar{\imath})}$ [SINCE OTHERWISE A STABLE PAIR $(\bar{\imath}, M(\bar{\imath}))$ WAS REJECTED BEFORE (i,j)]

Some interesting features of GS algorithm

- It outputs a stable matching; in particular, no student-school pair has incentive to find a deal "under the table".
- It requires only one round of communication and runs in time linear in the input size.

  $\approx$ 3 minutes for instances of the size of the NYC high school market.
- Among all stable matchings, it finds one that is that all students prefer (student-optimal).
- The algorithm is strategy-proof for students.

Stability, beyond student-optimality

There are instances with exponentially many stable matchings.

$$I^1 \;=\; \begin{array}{c|cc} 1 & 1 & 2 \\ 2 & 2 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}$$

$$I^2 \;=\; \begin{array}{c|} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad\qquad\qquad \begin{array}{c|} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad\qquad .$$

Stability, beyond student-optimality

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$$I^2 \; = \; \begin{array}{c|cccc} 1 & 1 & 2 & & \\ 2 & 2 & 1 & & \\ 3 & & & 1 & 2 \\ 4 & & & 2 & 1 \end{array} \qquad \begin{array}{c|cccc} 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \quad .$$

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$$I^2 \;=\; \begin{array}{c|cccc} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{array} \qquad \begin{array}{c|c} 1 & \\ 2 & \\ 3 & \\ 4 & \end{array} \qquad .$$

Stability, beyond student-optimality

There are instances with exponentially many stable matchings.

$$
I^1 \; = \quad \begin{array}{c|cc} 1 & 1 & 2 \\ 2 & 2 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}
$$

$$
I^2 \; = \quad \begin{array}{c|cccc} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{array} \qquad\qquad \begin{array}{c|cccc} 1 & & & 2 & 1 \\ 2 & & & 1 & 2 \\ 3 & 2 & 1 & & \\ 4 & 1 & 2 & & \end{array} .
$$

Stability, beyond student-optimality

There are instances with exponentially many stable matchings.

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$$I^2 = \begin{array}{c|cccc} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{array} \qquad \begin{array}{c|cccc} 1 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 & 3 \\ 4 & 1 & 2 & 3 & 4 \end{array}.$$

Stability, beyond student-optimality

There are instances with exponentially many stable matchings.

$$I^1 \;=\; \begin{array}{c|cc} 1 & 1 & 2 \\ 2 & 2 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & 2 & 1 \\ 2 & 1 & 2 \end{array}$$

$$I^2 \;=\; \begin{array}{c|cccc} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{array} \qquad\qquad \begin{array}{c|cccc} 1 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 & 3 \\ 4 & 1 & 2 & 3 & 4 \end{array}.$$

Exercise 1. Let $k \in \mathbb{N}$. I^k has $n := 2^{k+1}$ agents and $\geq 2^{\frac{n}{2}-1}$ stable matchings.

STABILITY, BEYOND STUDENT-OPTIMALITY

There are instances with exponentially many stable matchings.

What if we want to find a stable matching other than the student-optimal?

- We may care about both sides of the market.
- We may want a specific student-school pair (not) to be matched.

The discrete optimizer's approach:

[OPT-SM]

- Let E be the possible student-school pairs, and define $w : E \to \mathbb{Z}$;
- Find a stable matching that minimizes $\quad w(M) : M \in \mathcal{S}.$ set of stable matchings

Ordering matchings by student preferences

For matchings M, M', we write

$$M \succeq M'$$

if $M(i) \geq_i M'(i)$ for every student i ("i weakly prefers M to M'").

1	$\boxed{A}$	{B}	C	D
2	$\boxed{B}$	{A}	D	C
3	{C}	$\boxed{D}$	A	B
4	{D}	$\boxed{C}$	B	A

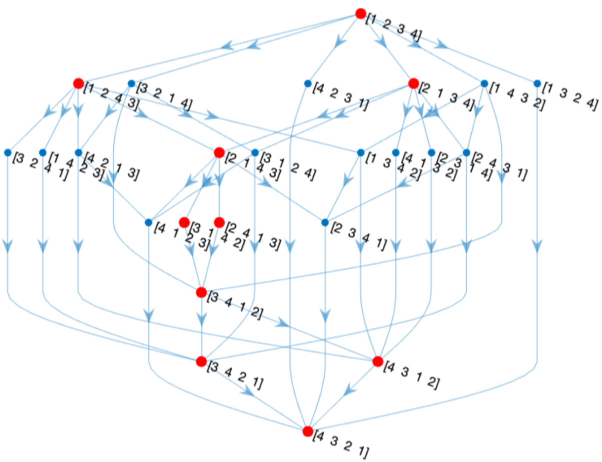
A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

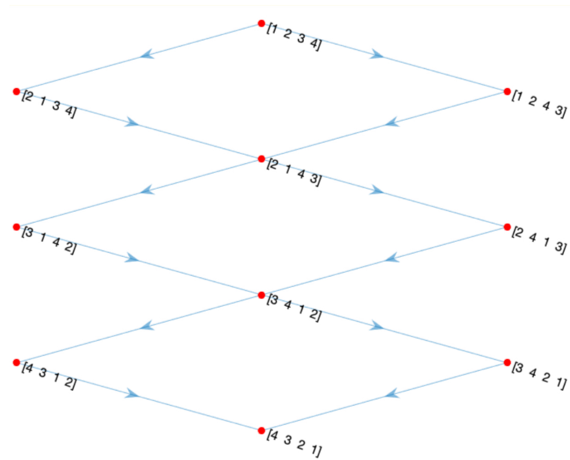
Red $\succsim$ {} $\square$ $\succsim$ Blue

Poset of matchings: examples

1	A	B	C	D
2	B	A	D	C
3	C	D	A	B
4	D	C	B	A

A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

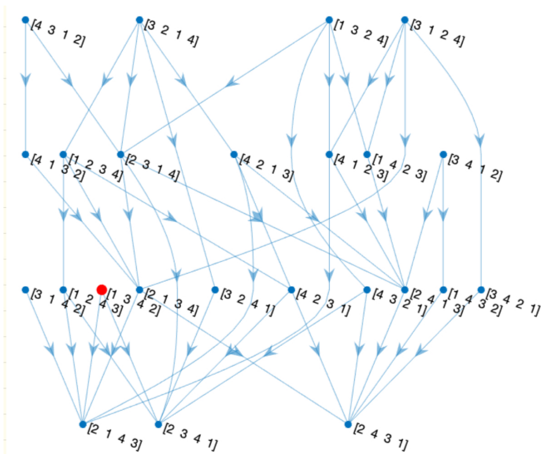




Poset of matchings: examples

1	C	A	D	B
2	B	C	A	D
3	B	A	C	D
4	B	D	C	A

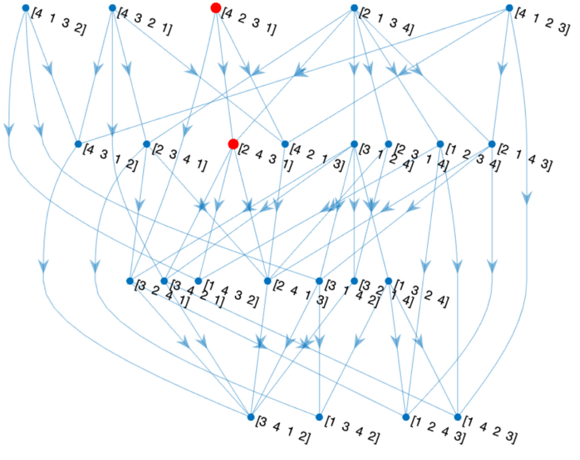
A	1	3	2	4
B	4	2	3	1
C	2	3	1	4
D	4	3	1	2

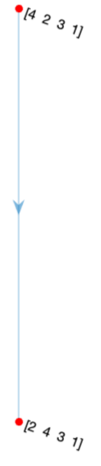


POSET OF MATCHINGS: EXAMPLES

1	D	B	C	A
2	A	C	D	B
3	C	B	D	A
4	D	A	C	B

A	4	3	2	1
B	3	1	4	2
C	3	4	1	2
D	2	1	4	3

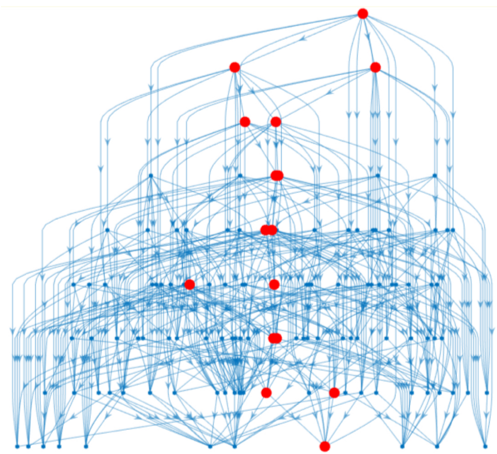


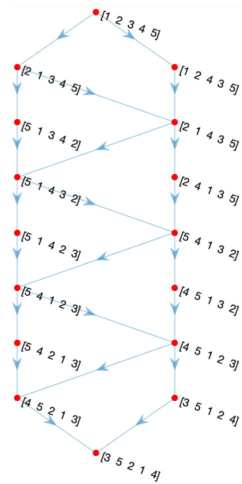


Posets of matchings: examples

1	A	B	E	D	C
2	B	A	D	E	C
3	C	D	A	B	E
4	D	C	B	A	E
5	E	B	C	D	A

A	5	4	3	2	1
B	3	4	5	1	2
C	2	1	5	4	3
D	5	1	2	3	4
E	4	3	2	1	5





Meet and join of stable matchings

Let M, M' be stable matchings. Define

$$M \vee M' = M^{\uparrow}$$

be the set of pairs where each student is assigned to their favorite partner between M, M'.

Lemma. $M^{\uparrow}$ is a stable matching. (EXERCISE 1.1)

Meet and join of stable matchings

Let M, M' be stable matchings. Define

$$M \vee M' = M^{\uparrow}$$

be the set of pairs where each student is assigned to their favorite partner between M, M'.

$M^{\uparrow} \succeq \begin{matrix} M \\ M' \end{matrix} \succeq M^{\downarrow}$

Lemma. $M^{\uparrow}$ is a stable matching.
Similarly, define

$$M \wedge M' = M^{\downarrow}$$

by assigning to each student their worse partner between M, M'.
Lemma. $M^{\downarrow}$ is a stable matching.

Meet and join of stable matchings

Let M, M' be stable matchings. Define

$$M \vee M' = M^{\uparrow}$$

be the set of pairs where each student is assigned to their favorite partner between M, M'.

Lemma. $M^{\uparrow}$ is a stable matching.

Similarly, define

$$M \wedge M' = M^{\downarrow}$$

by assigning to each student their worse partner between M, M'.

Lemma. $M^{\downarrow}$ is a stable matching.

Corollary. $M^{\uparrow}$ is the join of M, M', i.e.,

- $M^{\uparrow} \succeq M, M'$ and
- $\overline{M} \succeq M, M' \Rightarrow \overline{M} \succeq M^{\uparrow}$.

$M^{\downarrow}$ is the meet of M, M'.

THE LATTICE OF STABLE MATCHINGS

A poset where each pair of elements has a meet and a join is called a Lattice.

$\Rightarrow$ **Corollary.** $(\mathcal{S}, \succeq)$ is a lattice.

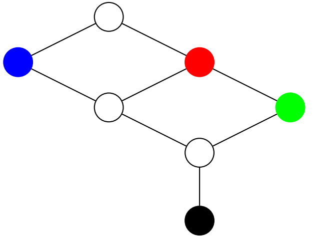
Exercise 2. The lattice $(\mathcal{S}, \succeq)$ is distributive, i.e., the operations of meet and join distribute over each other:

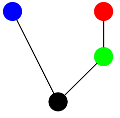
$$(M \vee M') \wedge M'' = (M \wedge M'') \vee (M' \wedge M'')$$
$$(M \wedge M') \vee M'' = (M \vee M'') \wedge (M' \vee M'')$$

Birkhoff's representation theorem

Birkhoff's representation theorem. Let $(L, \succeq)$ be a distributive lattice. There exists $R \subseteq L$ and a bijection ψ between elements of L and upper closed sets of $(R, \succeq)$.

- $S \subseteq R$ is a upper closed set of $(R, \succeq)$ if $e \in R, e' \succeq e \;\Rightarrow e' \in S$;
- $(R, \succeq)$ is the representation poset;
- Often, $|R| \ll |L|$.





Can we use Birkhoff's theorem to solve `[OPT-SM]`?

ROADMAP TO THE SOLUTION OF [OPT-SM]

(a) Transform [OPT-SM] to a linear optimization problem over the UCS of the representation poset $(R, \succeq)$ of $(\mathcal{S}, \succeq)$.

(b) Show that $|R| =$poly(n). → # agents of the instance

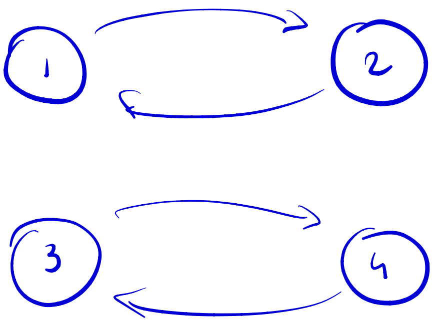
(c) Find an UCS of minimum weight in $(R, \succeq)$ in time poly$(|R|)$.

Rotation digraph

How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	$\boxed{A}$	B	D	C		A	3	4	2	$\boxed{1}$
2	$\boxed{B}$	A	C	D		B	4	3	1	$\boxed{2}$
3	$\boxed{C}$	D	B	A		C	1	2	4	$\boxed{3}$
4	$\boxed{D}$	C	A	B		D	2	1	3	$\boxed{4}$



Rotation digraph

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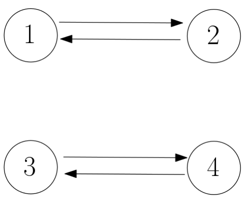
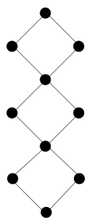
1	$\boxed{A}$	B	D	C
2	$\boxed{B}$	A	C	D
3	$\boxed{C}$	D	B	A
4	$\boxed{D}$	C	A	B

A	3	4	2	$\boxed{1}$
B	4	3	1	$\boxed{2}$
C	1	2	4	$\boxed{3}$
D	2	1	3	$\boxed{4}$





$$\rho_1 = 1A, 1B, 2B, 2A$$

$$\rho_1^- = \{1A, 2B\}$$
$$\rho_1^+ = \{1B, 2A\}$$

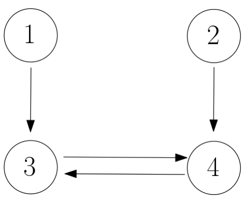
Rotation digraph

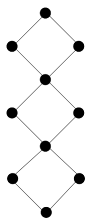
How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	A	$\boxed{B}$	D	C
2	B	$\boxed{A}$	C	D
3	$\boxed{C}$	D	B	A
4	$\boxed{D}$	C	A	B

A	3	4	$\boxed{2}$	1
B	4	3	$\boxed{1}$	2
C	1	2	4	$\boxed{3}$
D	2	1	3	$\boxed{4}$





$$\rho_2 = 3C, 3D, 4D, 4C$$

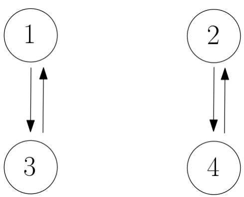
$$\rho_2^- = \{3C, 4D\}$$
$$\rho_2^+ = \{3D, 4C\}$$

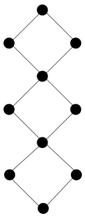
ROTATION DIGRAPH

How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	A	$\boxed{B}$	D	C		A	3	4	$\boxed{2}$	1
2	B	$\boxed{A}$	C	D		B	4	3	$\boxed{1}$	2
3	C	$\boxed{D}$	B	A		C	1	2	$\boxed{4}$	3
4	D	$\boxed{C}$	A	B		D	2	1	$\boxed{3}$	4





$$\rho_3 = 1B, 1C, 3C, 3B$$

$$\rho_3^- = \{1B, 3C\}$$
$$\rho_3^+ = \{1C, 3B\}$$

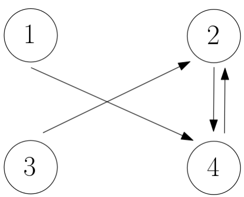
Rotation digraph

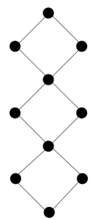
How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	A	B	$\boxed{D}$	C
2	B	$\boxed{A}$	C	D
3	C	D	$\boxed{B}$	A
4	D	$\boxed{C}$	A	B

A	3	4	$\boxed{2}$	1
B	4	$\boxed{3}$	1	2
C	1	2	$\boxed{4}$	3
D	2	$\boxed{1}$	3	4





$$\rho_4 = 2A, 2D, 4D, 4A$$

$$\rho_4^- = \{2A, 4D\}$$
$$\rho_4^+ = \{2D, 4A\}$$

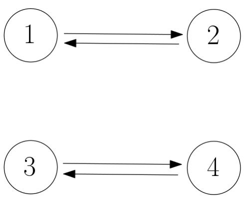
ROTATION DIGRAPH

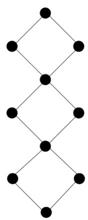
How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	A	B	$\boxed{D}$	C
2	B	A	$\boxed{C}$	D
3	C	D	$\boxed{B}$	A
4	D	C	$\boxed{A}$	B

A	3	$\boxed{4}$	2	1
B	4	$\boxed{3}$	1	2
C	1	$\boxed{2}$	4	3
D	2	$\boxed{1}$	3	4





$$\rho_5 = 1D, 1C, 2C, 2D$$

$$\rho_5^- = \{1D, 2C\}$$
$$\rho_5^+ = \{1C, 2D\}$$

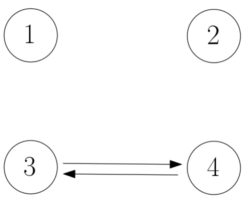
Rotation digraph

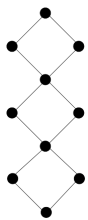
How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	A	B	D	$\boxed{C}$
2	B	A	C	$\boxed{D}$
3	C	D	$\boxed{B}$	A
4	D	C	$\boxed{A}$	B

A	3	$\boxed{4}$	2	1
B	4	$\boxed{3}$	1	2
C	$\boxed{1}$	2	4	3
D	$\boxed{2}$	1	3	4





$$\rho_6 = 3B, 3A, 4A, 4B$$

$$\rho_6^- = \{3B, 4A\}$$
$$\rho_6^+ = \{3A, 4B\}$$

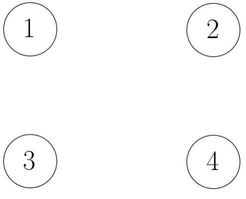
Rotation digraph

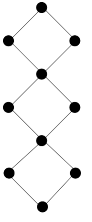
How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

1	A	B	D	$\boxed{C}$
2	B	A	C	$\boxed{D}$
3	C	D	B	$\boxed{A}$
4	D	C	A	$\boxed{B}$

A	$\boxed{3}$	4	2	1
B	$\boxed{4}$	3	1	2
C	$\boxed{1}$	2	4	3
D	$\boxed{2}$	1	3	4





Rotation digraph

How do we "move" within the lattice of stable matchings?

Rotation digraph D_M

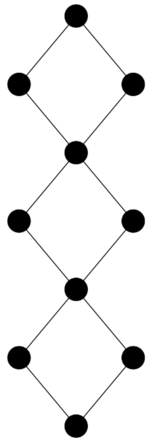
1	A	B	D	$\boxed{C}$	A	$\boxed{3}$	4	2	1
2	B	A	C	$\boxed{D}$	B	$\boxed{4}$	3	1	2
3	C	D	B	$\boxed{A}$	C	$\boxed{1}$	2	4	3
4	D	C	A	$\boxed{B}$	D	$\boxed{2}$	1	3	4

Theorem. Let M be stable matching, ρ rotation in D_M. Then:

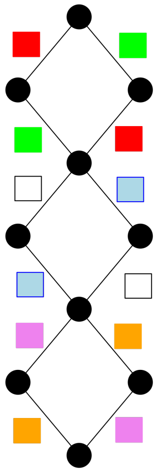
- $M' := M/\rho$ immediately follows M in $(\mathcal{S}, \succeq)$.

Theorem. We can "move" from the student-optimal to the school-optimal stable matching by iteratively finding and eliminating rotations.

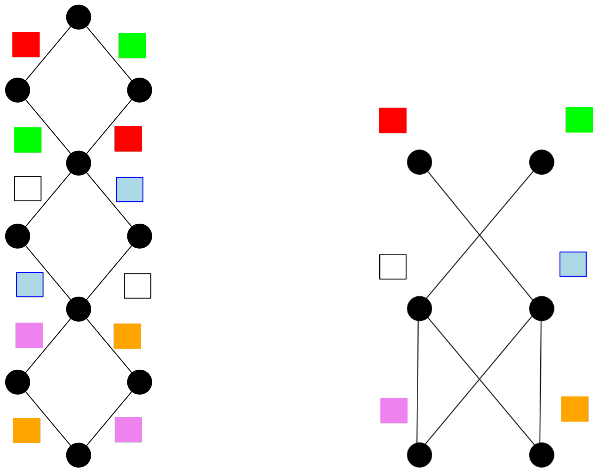
A closer look at rotations



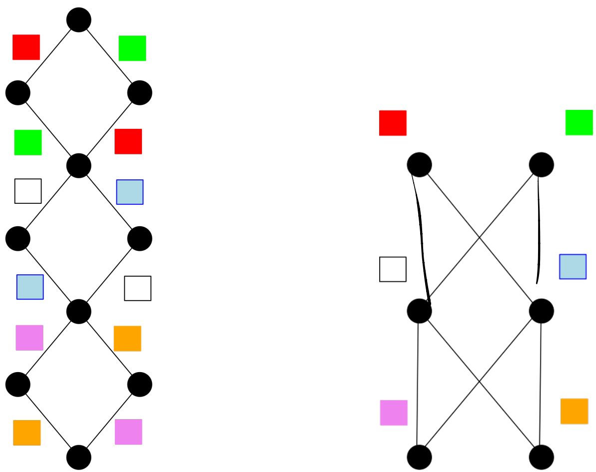
A closer look at rotations



A closer look at rotations



A CLOSER LOOK AT ROTATIONS



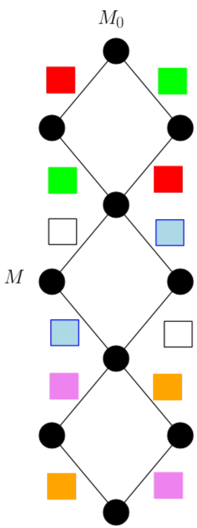
Theorem.

- The poset of rotations is a representation poset for $(\mathcal{S}, \succ)$.
- The bijection between stable matching and UCS of rotations is as follows:

$$M = M_0/\rho_1/\rho_2/\ldots/\rho_k \quad \leftrightarrow \quad \{\rho_1, \rho_2, \ldots, \rho_k\}$$

LATTICE OF STABLE MATCHINGS AND ROTATIONS: EXAMPLE

1	$\boxed{A}$	B	{D}	C		A	3	4	{2}	$\boxed{1}$
2	$\boxed{B}$	{A}	C	D		B	4	{3}	1	$\boxed{2}$
3	$\boxed{C}$	D	{B}	A		C	1	2	{4}	$\boxed{3}$
4	$\boxed{D}$	{C}	A	B		D	2	{1}	3	$\boxed{4}$



$$= \rho_1 = 1A, 1B, 2B, 2A;$$

$$= \rho_2 = 3C, 3D, 4D, 4C;$$

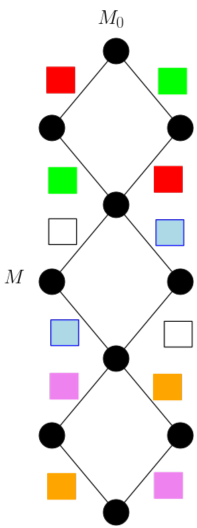
$$= \rho_3 = 1B, 1D, 3D, 3B;$$

$$M = M^0 \cup \{\cancel{1B}, 2A, \cancel{3D}, 4C, 1D, 3B\} \setminus \{\cancel{1A}, \cancel{2B}, \cancel{3C}, \cancel{4D}, \cancel{1B}, \cancel{3D}\}.$$

~~1A~~ 2A
~~2B~~ 1D
~~3C~~ 3B
~~4D~~ 4C

Lattice of stable matchings and rotations: example

1	$\boxed{A}$	B	{D}	C		A	3	4	{2}	$\boxed{1}$
2	$\boxed{B}$	{A}	C	D		B	4	{3}	1	$\boxed{2}$
3	$\boxed{C}$	D	{B}	A		C	1	2	{4}	$\boxed{3}$
4	$\boxed{D}$	{C}	A	B		D	2	{1}	3	$\boxed{4}$



$= \rho_1 = 1A, 1B, 2B, 2A;$

$= \rho_2 = 3C, 3D, 4D, 4C;$

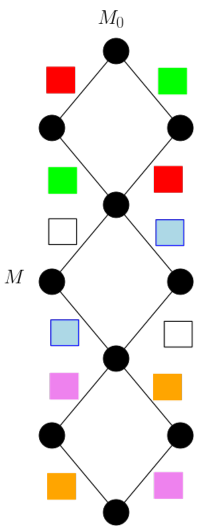
$\square = \rho_3 = 1B, 1D, 3D, 3B;$

$$M = M^0 \cup \{1B, 2A, 3D, 4C, 1D, 3B\} \setminus \{1A, 2B, 3C, 4D, 1B, 3D\}.$$

$$w(M) = w(M^0) + w(\rho_1^+) - w(\rho_1^-) + w(\rho_2^+) - w(\rho_2^-) + w(\rho_3^+) - w(\rho_3^-).$$

LATTICE OF STABLE MATCHINGS AND ROTATIONS: EXAMPLE

1	$\boxed{A}$	B	{D}	C	A	3	4	{2}	$\boxed{1}$
2	$\boxed{B}$	{A}	C	D	B	4	{3}	1	$\boxed{2}$
3	$\boxed{C}$	D	{B}	A	C	1	2	{4}	$\boxed{3}$
4	$\boxed{D}$	{C}	A	B	D	2	{1}	3	$\boxed{4}$



$= \rho_1 = 1A, 1B, 2B, 2A;$

$= \rho_2 = 3C, 3D, 4D, 4C;$

$= \rho_3 = 1B, 1D, 3D, 3B;$

$$M = M^0 \cup \{1B, 2A, 3D, 4C, 1D, 3B\} \setminus \{1A, 2B, 3C, 4D, 1B, 3D\}.$$

$$w(M) = w(M^0) + \underbrace{w(\rho_1^+) - w(\rho_1^-)}_{:=w(\rho_1)} + \underbrace{w(\rho_2^+) - w(\rho_2^-)}_{:=w(\rho_2)} + \underbrace{w(\rho_3^+) - w(\rho_3^-)}_{:=w(\rho_3)}.$$

Back to the roadmap for [OPT-SM]

(a) Transform [OPT-SM] to a linear optimization problem over the UCS of the representation poset $(R, \succeq^*)$ of $(\mathcal{S}, \succeq)$.

$$\begin{aligned}
\min_{M \in \mathcal{S}} w^T M &= \min_{\{\rho_1,\ldots,\rho_k\} \text{ UCS of } (R,\succeq^*)} w(M^0/\rho_1/\ldots/\rho_k) \\
&= w(M^0) + \min_{\{\rho_1,\ldots,\rho_k\} \text{ UCS of } (R,\succeq^*)} \textstyle\sum_{j=1}^k w(\rho_j) \\
&= w(M^0) + \min_{X \text{ UCS of } (R,\succeq^*)} \textstyle\sum_{\rho \in X} w(\rho)
\end{aligned}$$

(b) Show that $|R| = poly(n)$.

Lemma. $(i,j) \in \rho^+$ (resp. ρ^-) for at most one rotation ρ
$\Rightarrow |R| = O(n^2)$.

(c) Find an UCS of minimum weight in $(R, \succeq^*)$ in time poly$(|R|)$.

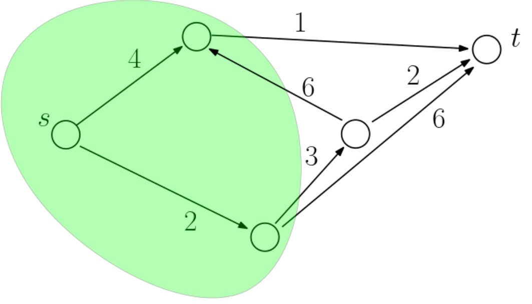
MINIMUM $s - t$ CUT PROBLEM

Given a digraph $G(V, E)$, nodes $s \neq t \in V$, capacities $u : E \to \mathbb{N}$.

- $X \subseteq V$ is an $s - t$ cut if $s \in X, t \notin X$;
- The capacity of an $s - t$ cut X is defined as

$$u(\delta(X)) = \sum_{e=(u,v)\in E: u, in X, v \notin X} u(e).$$

- The min $s - t$ cut problem aims at finding an $s - t$ cut of minimum capacity.



Minimum $s - t$ cut problem

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$$u(\delta(X)) = \sum_{e=(u,v)\in E: u, in X, v \notin X} u(e).$$

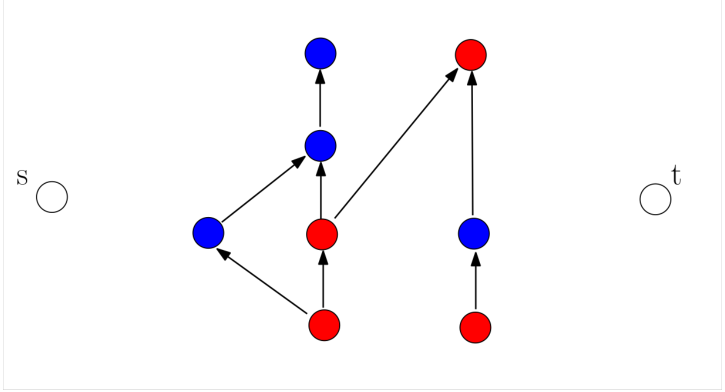
- The min $s - t$ cut problem aims at finding an $s - t$ cut of minimum capacity.
- A minimum $s - t$ cut can be find in time $poly(|V|)$.

Theorem. The problem of finding an UCS of minimum weight can be polynomially reduced to min $s - t$ cut.

REDUCING MIN-WEIGHT UCS TO MIN $s-t$ CUT

Poset $(R, \succeq)$, weights $w : R \to \mathbb{Z}$, with

$w(\color{blue}\bullet) \geq 0$, $w(\color{red}\bullet) < 0$.



$u(\uparrow) = +\infty$

1. Let $X \subseteq R$. $\quad X \cup \{s\}$ $s-t$ cut with $u(X) < +\infty \Leftrightarrow X$ is an UCS.

Reducing min-weight UCS to min $s-t$ cut $X \subseteq R,\ X$ ucs

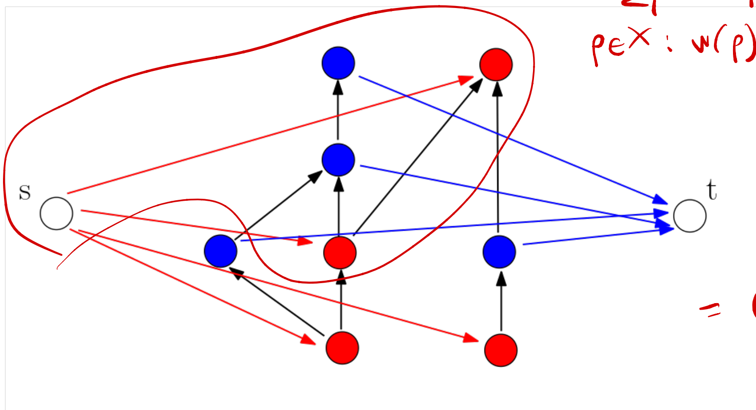
Poset $(R, \succeq)$, weights $w : R \to \mathbb{Z}$, with

$w(\color{blue}{\bullet}) \geq 0,\ w(\color{red}{\bullet}) < 0.$

$$u(\delta(X)) = \sum_{p \in X:\, w(p) \geq 0} w(p) \quad \underbrace{- \sum_{p \in R \setminus X;\, w(p) < 0} (w(p))}_{=\ \boxed{\sum_{p \in R:\, w(p)<0} w(p)} - \sum_{p \in X:\, w(p) < 0} w(p)}$$

$<$

$$= C + w(X)$$



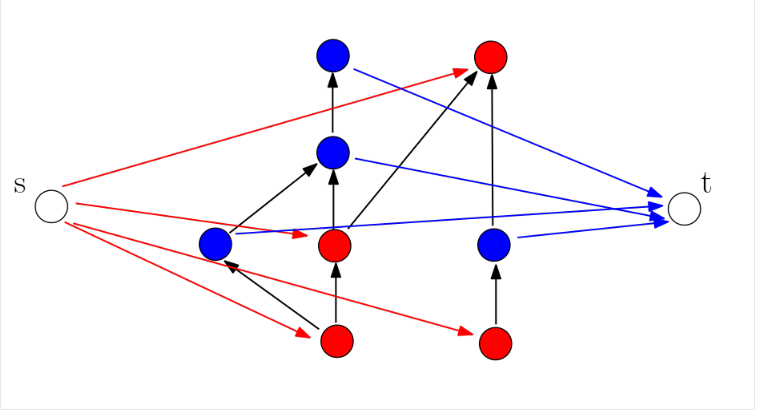
$u(\uparrow) = +\infty, \quad u(\text{(s)} \to \color{red}{\bullet}) = -w(\color{red}{\bullet}), \quad u(\color{blue}{\bullet} \to \text{(t)}) = w(\color{blue}{\bullet}),$

1. Let $X \subseteq R$. $\quad X \cup \{s\}$ $s-t$ cut with $u(X) < +\infty \Leftrightarrow X$ is an UCS.

REDUCING MIN-WEIGHT UCS TO MIN $s-t$ CUT

Poset $(R, \succeq)$, weights $w : R \to \mathbb{Z}$, with

$$w(\textcolor{blue}{\bullet}) \geq 0,\ w(\textcolor{red}{\bullet}) < 0.$$



$$u(\uparrow) = +\infty, \quad u(s \to \textcolor{red}{\bullet}) = -w(\textcolor{red}{\bullet}), \quad u(\textcolor{blue}{\bullet} \to t) = w(\textcolor{blue}{\bullet}),$$

1. Let $X \subseteq R$. $\quad X \cup \{s\}$ $s-t$ cut with $u(X) < +\infty \Leftrightarrow X$ is an UCS.

2. $X \subseteq R$ UCS $\Rightarrow u(X \cup \{s\}) = C + w(X)$.

The algorithm for [OPT-SM]

1. Use the Gale-Shapley algorithm to compute M_0;
2. Compute the poset of rotation $(R, \succeq)$ and, for each $\rho \in R$, $w(\rho)$;
3. Compute an UCS $\{\rho_1, \ldots, \rho_k\}$ of minimum weight in $(R, \succeq)$;
4. Return $M_0/\rho_1/\ldots/\rho_k$.

Theorem. [OPT-SM] can be solved in time $O(n^4 \log n)$.

The red–blue unstable matching problem

Exercise 3. Give a polynomial-time algorithm for the following problem.

Given: An instance I of the marriage problem with weights w on the edges E ("blue"), plus an additional set F of edges ("red") with weights w.
Find: Among those that are stable in I, a matching M maximizing

$$w(M) - w(\text{edges from } F \text{ that block } M).$$

Auxiliary facts that may help:

- ij is in some stable matching (i.e., it is a stable pair) iff it is contained in the student-optimal stable matching, or in some rotation;
- For a student-school pair (i, j) that is not in any stable matching, there exists at most one rotation ρ and schools j', j'' so that: $(i, j') \in \rho^-$, $(i, j'') \in \rho^+$, and $j' >_i j >_i j''$.
- In each sequence of matching obtained starting from the student-optimal stable matching and iteratively eliminating rotations until the school-optimal, we rotate all rotations.

THE ROOMMATE CASE

Dropping the bipartiteness assumption from the marriage model, we obtain the roommate model.

In the roommate model, a stable matching may not exist.

$$\begin{array}{c|cc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$

Good news:

- One can still associate a poset of (differently defined) rotations that represents the set of stable matchings;
- It can be decided in polynomial time if a stable matching exists.

Bad news:

- Solving `[OPT-SM]` in the roommate model is NP-Hard.

APPLICATION TO NYC HIGH SCHOOL MATCH

With the pre-2003 algorithm, 30.000 students were not assigned any of the schools they chose.

Choice	Student-optimal stable matching (1)
1	32,701.5 (58.4)
2	14,382.6 (50.9)
3	9,208.6 (46.0)
4	5,999.8 (41.4)
5	3,883.4 (33.8)
6	2,519.5 (28.4)
7	1,654.6 (24.1)
8	1,034.8 (22.1)
9	716.7 (17.4)
10	485.6 (15.1)
11	316.3 (12.3)
12	211.2 (10.4)
Unassigned	5,613.4 (26.5)

34% Got Their First City High School Choice

By David M. Herszenhorn

May 6, 2004

The City Department of Education said yesterday that 33.6 percent of students who applied to public high schools this year were admitted to their top choice, up from 26.7 percent who got their first choice last year.

Officials attributed the improvement to a new admissions process modeled after the matching system used to assign medical school graduates to hospital residency programs.

Dec. 5, 2014

NEW YORK

How Game Theory Helped Improve New York City's High School Application Process

As bad as many students and parents may think New York's method of matching students and schools is, it was much worse before economists from Duke, M.I.T. and Stanford tackled it.

By Tracy Tullis

Highlights from Part I

- In the marriage model, Gale and Shapley's algorithm outputs in linear time a stable matching with many interesting features;
- Stable matchings in the marriage model have a lot of structure, that can be leveraged on to deduce efficient algorithms;
- In particular, the lattice of stable matchings and the poset of rotations are useful objects.
- Stable matchings are the state-of-the-art concept for many applications.

Bibliography for part I

The marriage model and the stability concept

- Gale, David, and Lloyd S. Shapley. College admissions and the stability of marriage. The American Mathematical Monthly 69.1 (1962): 9-15.

Classical theory of the Marriage and Roommate models

- Gusfield, Dan, and Robert W. Irving. The stable marriage problem: structure and algorithms. *MIT press*, 1989.
- Roth, Alvin E., and Marilda Sotomayor. Two-sided matching. Handbook of game theory with economic applications 1 (1992): 485-541.

Application to the NYC high school match

- Abdulkadiroglu, Atila, Parag A. Pathak, and Alvin E. Roth. The New York City high school match. American Economic Review 95.2 (2005): 364-367.
- Abdulkadiroglu, A., Pathak, P. A., and Roth, A. E. (2009). Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match. American Economic Review, 99(5), 1954-1978.

Finding upper closed sets of minimum weight

- Picard, Jean-Claude. Maximal closure of a graph and applications to combinatorial problems. Management science 22.11 (1976): 1268-1272.

Bibliography for part I, continued

The Red-Blue unstable matching problem

- Faenza, Y., Mourtos, I., Samaris, M., and Sethuraman, J. (2021). (Un) stable matchings with blocking costs. Operations Research Letters, 49(5), 655-662.

Order polytope

- Stanley, Richard P. Two poset polytopes. Discrete Computational Geometry 1.1 (1986): 9-23.

The stable matching polytope

- Aprile, Manuel, Alfonso Cevallos, and Yuri Faenza. On 2-level polytopes arising in combinatorial settings. SIAM Journal on Discrete Mathematics 32.3 (2018): 1857-1886.
- Roth, Alvin E., Uriel G. Rothblum, and John H. Vande Vate. Stable matchings, optimal assignments, and linear programming. Mathematics of Operations Research 18.4 (1993): 803-828.
- Vande Vate, John H. Linear programming brings marital bliss. Operations Research Letters 8.3 (1989): 147-153.

Some drawbacks of the classical model & of stability

Stability is a very stringent condition:

- It disqualifies many "good" and "fair" solutions. For instance, (Abdulkadiroğlu, Pathak, and Roth 09) showed empirically that by forgoing stability, we can obtain a matching that is much "better" for students.
- It may leave many seats empty: the cardinality of a stable matching can be half that of a maximum-size matching.

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- It may leave many seats empty: the cardinality of a stable matching can be half that of a maximum-size matching.

The marriage model cannot always model preferences accurately:

- Ties may be present in rankings;
- Strict and non-strict rankings cannot model other goals a school may want to achieve, like diversity in school cohorts.

Part II:
Changing the output

(a) Pareto-optimality & von Neumann-Morgenstern Stability

Pareto-Optimality

1	$\boxed{A}$	B	C	D	A	4	3	2	1
2	$\boxed{B}$	D	A	C	B	3	4	1	2
3	$\boxed{C}$	B	A	B	C	2	1	4	3
4	C	$\boxed{D}$	A	B	D	1	2	3	4

We defined $M \succeq M'$ if all students weakly prefer M to M'.

We say that M Pareto-dominates M' (for student).

M is Pareto-optimal (for students) if it is not Pareto-dominated, i.e., if

$$\nexists M' \neq M \text{ such that } M'(i) \geq_i M(i) \text{ for all students } i.$$

Stability vs. Pareto-Optimality

There exist instances where no stable matching is Pareto-optimal.

1	A	B	C		A	2	1	3
2	B	A	C		B	1	3	2
3	B	C	A		C	2	1	3

Stability vs. Pareto-Optimality

There exist instances where no stable matching is Pareto-optimal.

1	A	B	C		A	2	1	3
2	B	A	C		B	1	3	2
3	B	C	A		C	2	1	3

Exercise 4. Give an infinite family of marriage instances with n agents such that there is a matching M that Pareto-dominates the student-optimal matching M_0 and moreover, for $\Theta(n)$ students i, the rank of $M(i)$ is $\Theta(n)$ positions better than the rank of $M_0(i)$.

Pareto-Optimality

1	$\boxed{A}$	B	C	D	A	4	3	2	1
2	$\boxed{B}$	D	A	C	B	3	4	1	2
3	$\boxed{C}$	B	A	B	C	2	1	4	3
4	C	$\boxed{D}$	A	B	D	1	2	3	4

We defined $M \succeq M'$ if all students weakly prefer M to M'.

We say that M Pareto-dominates M' (for student).

M is Pareto-optimal (for students) if it is not Pareto-dominated, i.e., if

$$\nexists M' \neq M \text{ such that } M'(i) \geq_i M(i) \text{ for all students } i.$$

Can we find Pareto-optimal matchings that have additional features?

THE TOP TRADING CYCLE ALGORITHM

1	A	B	C	D	E	F	G
2	B	G	A	D	F	C	E
3	C	F	A	G	B	E	D
4	D	G	B	F	C	A	E
5	B	A	G	C	D	E	F
6	D	F	E	C	B	G	A
7	A	G	C	F	D	B	E

A	2	1	7	5	6	3	4
B	3	4	6	5	1	2	7
C	1	7	5	3	4	2	6
D	6	3	6	1	7	2	4
E	6	4	7	3	1	5	2
F	5	2	3	7	4	6	1
G	2	4	1	6	3	5	7

Theorem. The top trading cycle is strategy-proof for students and outputs a Pareto-optimal matching.

Theorem. Among all strategy-proof mechanisms that output a Pareto-Optimal matching, the top trading cycle produces a matching that has an inclusionwise minimal set of blocking pairs.

Theorem. (informal) In large random markets (i.e., # agents $\to +\infty$), TTC does not have less blocking pairs that the Random Serial Dictatorship.

More results on Pareto-optimal matchings in the marriage model

Theorem. A Pareto-optimal matchings of maximum size can be found efficiently.

- Compute a matching of maximum cardinality;
- Iteratively switch every student to an unmatched school they prefer to their current match;
- Iteratively run TTC on a suitably modified instance.

Exercise 5. Prove the previous theorem.

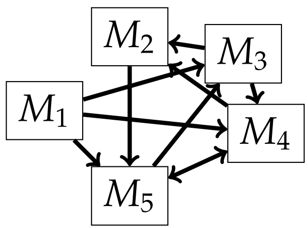
Theorem. It is NP-hard to find a Pareto-optimal matching of minimum weight.

A NEW PERSPECTIVE ON STABILITY

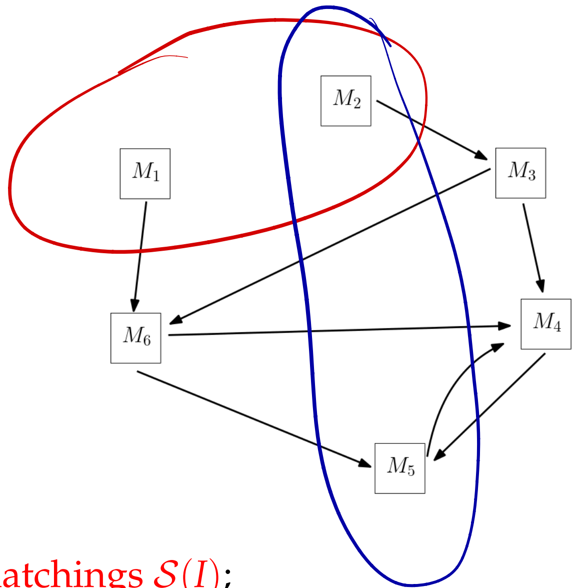
1	A	B	C
2	B	A	
3	A	C	

A	2	3	1
B	1	2	
C	3	1	

#	maximal matching	blocking pairs
M_1	$1B, 2A, 3C$	$\emptyset$
M_2	$1A, 2B, 3C$	$3A$
M_3	$1B, 3A$	$2A$
M_4	$1C, 2B, 3A$	$1B$
M_5	$1C, 2A$	$1B, 2B, 3C$

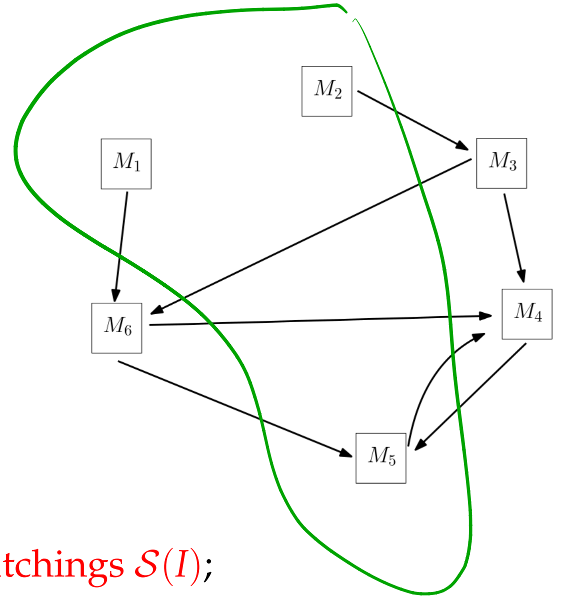


SETS



- Set of stable matchings $\mathcal{S}(I)$;
- Internally stable set;
- Externally stable set;
- Internally stable + Externally stable = vNM stable;

SETS



- Set of stable matchings $\mathcal{S}(I)$;
- Internally stable set;
- Externally stable set;
- Internally stable + Externally stable = vNM stable;

vNM stable sets in game theory

Von Neumann-Morgenstern stability in abstract & cooperative games:

- First solution concept for games with > 2 players;
- Their goal was to codify a “collection of acceptable behaviours in a society: None is clearly preferred to any other, but for each unacceptable behaviour there is a preferred alternative”;
- Popular concept at first (e.g., (Shubik, 82) cites > 100 papers on the topic), then fell out of fashion, mainly because:
  - vNM ss may not exist or be arbitrarily many;
  - In general, they are considered “hard to work with”.

Results on vNM stable sets in the marriage model

In the marriage model:

- There is a unique vNM stable set $\mathcal{V}$;
- There is a matching $M \in \mathcal{V}$ that is Pareto-Optimal (for students);
- M also Pareto-dominates (for students) all stable matchings;
- M can be found in time $O(n^2)$ and a min-weight matching in $\mathcal{V}$ can be found in time $O(n^4 \log n)$.

So, in the marriage case:

- The vNM stable set has nice properties it fails to have in general games;
- The vNM stable set extends the set of stable matchings while keeping its algorithmic tractability.

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vNM STABLE MATCHINGS ARE STABLE MATCHINGS IN DISGUISE

Obs. $\mathcal{V}$ vNM stable $\Rightarrow \mathcal{V} \supseteq \mathcal{S}$.

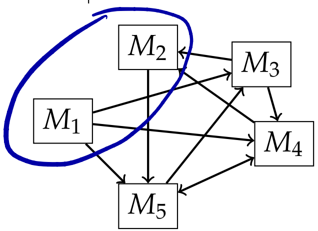
Lemma. Let $\mathcal{V}$ be a vNM stable set of a marriage instance I. Let $I_{\mathcal{V}}$ be the subinstance containing all and only the edges in some matching from $\mathcal{V}$. Then

$$\mathcal{V} = \mathcal{S}(I_{\mathcal{V}}).$$

1	A	B	$\cancel{C}$
2	B	A	
3	$\cancel{A}$	C	

A	2	$\cancel{3}$	1
B	1	2	
C	3	$\cancel{1}$	

#	maximal matching
M_1	$1B, 2A, 3C$
M_2	$1A, 2B, 3C$
M_3	$1B, 3A$
M_4	$1C, 2B, 3A$
M_5	$1C, 2A$



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$$\Downarrow$$

The problem becomes: decide which edges are in $I_{\mathcal{V}}$.

1	*	*	*	*	*
2	*	*	*	*	*
3	*	*	*	*	*
A	*	*	*	*	*
B	*	*	*	*	*
C	*	*	*	*	*

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2	*	*	*	*	*
3	*	*	*	*	*
A	*	*	*	*	*
B	*	*	*	*	*
C	*	*	*	*	*

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1	$*$	$*$	$\color{green}{*}$	$*$	$\color{green}{*}$
2	$*$	$\color{green}{*}$	$*$	$*$	$*$
3	$\color{green}{*}$	$*$	$\color{green}{*}$	$\color{green}{*}$	$*$
A	$*$	$\color{green}{*}$	$*$	$*$	$*$
B	$*$	$*$	$\color{green}{*}$	$\color{green}{*}$	$*$
C	$\color{green}{*}$	$\color{green}{*}$	$*$	$\color{green}{*}$	$*$

$\Rightarrow$

1	$\color{red}{*}$	$\color{red}{*}$	$\color{green}{*}$	$\color{green}{*}$	$\color{green}{*}$
2	$\color{red}{*}$	$\color{green}{*}$	$\color{green}{*}$	$\color{red}{*}$	$\color{red}{*}$
3	$\color{green}{*}$	$\color{red}{*}$	$\color{green}{*}$	$\color{green}{*}$	$\color{red}{*}$
A	$\color{red}{*}$	$\color{green}{*}$	$\color{red}{*}$	$\color{red}{*}$	$\color{red}{*}$
B	$\color{red}{*}$	$\color{red}{*}$	$\color{green}{*}$	$\color{green}{*}$	$\color{green}{*}$
C	$\color{green}{*}$	$\color{green}{*}$	$\color{red}{*}$	$\color{green}{*}$	$\color{red}{*}$

FINDING (IL)LEGAL EDGES VIA ROTATIONS: ROTATE-AND-REMOVE

1	A	$\boxed{B}$	D	C	E
2	$\boxed{A}$	B	D	C	E
3	$\boxed{C}$	D	B	A	E
4	$\boxed{D}$	A	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	4	$\boxed{2}$	1
B	5	3	4	$\boxed{1}$	2
C	5	1	2	4	$\boxed{3}$
D	5	1	2	3	$\boxed{4}$
E	2	3	$\boxed{5}$	1	4

1. Using Gale and Shapley, find the student-optimal sm M.
2. Starting from M, eliminate rotations as to find the school-optimal sm M' and mark stable edges found on the way legal.

Finding (il)legal edges via rotations: Rotate-and-remove

1	A	$\boxed{B}$	D	C	E
2	A	B	$\boxed{D}$	C	E
3	$\boxed{C}$	D	B	A	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	$\boxed{4}$	2	1
B	5	3	4	$\boxed{1}$	2
C	5	1	2	4	$\boxed{3}$
D	5	1	$\boxed{2}$	3	4
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1	A	B	$\boxed{D}$	C	E	A	5	3	$\boxed{4}$	2	1
2	A	B	D	$\boxed{C}$	E	B	5	$\boxed{3}$	4	1	2
3	C	D	$\boxed{B}$	A	E	C	5	1	$\boxed{2}$	4	3
4	D	$\boxed{A}$	B	C	E	D	5	$\boxed{1}$	2	3	4
5	$\boxed{E}$	B	C	D	A	E	2	3	$\boxed{5}$	1	4

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1	A	B	$\boxed{D}$	C	E
2	A	B	D	$\boxed{C}$	E
3	C	D	$\boxed{B}$	A	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	2
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1	A	B	$\boxed{D}$	C	E	A	5	3	$\boxed{4}$	2	1
2	A	B	D	$\boxed{C}$	E	B	5	$\boxed{3}$	4	1	2
3	C	D	$\boxed{B}$	A	E	C	5	1	$\boxed{2}$	4	3
4	D	$\boxed{A}$	B	C	E	D	5	$\boxed{1}$	2	3	4
5	$\boxed{E}$	B	C	D	A	E	2	3	$\boxed{5}$	1	4

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2	A	B	D	$\boxed{C}$	E
3	C	D	$\boxed{B}$	A	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	2
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1	A	B	$\boxed{D}$	C	E	A	5	3	$\boxed{4}$	2	1
2	A	B	D	$\boxed{C}$	E	B	5	$\boxed{3}$	4	1	2
3	C	D	$\boxed{B}$	A	E	C	5	1	$\boxed{2}$	4	3
4	D	$\boxed{A}$	B	C	E	D	5	$\boxed{1}$	2	3	4
5	$\boxed{E}$	B	C	D	A	E	2	3	$\boxed{5}$	1	4

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2	A	B	D	$\boxed{C}$	E	B	5	$\boxed{3}$	4	1	2
3	C	D	$\boxed{B}$	A	E	C	5	1	$\boxed{2}$	4	3
4	D	$\boxed{A}$	B	C	E	D	5	$\boxed{1}$	2	3	4
5	$\boxed{E}$	B	C	D	A	E	2	3	$\boxed{5}$	1	4

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1	A	B	$\boxed{D}$	C	E
2	A	~~B~~	D	$\boxed{C}$	E
3	C	~~D~~	$\boxed{B}$	A	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	~~2~~
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	~~3~~	4
E	2	3	$\boxed{5}$	1	4

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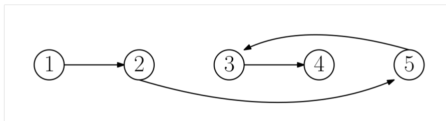
2	A	~~B~~	D	$\boxed{C}$	E
3	C	~~D~~	$\boxed{B}$	A	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	~~2~~
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	~~3~~	4
E	2	3	$\boxed{5}$	1	4



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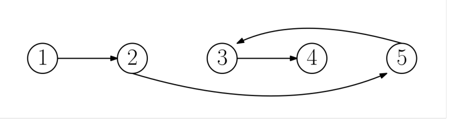
1	A	B	$\boxed{D}$	C	E
2	A	~~B~~	D	$\boxed{C}$	E
3	C	~~D~~	$\boxed{B}$	A	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	3	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	~~2~~
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	~~3~~	4
E	2	3	$\boxed{5}$	1	4



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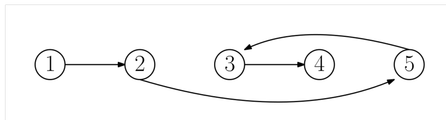
Finding (il)legal edges via rotations: Rotate−and−remove

1	A	B	$\boxed{D}$	C	E
2	A	$\cancel{B}$	D	$\boxed{C}$	E
3	C	$\cancel{D}$	$\boxed{B}$	$\cancel{A}$	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	$\cancel{3}$	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	$\cancel{2}$
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	$\cancel{3}$	4
E	2	3	$\boxed{5}$	1	4



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2	A	~~B~~	D	$\boxed{C}$	E
3	C	~~D~~	$\boxed{B}$	~~A~~	E
4	D	$\boxed{A}$	B	C	E
5	$\boxed{E}$	B	C	D	A

A	5	~~3~~	$\boxed{4}$	2	1
B	5	$\boxed{3}$	4	1	~~2~~
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	~~3~~	4
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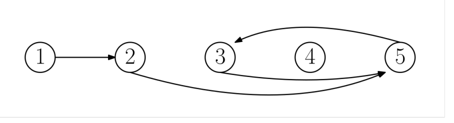

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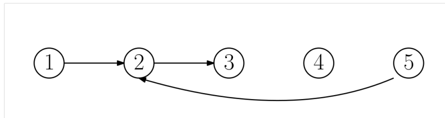
FINDING (IL)LEGAL EDGES VIA ROTATIONS: ROTATE−AND−REMOVE

1	A	B	$\boxed{D}$	C	E
2	A	$\cancel{B}$	D	$\boxed{C}$	E
3	C	$\cancel{D}$	B	$\cancel{A}$	$\boxed{E}$
4	D	$\boxed{A}$	B	C	E
5	E	$\boxed{B}$	C	D	A

A	5	$\cancel{3}$	$\boxed{4}$	2	1
B	$\boxed{5}$	3	4	1	$\cancel{2}$
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	$\cancel{3}$	4
E	2	$\boxed{3}$	5	1	4



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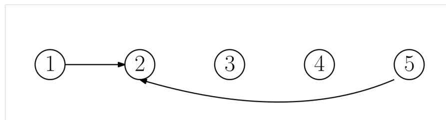
Finding (il)legal edges via rotations: Rotate-and-remove

1	A	B	$\boxed{D}$	C	E
2	A	~~B~~	D	$\boxed{C}$	~~E~~
3	C	~~D~~	B	~~A~~	$\boxed{E}$
4	D	$\boxed{A}$	B	C	E
5	E	$\boxed{B}$	C	D	A

A	5	~~3~~	$\boxed{4}$	2	1
B	$\boxed{5}$	3	4	1	~~2~~
C	5	1	$\boxed{2}$	4	3
D	5	$\boxed{1}$	2	~~3~~	4
E	~~2~~	$\boxed{3}$	5	1	4



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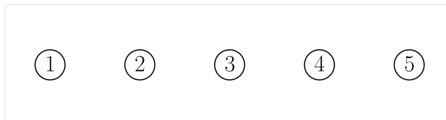
FINDING (IL)LEGAL EDGES VIA ROTATIONS: ROTATE-AND-REMOVE

1	A	B	$\boxed{D}$	~~C~~	~~E~~
2	A	~~B~~	D	$\boxed{C}$	~~E~~
3	C	~~D~~	B	~~A~~	$\boxed{E}$
4	D	$\boxed{A}$	~~B~~	~~C~~	~~E~~
5	E	$\boxed{B}$	~~C~~	~~D~~	~~A~~

A	~~5~~	~~3~~	$\boxed{4}$	2	1
B	$\boxed{5}$	3	~~4~~	1	~~2~~
C	~~5~~	~~1~~	$\boxed{2}$	~~4~~	3
D	~~5~~	$\boxed{1}$	2	~~3~~	4
E	~~2~~	$\boxed{3}$	5	~~1~~	~~4~~



1. Using Gale and Shapley, find the student-optimal sm M.
2. Starting from M, eliminate rotations as to find the school-optimal sm M' and mark stable edges found on the way legal.

   Lemma Let e be an edge between two legal edges that is not marked as legal. Then any matching containing e is blocked by an edge of I_{legal}. We call an edge with the latter property illegal.

   Lemma. Illegal edges can be deleted without changing the vNM stable set.
3. **Lemma.** If there exists $(a, next_{M'}(a)) \in A(D_M)$ such that $next_{M'}(a)$ is a sink, then $a, s_M(a)$ is illegal. Delete it and go to 2.

THE ALGORITHM FOR FINDING THE VNM STABLE SET

1. Run `Rotate-and-remove`.
2. Switch the roles of students and schools, run `Rotate-and-remove`.
3. Output the instance restricted to edges marked as legal.

Theorem. The algorithm above can be implemented as to run in time $O(n^2)$ and outputs an instance I' so that $\mathcal{S}(I')$ is a vNM stable set. Moreover, the vNM stable set is unique.

Then we can use:

- Gale-Shapley algorithm to output a student optimal matching in the vNM stable set $\mathcal{V}$;
- The algorithm for `[OPT-SM]` to output a min-weight matching from $\mathcal{V}$.

Results on vNM stable sets in the marriage model

In the marriage model:

- There is a unique vNM stable set $\mathcal{V}$;
- There is a matching $M \in \mathcal{V}$ that is Pareto-Optimal (for students):
- M also Pareto-dominates (for students) all stable matchings:
- M can be found in time $O(n^2)$ and a min-weight matching in $\mathcal{V}$ can be found in time $O(n^4 \log n)$.

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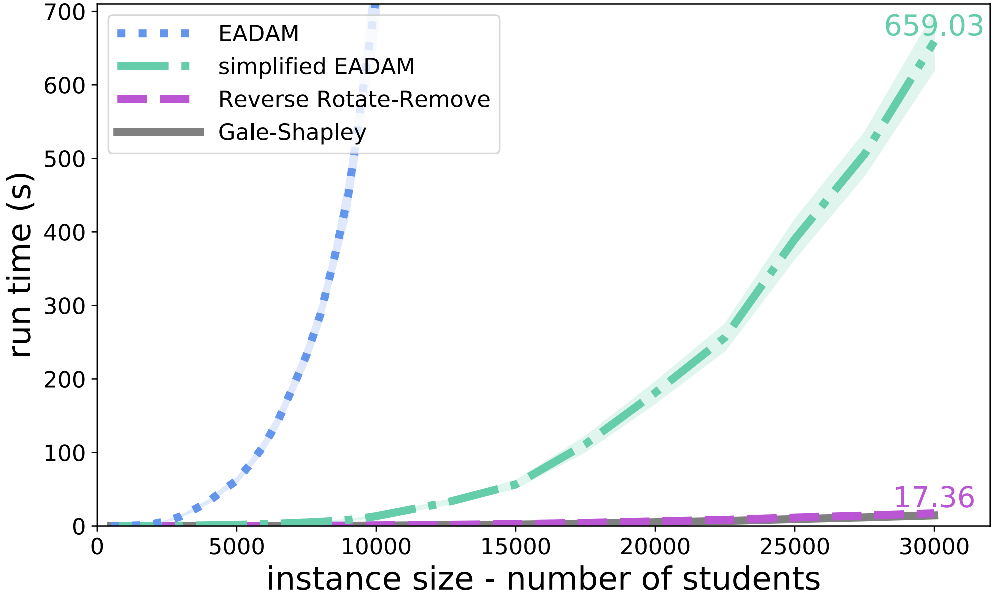
Pareto-optimality of the vNM student-optimal matching

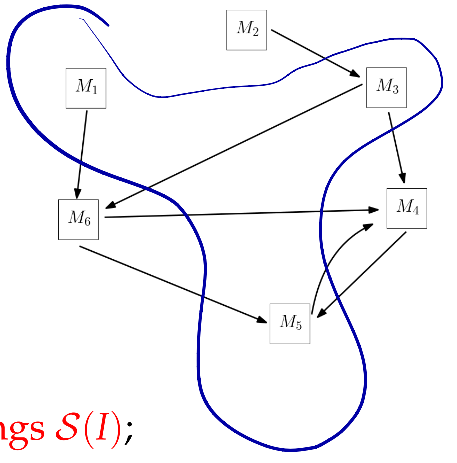
Theorem. The student-optimal matching in the von Neumann-Morgenstern stable set $\mathcal{V}$:

- Is Pareto-optimal for students.
- Because of the lattice structure, it Pareto-dominates all other matchings from $\mathcal{V}$ – in particular, all stable matchings.

Finding the student-optimal matching in the vNM set

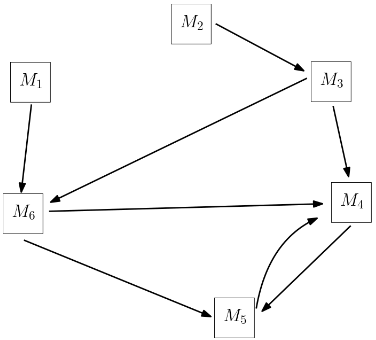


Back to Sets



- Set of stable matchings $\mathcal{S}(I)$;
- Internally stable set;
- Externally stable set;
- Internally stable + Externally stable = vNM stable;

Back to Sets



- Set of stable matchings $\mathcal{S}(I)$;
- Internally stable set;
- Externally stable set;
- Internally stable + Externally stable = vNM stable;
- Inclusionwise maximal among internally stable = internally closed.

Finding an internally closed set containing a given set of matchings

~~Given: a set of matchings $\mathcal{M}$ of an instance I~~.
~~Find: an internally closed set of matching $\mathcal{M}' \supseteq \mathcal{M}$~~

Lemma. Let $\mathcal{M}$ be an internally closed set of an instance I. Let $I_{\mathcal{M}}$ be the subinstance containing all and only the edges in some matching from $\mathcal{M}$. Then

$$\mathcal{M} = \mathcal{S}(I_{\mathcal{M}}).$$

Given: a roommate instance I and a subinstance I_0 of I.
Find: an instance $I' \subseteq I$ with $\mathcal{S}(I_0) \subseteq \mathcal{S}(I')$ such that $\mathcal{S}(I')$ is internally closed.

1	*	*	*	*	*
2	*	*	*	*	*
3	*	*	*	*	*
A	*	*	*	*	*
B	*	*	*	*	*
C	*	*	*	*	*

$\Rightarrow$

1	*	*	*	*	*
2	*	*	*	*	*
3	*	*	*	*	*
A	*	*	*	*	*
B	*	*	*	*	*
C	*	*	*	*	*

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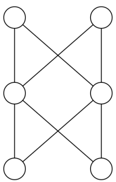
2	*	*	*	*	*
3	*	*	*	*	*
A	*	*	*	*	*
B	*	*	*	*	*
C	*	*	*	*	*

$\Rightarrow$

1	*	*	*	*	*
2	*	*	*	*	*
3	*	*	*	*	*
A	*	*	*	*	*
B	*	*	*	*	*
C	*	*	*	*	*

Reinterpreting the algorithm to find a vNM stable set

To find a vNM stable set we iteratively "append" new $\succeq$-maximal and $\succeq$-minimal rotations to $(R, \succeq)$.

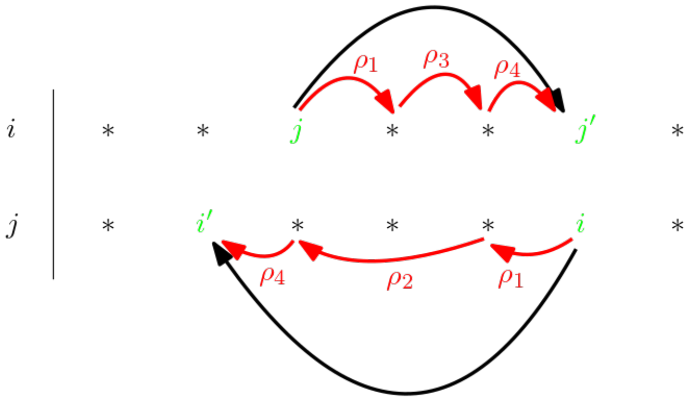


To compute an internally closed set, we may also need to "dissect" a rotation, i.e., replace it with a poset of new rotations.

How to dissect a rotation (sketch)

Let ρ be a rotation.

- Student $i \in \rho$ is matched to a worse school after the elimination of ρ.
- School $j \in \rho$ will be matched to a better student after the elimination of ρ.

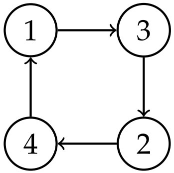


We dissect a rotation by looking for cycles in a generalized rotation digraph.

How to dissect a rotation (sketch)

1	D, A, C
2	B, A
3	C, B
4	A, D
A	$2, 1, 4$
B	$3, 2$
C	$1, 3$
D	$4, 1$

1	D, C
2	B, A
3	C, B
4	A, D
A	$2, 4$
B	$3, 2$
C	$1, 3$
D	$4, 1$



$$M \xrightarrow{\rho} M'$$

How to dissect a rotation (sketch)

1	D, A, C
2	B, A
3	C, B
4	A, D
A	$2, 1, 4$
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2	B, A
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C	$1, 3$
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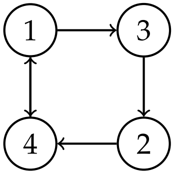
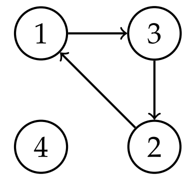


$$M \xrightarrow{\rho} M'$$

How to dissect a rotation (sketch)

1	$D, \boxed{A}, C$
2	$\boxed{B}, A$
3	$\boxed{C}, B$
4	$A, \boxed{D}$
A	$2, \boxed{1}, 4$
B	$3, \boxed{2}$
C	$1, \boxed{3}$
D	$\boxed{4}, 1$

1	D, C
2	B, A
3	C, B
4	A, D
A	$2, 4$
B	$3, 2$
C	$1, 3$
D	$4, 1$



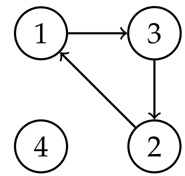
$$M \xrightarrow{\rho} M'$$

$$M \xrightarrow{\rho_1} \boxed{M''}$$

How to dissect a rotation (sketch)

1	D, $\boxed{A}$, C
2	$\boxed{B}$, A
3	$\boxed{C}$, B
4	A, $\boxed{D}$
A	2, $\boxed{1}$, 4
B	3, $\boxed{2}$
C	1, $\boxed{3}$
D	$\boxed{4}$, 1

1	D, C
2	B, A
3	C, B
4	A, D
A	2, 4
B	3, 2
C	1, 3
D	4, 1



$$M \xrightarrow{\rho} M'$$

$$M \xrightarrow{\rho_1} \boxed{M''} \xrightarrow{\rho_2} M'$$

Finding an internally closed set of matchings

Given: a roommate instance I and a subinstance I_0 of I.
Find: an instance I' with $I_0 \subseteq I' \subseteq I$ such that $\mathcal{S}(I')$ is internally closed.

1 Start from $I' = I_0$.

2 Compute the rotation poset $(R, \succeq)$ associated to I'.

3a Try to "append" a rotation to $(R, \succeq)$

- If successful, add the corresponding edges go to I' and go to 2.

3b Try to "dissect" rotations from $(R, \succeq)$.

- If successful, add the corresponding edges go to I' and go to 2.

4 Output I'.

Theorem. The algorithm above outputs $I' \subseteq I$ such that $\mathcal{S}(I') \supseteq \mathcal{S}(I_0)$ and $\mathcal{S}(I')$ is internally closed in time $O(n^4)$.

THE ROOMMATE CASE

Recall that, in the roommate case, stable matchings can also be represented via a rotation poset (defined differently).

We can still find a rotation-based condition for enlarging certain internally stable sets.

Theorem. Let I be a roommate instance with $\mathcal{S}(I) \neq \emptyset$. $\mathcal{S}(I_{\mathcal{S}})$ is internally closed iff its rotation poset does not have stitched rotations.

Theorem. Deciding if I has a stitched rotation is NP-Hard.

Corollary. Deciding if a given $I' \subseteq I$ is such that $\mathcal{S}(I')$ is internally closed is co-NP-Hard. Deciding if $I' \subseteq I$ is such that $\mathcal{S}(I')$ is vNM stable is co-NP-Hard. Finding a vNM stable set of I (or deciding none exists) is co-NP-Hard.

Highlights from Part II(b)

In the marriage model:

- Pareto-optimality is an interesting concept alternative to stability;
- However, with the exception of a few (still relevant) Pareto-optimal matchings, not much is known about this class;
- The vNM Stable sets of matchings contain both the set of stable matchings and one Pareto-optimal matching; also, it is algorithmically as tractable as the family of stable matchings
- Internally closed sets allow us to extend sets of pairwise-non blocking matchings in a principled manner;
- For vNM stable and internally closed sets of matchings, the lattice of stable matchings and the poset of rotations are crucial tools.

Bibliography for part II (a) (Pareto-Opt & vNM stability)

The Top Trading cycle algorithm in the house allocation problem

- Attributed to David Gale in: Shapley, Lloyd, and Herbert Scarf. On cores and indivisibility. Journal of mathematical economics 1.1 (1974): 23-37.

Top Trading cycle in the marriage model

- Roth, Alvin E. Incentive compatibility in a market with indivisible goods. Economics letters 9.2 (1982): 127-132.
- Abdulkadiroglu, A., Che, Y. K., Pathak, P. A., Roth, A. E., Tercieux, O. (2017). Minimizing justified envy in school choice: the design of New Orleans' OneApp (No. w23265). National Bureau of Economic Research.
- Che, Yeon-Koo, and Olivier Tercieux. Efficiency and stability in large matching markets. Journal of Political Economy 127.5 (2019): 2301-2342.

Max cardinality / max weight Pareto-optimal matchings

- Abraham, D. J., Cechlárová, K., Manlove, D. F., & Mehlhorn, K. (2005). Pareto optimality in house allocation problems. Proceedings of ISAAC 2004.

Bibliography for part II (a) (Pareto-Opt & vNM stability), continued

von Neumann-Morgenstern stable sets in game theory

- Von Neumann, John, and Oskar Morgenstern. Theory of games and economic behavior. Theory of games and economic behavior. Princeton university press, 2007 (reprint).
- Lucas, William F. von Neumann-Morgenstern stable sets. Handbook of game theory with economic applications 1 (1992): 543-590.

von Neumann-Morgenstern stable sets of matchings in the marriage model

- Ehlers, Lars. Von Neumann–Morgenstern stable sets in matching problems. Journal of Economic Theory 134.1 (2007): 537-547.
- Wako, Jun. A polynomial-time algorithm to find von Neumann-Morgenstern stable matchings in marriage games. Algorithmica 58 (2010): 188-220.
- Faenza, Yuri, and Xuan Zhang. Legal assignments and fast EADAM with consent via classic theory of stable matchings. Operations Research 70.3 (2022): 1873-1890.

Internally Closed sets and vNM stability in the roommate model

- Faenza, Yuri, Clifford Stein, and Jia Wan. Internal Closedness and von Neumann-Morgenstern Stability in Matching Markets. arXiv preprint arXiv:2211.17050 (2022).