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ICERM NR Community PhD School (I)

Introduction to Numerical Relativity (Theory)

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References:

- Miguel Alcubierre 2008 "Introduction to 3+1 NR"
- Baumgarte & Shapiro "Numerical relativity" 2010
- → "NR: starting from scratch" 2021
- Masaru Shibata 2015 "Numerical Relativity"
- E. Gourgoulhon 2007 online lecture notes gr-qc/0703035

Notation:

- $\mu,\nu,\ldots = 0,\ldots,3$ ST indices
- $i,j,k,\ldots = 1,2,3$ spatial indices
- $g_{\mu\nu}$ – Lorentzian (ST) metric with $(-+++)$

  ${}^{(4)}R^{\mu}{}_{\nu\sigma\rho}$ – 4D Riemann wrt g
- γ_{ij} – Euclidean (spatial) metric $(+++)$

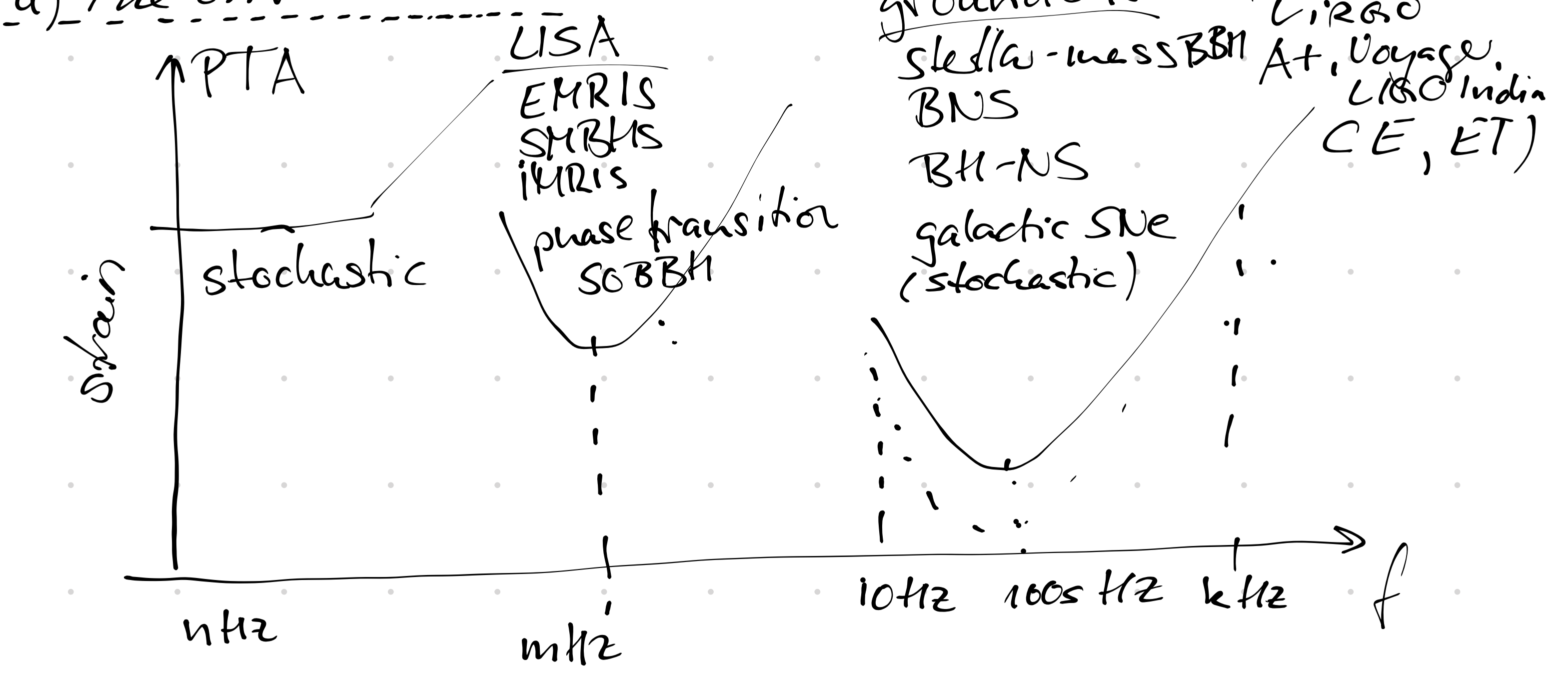
  $R^{i}{}_{jkl}$ – 3D Riemann wrt γ

0) Introduction

· What do we (typically) mean by "NR"?

0.1) NR in the context of GW science (astro, cosmology, fundamental physics)

a) The GW universe



b) Stages of compact binary evolution

INSPIRAL

$v/c \ll 1$, $\Phi/c^2 \ll 1$

- Post-Newtonian expansion in (v/c)
- Post-Minkowski expansion in G (connect to scattering amplitudes)

MERGER

$v/c \sim \mathcal{O}(0.5)$

Numerical Relativity (NR)

RINGDOWN

↳ perturbations around Kerr (Ψ_4) or Schwarzschild (h)

↳ NR

COMBINE to inspiral-merger-ringdown (IMR)

↳ Phenom

↳ effective one body (EOB)

↳ NR surrogates

Other applications

- numerical cosmology
- tests of gravity & NR beyond GR
- DM / BSM particle physics
- Holography / AdS/CFT
- higher d gravity
- stability

"Ingredients" / Outline

0) Theoretical model

today: EEs in $D=4$, vacuum, asympt. flat.

$$G_{\mu\nu} = {}^{(4)}R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} {}^{(4)}R = 0$$

$$\Leftrightarrow {}^{(4)}R_{\mu\nu} = 0$$

1) Spacetime decomposition

2) 3+1 decompositions of EEs ← xTensor

3) Initial data

4) Evolution eqs
(well-posedness, BSSN)

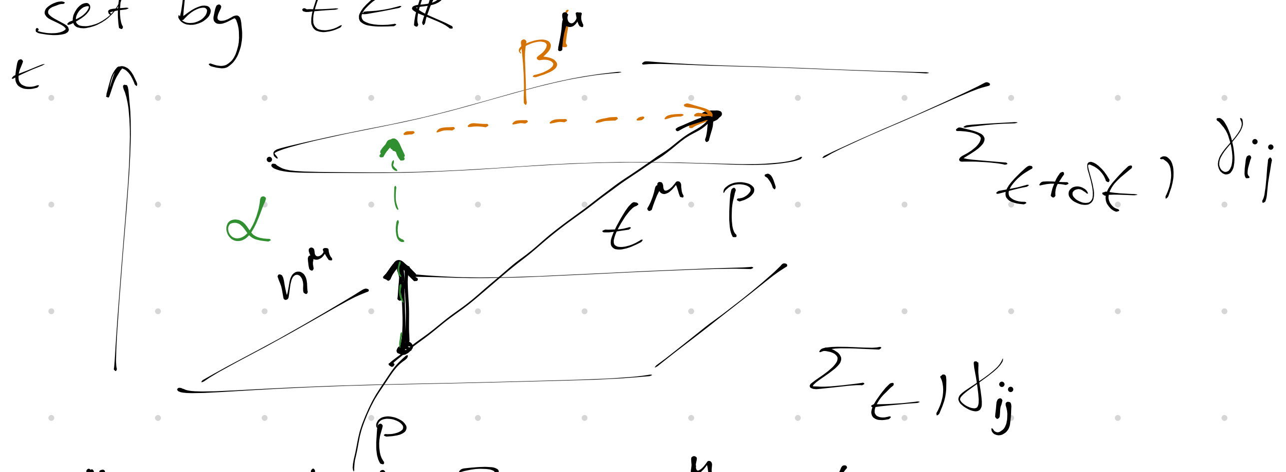
today

5) Gauge conditions, treatment of BHs

6) Observables (GWs, horizon) Deirdre

1) 3+1 decomposition of ST

- foliate 4D manifold $(\mathcal{M}, g)$ into (spacelike) hypersurfaces (Σ_t, γ) with levels set by $t \in \mathbb{R}$



n^μ: normal to Σ_t; $n_\mu n^\mu = -1$

α: lapse fct., propertime between hypersurfaces measured by observer along n^μ

β^i: shift vector, relative velocity between normal obs. & lines of const spatial coords; $\beta^\mu n_\mu = 0$

$$t^\mu = \alpha n^\mu + \beta^\mu$$

γ_{ij}: induced, spatial metric measured $dl^2 = \gamma_{ij} dx^i dx^j$

- $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$$= -(\alpha^2 - \underbrace{\gamma_{ij}\beta^i\beta^j}_{\beta^2}) dt^2 + 2\gamma_{ij}\beta^i dt\, dx^j + \gamma_{ij} dx^i dx^j$$

$$(g_{\mu\nu}) = \left(\begin{array}{c|c} -\alpha^2 + \beta^2 & \beta_i \\ \hline \beta_j & \gamma_{ij} \end{array}\right)$$

- relate $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ ← related $g_{\mu\nu}$

γ defines a projection op. $\perp = \gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$

↳ decompose any tensor:

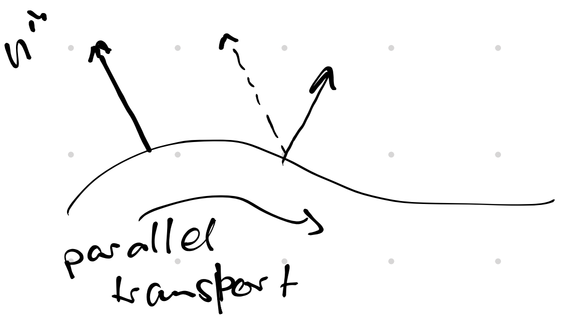
$$V^\mu = \mathcal{V} n^\mu + \nu^\mu \quad \text{with} \quad \mathcal{V} = -V^\mu n_\mu$$
$$\nu^\mu = \gamma^\mu{}_\nu V^\nu$$

• covariant derivative

$$D_i \simeq \perp (\nabla_\mu)_i \,, \quad D_i \gamma_{jk} = 0 \;/\; \nabla_\mu \gamma_{\nu\rho} \neq 0 \;!$$

• Ricci curvature (intrinsic) $(D_m D_n - D_n D_m) v^k = R^k{}_{\ell m n} v^\ell$

Extrinsic curvature $K_{\mu\nu}$ (K_{ij})



$K_{\mu\nu}$ - measure for change of normal vector

$$K_{\mu\nu} = - \gamma^\kappa_\mu \nabla_\kappa n_\nu$$

↑ sign convention.

↳ one can show: $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}$, $t^\mu = \alpha n^\mu + \beta^\mu$

$$= -\frac{1}{2\alpha} (\partial_t - \mathcal{L}_\beta) \gamma_{\mu\nu}$$

$$\rightarrow \boxed{(\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\alpha K_{ij}}$$ kinematic evol eq.

2) 3+1 Decomposition of EES

↳ dynamics in GR

$$^{(4)}R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

↳ Projection with γ, Contractions with n

a) Projections of Riemann → Gauss-Codazzi-Mainardi eqs

b) Projections of energy-momentum tensor $T_{\mu\nu}$

(i) energy density $\rho = T_{\mu\nu} n^\mu n^\nu$

(ii) energy flux $j_i = -\gamma^\mu{}_i T_{\mu\nu} n^\nu$

(iii) spatial stress $S_{ij} = \gamma^\mu{}_i \gamma^\nu{}_j T_{\mu\nu}$

c) put together to find projections of EES

NOTE: rewrite as

$$\mathcal{E}^{(1)}_{\mu\nu} = {}^{(4)}R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} {}^{(4)}R - 8\pi T_{\mu\nu} + g_{\mu\nu} \underbrace{\Lambda}_{\text{cosmological const}} = 0$$

trace-reversed

$$\mathcal{E}^{(2)}_{\mu\nu} = {}^{(4)}R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Lambda - 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) = 0$$

Found

(i) Hamiltonian constraint

$$0 = \mathcal{H} = 2\, \mathcal{E}^{(1)}_{\mu\nu} n^\mu n^\nu = R - K_{ij} K^{ij} + \underbrace{K}_{\gamma^{ij} K_{ij}}{}^2 - 16\pi \rho = 0$$

(ii) Momentum constraint

$$0 = \mathcal{M}_i = - \mathcal{E}^{(1)}_{\mu\nu} \gamma^\mu{}_i n^\nu = D^j K_{ij} - D_i K - 8\pi j_i = 0$$

$\Big\}$ constr. eqs.

(iii) $(\partial_t - \mathcal{L}_\beta) K_{ij} = - D_i D_j \alpha + \alpha \left(R_{ij} - 2 K_{ik} K^k{}_j + K K_{ij} \right)$

$\qquad - 8\pi \alpha \left(S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho) \right)$

together with $(\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\alpha K_{ij}$

$\Big\}$ evol eqs

3) Initial problem

Goal: prescribe $(\gamma_{ij}, K_{ij})\big|_{t=0}$ [vacuum!]

$\llcorner$ $\gamma_{ij}, K_{ij} \to 12$ indep. comp.
4 of them fixed by solving constraints

→ • conformal decomposition Lichnerowicz (1944)
York ('71, '72)

• conformal thin sandwich method (York '99)

• CTTK (Aurrekoetxea, Clough, Lim '22)

3.1. Conformal decomposition

i) Metric

$$\underline{\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}}$$

ψ - conformal factor ← (1) fixed by constraint

$\hat{\gamma}_{ij}$ - conformal metric ← (5) free
(consider as given
eg: $\hat{\gamma}_{ij} = \eta_{ij}$, $\det \hat{\gamma}_{ij} \equiv \hat{\gamma}$

ii) Extrinsic curvature

$$K_{ij} = \underbrace{A_{ij}}_{\text{tracefree}} + \frac{1}{3}\gamma_{ij} \underbrace{K}_{\text{trace}}$$

$K = \gamma^{ij} K_{ij}$ ← (1)

$\gamma^{ij} A_{ij} = 0$ ← (1)

transverse part of $\hat{A}_{ij}$ ← (1)

"free"

$$A_{ij} = \psi^{-2} \hat{A}_{ij}$$

$$\rightarrow \underline{K_{ij} = \psi^{-2} \hat{A}_{ij} + \frac{1}{3}\psi^4 \hat{\gamma}_{ij} K}$$

• longitudinal part of $\hat{A}_{ij}$ fixed by constraint (3) constraints in $(\psi, \hat{\gamma}_{ij}, K, \hat{A}_{ij})$ in vacuum

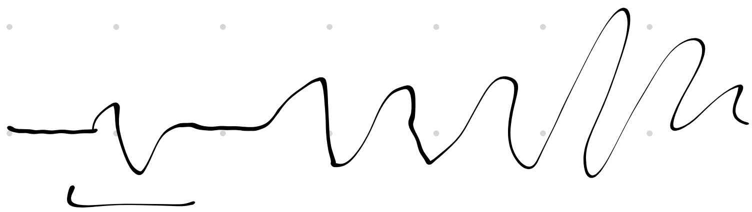
$$\mathcal{H} = \underline{\hat{\Delta}\psi} - \frac{1}{8}\psi\hat{R} + \frac{1}{8}\psi^{-7}\hat{A}_{ij}\hat{A}^{ij} - \frac{1}{12}\psi^{5}K^{2} = 0$$

$$\mathcal{M}_i = \underline{\hat{D}^j\hat{A}_{ij}} - \frac{2}{3}\psi^{6}\hat{D}_i K = 0 \qquad \hat{\Delta} = \hat{\gamma}^{ij}\hat{D}_i\hat{D}_j$$

<u>common choices for free components</u>

- conformally flat: $\hat{\gamma}_{ij} = \eta_{ij}$, $\hat{D}_i \longrightarrow \partial_i$ $(\sim)$
  $\hat{R} = 0$
  $\tilde{\Delta} = \eta^{ij}\partial_i\partial_j$
- maximal slicing: $K = 0$ $(\sim)$
- asymptotic flatness: $\lim_{r\to\infty}\psi = 1$

$\downarrow$



"junk"

$$\mathcal{M}_i = \underbrace{\hat{D}^j}_{\hat{\gamma}^{jk}\partial_k}\hat{A}_{ij} = 0$$

i) solve $\mathcal{M}_i$ for $\hat{A}_{ij}$

$$\mathcal{H} = \Delta_F\psi + \frac{1}{8}\psi^{-7}\hat{A}_{ij}\hat{A}^{ij} = 0$$

ii) solve $\mathcal{H}$ for ψ

$\Delta_F = \eta^{ij}\partial_i\partial_j$

<u>Example 1:</u> single, static BH ($\mathcal{L}_t\gamma_{ij} = 0$)
in asympt. flat ST

(i) time symmetry: $K_{ij} = 0 \longrightarrow K = 0 \Rightarrow \mathcal{M}_i = 0$ trivially
$\hat{A}_{ij} = 0$

(ii) conformal flatness (choice) $\hat{\gamma}_{ij} = \eta_{ij} \longrightarrow \hat{R}_{ij} = 0$
$\hat{\Delta} = \Delta_F$

(iii) bcs: asymptotic flatness
$\lim_{r\to\infty}\gamma_{ij} = \eta_{ij} \Rightarrow \lim_{r\to\infty}\psi = 1$

↳ need to solve Hamiltonian constr.

$$\mathcal{H} = \Delta_F \psi = 0 \quad \text{with bcs: } \psi \underset{r\to\infty}{\longrightarrow} 1$$

↳ simplest non-trivial $\boxed{\psi = 1 + \frac{k}{r}}$ ← const

$$\underline{ds^2 = -\alpha^2 dt^2 + \psi^4 \underbrace{\eta_{ij}}\, dx^i dx^j} \qquad \beta^i = 0$$

specify: $\underline{\psi = 1 + \frac{M}{2r}}$, $\alpha^2 = \frac{(1 - M/2r)^2}{(1 + M/2r)^2}$

is Schwarzschild metric in isotropic coords (t, r, θ, φ) / $r_S = r\,\psi^2$ /

Schwarzschild radial coord $(t, r_S, \theta, \varphi)$ — isotropic radial coord

identify $k = \frac{M}{2}$

Example 2:

ID for N BHs w/o momenta - Brill-Lindquist data (1963)

$K_{ij} = 0 \rightarrow \mathcal{M}_i = 0$ trivially

· $\mathcal{H} = \Delta_F \psi = 0$ for N BHs

↳ Laplace eq is linear
new sols via superposition

$$\psi = 1 + \sum_{(a)=1}^{N} \frac{m_{(a)}}{2|r - r_{(a)}|}$$

$m_{(a)}$ bare mass of (a)-th BH
$r_{(a)}$ - location of (a)-th BH

e.g. for headon of 2 BHs

Example 3: Bowen - York ID / Brandt - Brügmann '97

$\mathcal{L}\ \mathcal{M}_i = \hat{D}^j \hat{A}_{ij} = 0 \quad \leftarrow$ linear eqs

analytic sol for $\hat{A}_{ij}$

$$\hat{A}^{(a)}_{ij} = \frac{3}{2r^2}\left\{ q_i P_j + q_j P_i + q_k P^k \left(q_i q_j - \gamma_{ij} \right) \right\} - \frac{3}{r^3}\left\{ \epsilon_{ilk} q_j + \epsilon_{jlk} q_i \right\} S^l S^k$$

for single BH

P_i, S^i - constvecs

P_i: linear ADM mom

S^i: angular (ADM) mom.

q_i - unit radial vector

for N-BHs

$$\hat{A}_{ij} = \sum_{(a)}^{N} \hat{A}^{(a)}_{ij}$$

$$\mathcal{H} = \Delta_F \psi + \# \hat{A}_{ij} \hat{A}^{ij} = 0$$

need to solve numerically

$$\psi_{BL} = 1 + \sum_a \frac{m_{(a)}}{2|r - r_{(a)}|}$$

ansatz $\psi = \psi_{BL} + u$

ψ_{BL}: singular (but analytically known)

u: regular

$$\rightarrow \mathcal{H} = 0 \Rightarrow \Delta_F u + \frac{1}{8\psi_{BL}^7}\left(1 + \frac{u}{\psi_{BL}}\right)^{-7} \hat{A}^{(BY)}_{ij} \hat{A}^{ij}_{(BY)} = 0$$

solve numerically for u

4) Formulation of EEs as well-posed initial value problem. (IVP)

ADM-York formulation: vacuum

evol eqs: $(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij})$$

Def for well posed IVP:

consider:
$$(*) \begin{cases} \partial_t f = A^p \partial_p f + Bf \\ f(t=0) = g \end{cases}$$

spatial derivs

f - vector of vars

A^p - principal matrix

A system of PDEs $(*)$ is said to be a well posed IVP if there exists a unique solution that depends continously on smooth initial data.
In particular, a system $(*)$ is well-posed if $\exists$ $k=\text{const}$, $a=\text{const}$, s.t. for all initial data we have

$$\|f(t,.)\| \leq k e^{a\cdot t} \|f(t=0,.)\|$$

recent reviews: • Sarbach & Tiglio LRR
• David Hilditch lecture notes 2013

Lay-person's version: analogy with wave equation

EEs in vacuum: (trace-reversed form)

$$^{(4)}R_{\mu\nu} = 0$$
$$= g^{\kappa\lambda}\partial_\kappa\partial_\lambda g_{\mu\nu} + \underbrace{g^{\kappa\lambda}\partial_\mu\partial_\lambda g_{\kappa\nu} + \ldots}$$

$\Box g_{\mu\nu}$ ✓

Wave eqn

$$\boxed{\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0}$$

choose harmonic gauge (Choquet-Bruhat)

$$\Box X^\mu = 0$$

show that mixed $\partial\partial$ are eliminated

$\rightarrow$ $\Box g^{\mu\nu} + \text{l.o.t} = 0$

EEs in vacuum
(leading order terms!) $(\beta^i = 0)$

$$\partial_t \gamma_{ij} \simeq -\alpha K_{ij}$$

$$\partial_t K_{ij} \simeq -D_i D_j \alpha + \alpha R_{ij} + \text{l.o.t.}$$

$$\simeq -\partial_i \partial_j \alpha + \alpha \Big\{ \underbrace{\gamma^{lm} \partial_l \partial_m \gamma_{ij}}\ \checkmark$$

$$\underline{+ \gamma^{lm} \partial_i \partial_m \gamma_{lj} + \partial\partial \ldots \Big\}}$$

wave equ.

introduce $\Pi = -\mathcal{L}_n \phi$
$$\simeq -\partial_t \phi$$

$\Box\phi = 0$ becomes

$$\partial_t \phi \simeq -\Pi$$

$$\partial_t \Pi \simeq -\gamma^{ij} D_i D_j \phi$$

cure? e.g. • generalized harmonic formulation (Pretorius '05)

→ • Baumgarte-Shapiro ('99) Shibata-Nakamura ('95) (BSSN) with moving puncture gauge (Baker et al '06, Campanelli et al '06)

• Z4 Z4c, CCZ4, ...

HERE: BSSN formulation

i) conformal decomposition of vars

• $W = \gamma^{-1/6}$, $\tilde{\gamma}_{ij} = W^2 \gamma_{ij}$ $\qquad \gamma \equiv \det \gamma_{ij}$

• $K = \gamma^{ij} K_{ij}$, $\tilde{A}_{ij} = W^2 A_{ij}$

versions
· $\chi = W^2$
· ϕ, $\chi = e^{\pm 4\phi}$

$\tilde{\Gamma}^i = \tilde{\gamma}^{kl} \tilde{\Gamma}^i_{kl} \underset{\tilde{\gamma}=1}{=} -\partial_j \tilde{\gamma}^{ij}$ conformal connection fct.

ii) constraint add.

$\mathcal{L}_n K$ – add Hamiltonian

$\partial_t \tilde{\Gamma}^i$ – add momentum constraint structure of eqs

$$\partial_t W \simeq \frac{1}{3}\alpha K W + ..$$

$$\partial_t \tilde{\gamma}_{ij} \simeq -2\alpha \tilde{A}_{ij} + \ldots$$

$$\partial_t K \simeq -\tilde{\gamma}^{ij}\hat{D}_i\hat{D}_j\alpha + \text{l.o.t.}$$ (use $\mathcal{H}$ to eliminate R)

$$\partial_t \tilde{A}_{ij} \simeq -[\hat{D}_i\hat{D}_j\alpha]^{TF} + \alpha W^2 R_{ij}^{TF}$$

R_{ij}: $\tilde{\gamma}^{kl}\partial_k\partial_l\tilde{\gamma}_{ij} + \partial_i\partial_j W + \hat{\Gamma}^i$

$$\partial_t \hat{\Gamma}^i \simeq \tilde{\gamma}^{kl}\partial_k\partial_l\beta^i + \ldots$$

+ Gauge (1+log slicing; Γ-driver shift)

$$\partial_t \alpha \simeq -2\alpha K$$

$$\partial_t \beta^i \simeq \hat{\Gamma}^i + \ldots$$

5) Gauge conditions (focus on lapse α)

- choices for lapse & shift
- Guideline / "wishlist":
  - simplify (BUT caution: $\alpha = 1$, $\beta^i = 0$: reach singularity in finite time)
  - high resolution
  - no blowing up → avoid singularity (phys coord) → evol. PDEs + gauge must be well-posed
  - practical: easy to implement.

5.1 Slicing conditions, ie, choices for the lapse

acceleration of observer along n^μ: $a_\mu = n^\nu \nabla_\nu n_\mu = \frac{1}{\alpha} D_\mu \alpha$

$a_\mu n^\mu = 0$

Geodesic slicing ("simplest")

$\alpha = 1$

$\llcorner\ a_\mu = \frac{1}{\alpha} D_\mu \alpha = 0$

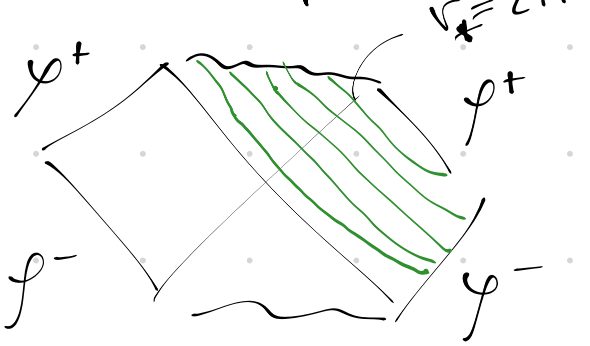
$\rightarrow$ normal observer follows timelike geodesics

eg in Schwarzschild: time to reach $r=0$ (from EH)

$t = \pi M$

$\rightarrow$ "focusing" of trajectory & reach $r=0$ in finite time

$\rightarrow$ crash in finite time



Maximal slicing

$\llcorner$ volume element to remain const $(\sqrt{-g})$

implies $\qquad \sqrt{-g} = \text{const}$

$$\mathcal{L}_n \sqrt{-g} = \dots = \sqrt{-g}\, \underbrace{\nabla_\mu n^\mu}_{=-K} = -\sqrt{-g}\, K = 0$$

$\rightarrow K = 0$

$\mathcal{L}_n K = 0$ (ie, K remains const during evol)

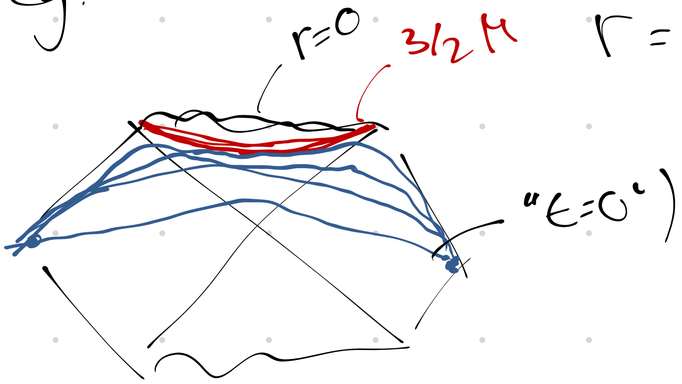
$$\mathcal{L}_n K = -\gamma^{ij} D_i D_j \alpha + \alpha(R + K^2) = 0$$

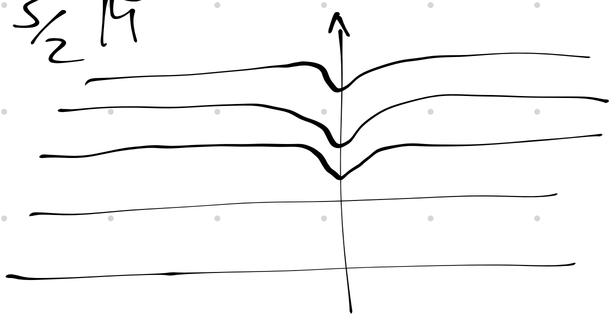
solution: $\alpha \sim e^{-R_0}$

↳ $\alpha \to 0$ as it approaches the singularity
("collapse of lapse")

↳ spatial hypersurfaces can <u>not</u> be arbitrarily close to phys. singularity
("singularity avoidance")

eg: in Schwarzschild: limiting surface is @ $r = 3/2\, M$





Hyperbolic slicing conditions

↳ numerically more efficient

Goal: keep singularity avoidance property of max. slicing cond.

General class: Bona-Masso family of slicing conds. [95]

$$(\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 f(\alpha)(K - K_0)$$

$f(\alpha)$ - pos., but otherwise arbitrary fct

K_0 - some background curvature

choices of $f(\alpha)$

(1) $f(\alpha) = 1$: harmonic slicing conditions

$$(\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 (K - K_0)$$

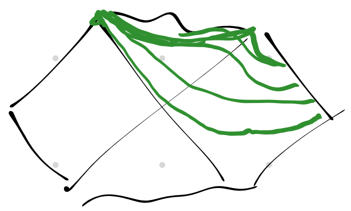
in 4D: $\Box x^\mu = 0$

(ii) $f(\alpha) = \frac{2}{\alpha}$: 1+ log slicing condition

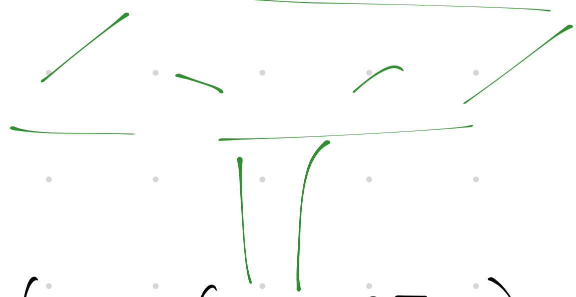
$$(\partial_t - \mathcal{L}_\beta)\alpha = -\underline{2\alpha(K-K_0)}$$

[note: integrate to $\alpha \simeq 1 + \ln \gamma$]

↳ singularity avoidance collapse



embedding: "trumpet"



- initial data for lapse (in CT)
  in practise set "pre collapsed lapse":

$$\alpha\big|_{t=0} = W = \gamma^{-1/6}$$

reduces gauge adjustment

typical choices for shift: Γ-driver

$$(\partial_t - \mathcal{L}_\beta)\beta^i = \underset{\text{const}}{B_\Gamma}\, \tilde{\Gamma}^i - \underset{\text{const, fct of } x^i}{\eta_\beta}\, \beta^i$$

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methods of line

$$\partial_t \underbrace{\begin{pmatrix} \gamma_{ij} \\ K_{ij} \\ \alpha \\ \beta^i \end{pmatrix}}_{u} = \mathrm{rhs}(u, \partial_i u, \partial_{ij} u)$$

finite difference, pseudo-spectral, ...

time integrator
eg Runge-Kutta 4th order