Billiards and the arithmetic of non-arithmetic groups

Curtis T McMullen
Harvard University

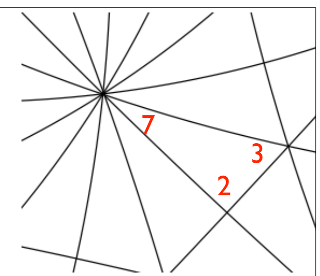
Geodesic planes

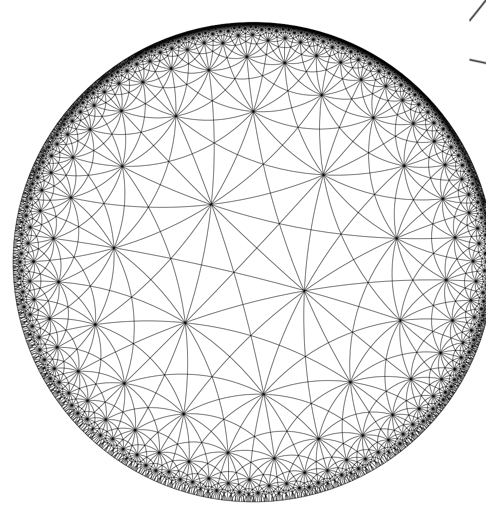
$$\mathbb{H} \longrightarrow \mathbb{H}^3/\Gamma = M^3$$
$$\longrightarrow \mathscr{M}_g = \mathscr{T}_g/\Gamma$$
$$\longrightarrow (\mathbb{H} \times \mathbb{H})/\Gamma$$

I. Triangle groups

Triangle groups

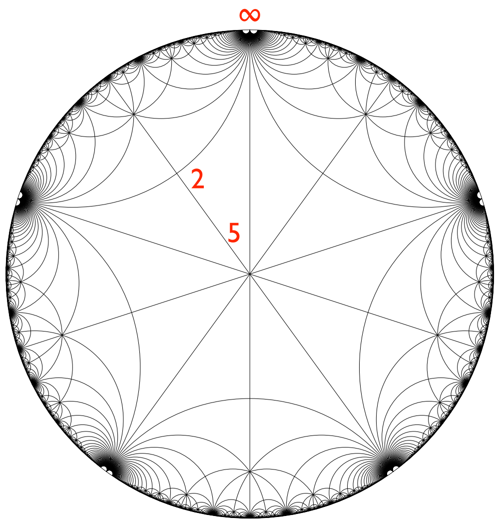
$\Delta(2,3,7) \subset SL_2(\mathbb{R})$





Triangle groups

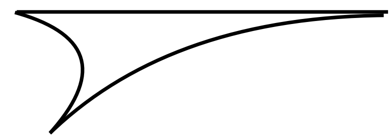
$\Delta(2,5,\infty) \subset SL_2(\mathbb{R})$



Triangle groups

$\Delta(p,q,\infty) \subset SL_2(\mathbb{R})$ *lattice*

π/p cusp



π/q $\mathbb{H}/\Delta$

invariant trace field

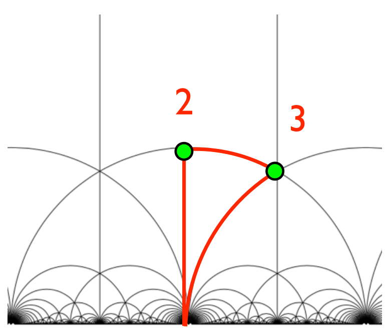
$$K_{pq} = \mathbb{Q}(\,\mathrm{Tr}(g^2) : g \in \Delta(p,q,\infty))$$

$$= \mathbb{Q}(\cos(2\pi/p),\ \cos(2\pi/q),\ \cos(\pi/p)\cos(\pi/q))$$

$\Delta(p,q,\infty)$ is arithmetic $\Leftrightarrow$ $K_{pq} = \mathbb{Q}$

Arithmetic case

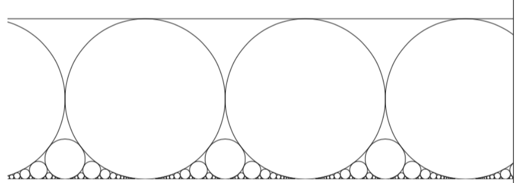
$$\Delta(2,3,\infty) = SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$



-1 0 1

$z \to z+1$

$z \to -1/z$



0 1/2 1 2

matrix entries = $\mathbb{Z}$

columns (a,b), gcd=1

cusp = $\mathbb{Q} \cup \{\infty\}$

Non-arithmetic case $\Delta(p,q,\infty)$

is more mysterious!

matrix entries = ?

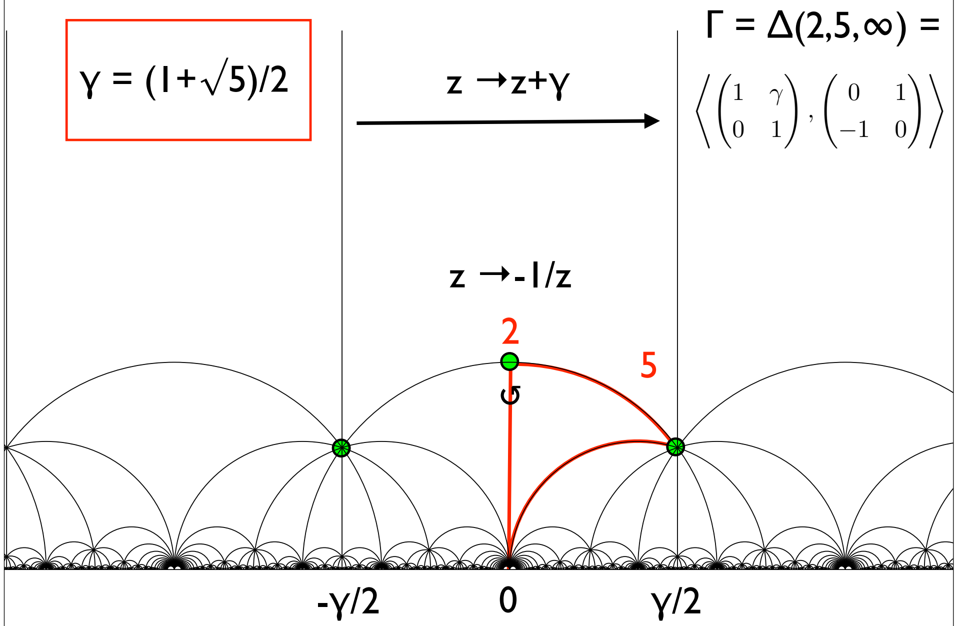
columns (a,b) ?

cusps = ? $\cup \{\infty\}$

Theorem

The cusps of $\Delta(p,q,\infty)$ coincide with $P^1(K_{pq})$ whenever $\deg(K_{pq}/\mathbb{Q}) = 2$, and satisfy quadratic height bounds.

The golden Hecke group

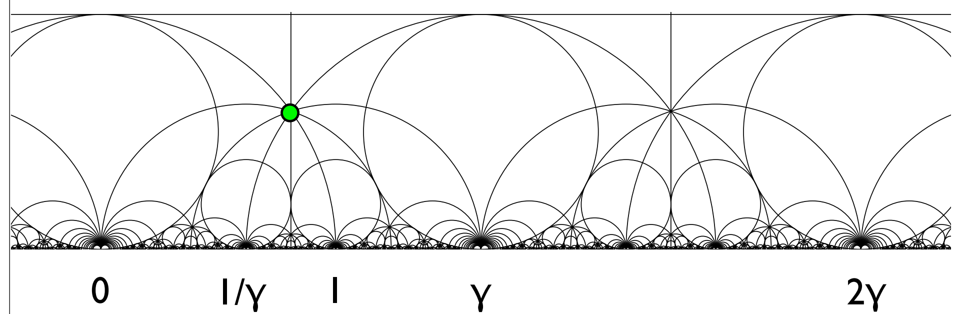


Cor

The cusps of Γ coincide with $K = \mathbb{Q}(\sqrt{5}) \cup \{\infty\}$.

Leutbecher, 1970s

5 packing



Golden Continued Fractions

Cor

Every x in $\mathbb{Q}(\sqrt{5})$ can be expressed as a *finite* golden continued fraction:

$$x = [a_1, a_2, a_3, \ldots, a_N] = a_1\gamma + \cfrac{1}{a_2\gamma + \cfrac{1}{a_3\gamma + \cdots + \cfrac{1}{a_N\gamma}}},$$

with a_i in $\mathbb{Z}$.

Quadratic height bounds: N, max $a_i = O(1+h(x))$.

Golden Fractions

Cor

Every x in K= $\mathbb{Q}(\sqrt{5})$ *can be written uniquely as a `golden fraction' x = a/c, up to sign.*

a,c in $\mathcal{O} = \mathbb{Z}[\gamma] \subset K$ relatively prime

(a,c) column of a matrix in Γ

Quadratic height bounds: $h(a)+h(c) = O(1+h(x)^2)$.

h(n) = log n

Thin group perspective

$\Gamma = \Delta(2,5,\infty) \subset_{\infty} SL_2(\mathbb{Z}[\gamma])$

lattice $\cap$ $\qquad\qquad$ $\cap$ *lattice*

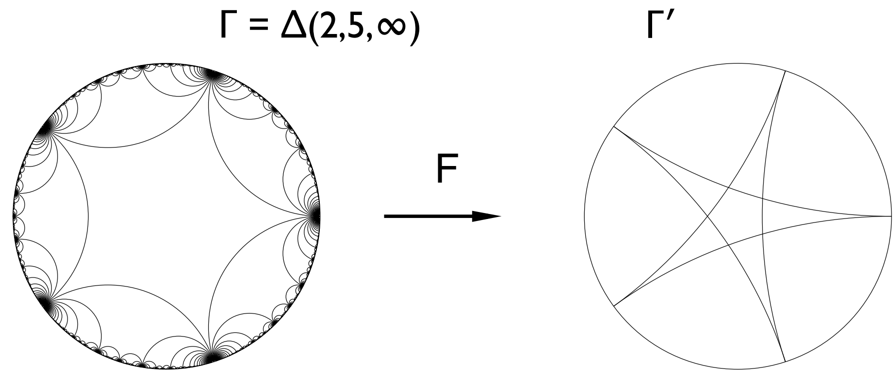
$SL_2(\mathbb{R}) \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

$V \subset X_K$

Galois symmetry is broken: Γ' is indiscrete

Γ acts as a sieve to select one of infinitely many expressions $x = (\gamma^k a)/(\gamma^k c)$.

Holomorphic pentagon-to-star map



$V \longrightarrow X_K$ covered by $\mathbb{H} \longrightarrow \mathbb{H}\times\mathbb{H}$

via $x \longrightarrow (x,F(x))$.

Curves on a Hilbert modular surface

cf. M, Möller-Viehweg

K = real quadratic field

$$X_K = (\mathbb{H} \times \mathbb{H}) / \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

$$V = \mathbb{H}/\Gamma \looparrowright X_K \qquad \text{geodesic curve}$$

Theorem Q

Either V is a Shimura curve, or the cusps of V coincide with $\mathbb{P}^1(K)$ and satisfy quadratic height bounds.

proof by descent

Triangle groups and Hilbert modular varieties

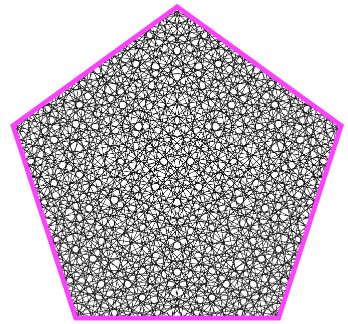
Theorem. Every $\Delta(p,q,\infty)$ comes from a geodesic curve V in a Hilbert modular variety X_K.

Cohen and Wolfart
Bouw and Möller

Cor. All previous results follow from Theorem Q.

II. Billiards

Billiards in a regular pentagon

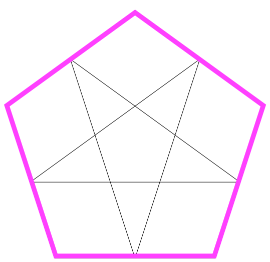


A dense set of slopes are periodic.

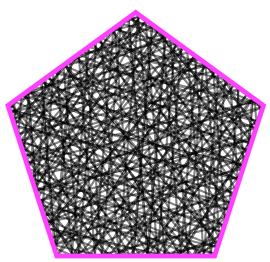
Which ones?

How do the periodic trajectories behave?

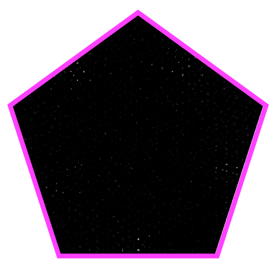
Slopes and lengths



s
L(s) = 5



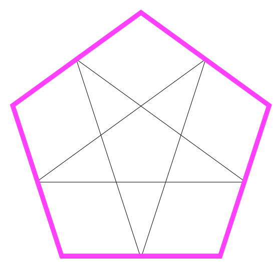
4s
L(s) = 469



20s
L(s) = 2338

6765s
L(6765s) = 1.734×10^{25}

Slopes, lengths and heights



s

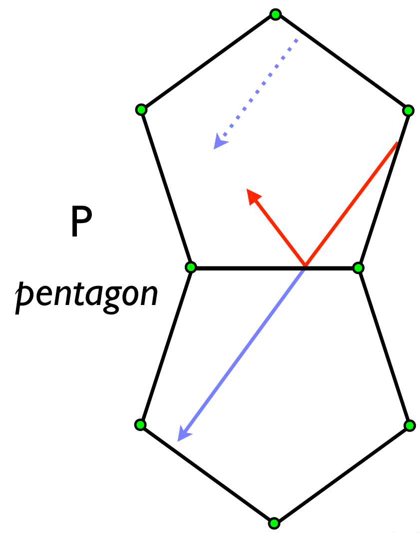
Theorem
The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})s$, and $\log L(xs) = O(h(x)^2)$.

exponent 2 is sharp

Another instance of quadratic height bounds.

III. Teichmüller curves

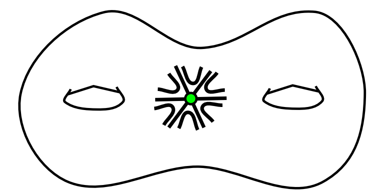
Billiards and Riemann surfaces



$(X,\omega) = (P,dz)$ / gluing

X has genus 2

ω has just one zero!



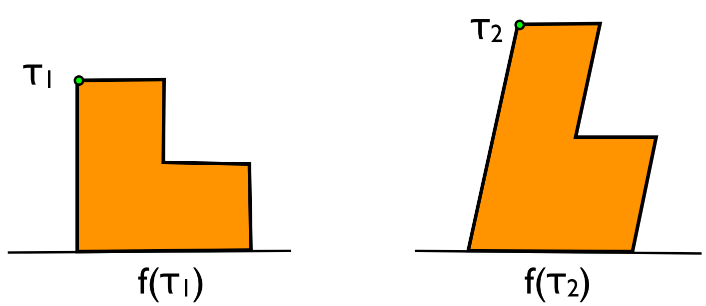
billiards ⟹ geodesics on $(X,|\omega|)$

Moduli space $\Omega\mathcal{M}_g$

Dynamical:
$SL_2(\mathbb{R})$ acts on $\Omega\mathcal{M}_g$

Polygon for $A \cdot (X,\omega) = A \cdot$ (Polygon for (X,ω))

Complex geodesics $f : \mathbb{H} \longrightarrow \mathcal{M}_g$



Teichmüller curves

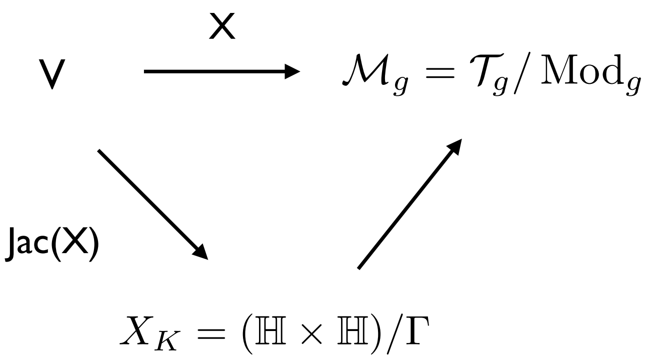
$SL(X,\omega)$ = stabilizer of (X,ω) in $SL_2(\mathbb{R})$

$SL(X,\omega)$ lattice ⇒ $SL_2(\mathbb{R})$ orbit of (X,ω) generates

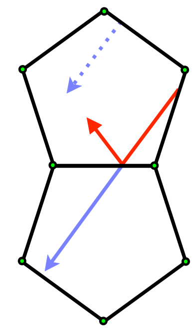
an isometrically immersed *Teichmüller curve:*

$$f : V = \mathbb{H} \,/\, SL(X,\omega) \rightarrow \mathcal{M}_g$$

Factorization through X_K



Pentagon revisited

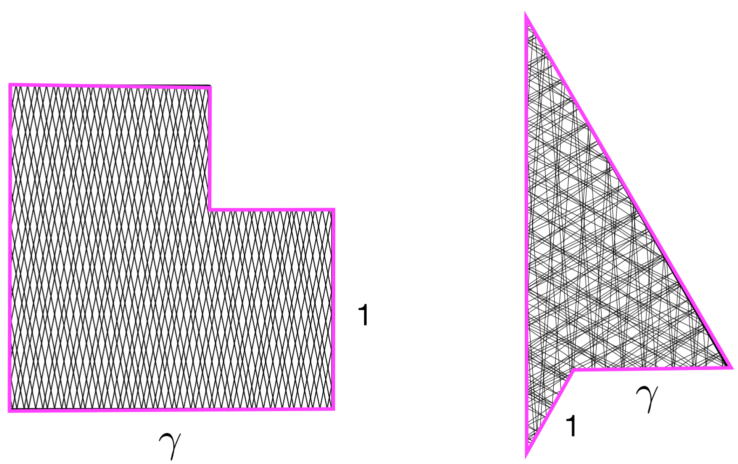


Theorem

In this case SL(X,ω) = Δ(2,5,∞).

Cor: *Results on billiards also follow from Theorem Q.*

Similarly for all families of optimal billiards



…since these are quadratic: *Eskin - Filip - Wright*

VI. Heights and Hilbert modular surfaces

Proof of Theorem Q

Curves on a Hilbert modular surface

$V = \mathbb{H}/\Gamma \hookrightarrow X_K$ geodesic curve

Theorem Q

Either V is a Shimura curve, or the cusps of V coincide with $\mathbb{P}^1(K)$ and satisfy quadratic height bounds.

proof by descent

Proof of Theorem Q

- Heights on $\mathbb{P}^1$(K)
- Abelian varieties with real multiplication by K
- Hilbert modular varieties
- Curves on Hilbert modular surfaces

Heights on $\mathbb{P}^n(K)$

$$H(x) = H(x_0 : x_1 : \cdots : x_n) = \prod_v \max_i |x_i|_v.$$

comparable to

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v, \qquad [a_0 : \cdots : a_n] = [x].$$

(a_i are integers)

only requires knowledge of integers and infinite places

Abelian varieties

$$A = \Omega(A)^*/H_1(A,\mathbb{Z}) \cong \mathbb{C}^g/L$$

Polarization = Hermitian inner product on

$$\Omega(A)^* \cong H_1(A,\mathbb{R}) \qquad \textit{such that}$$

$$[C,D] = -\operatorname{Im}\langle C,D\rangle$$

gives integral symplectic form on $H_1(A,\mathbb{Z})$.

Hodge norm:

$$\|C\|_A = \langle C,C\rangle^{1/2} \quad \text{on} \quad H_1(A,\mathbb{R})$$

Example: The Jacobian

X compact Riemann surface, genus g

Ω(X) = space of holomorphic 1-forms, ≃ $\mathbb{C}^g$

A = Jac(X) = $\Omega(X)^*/H_1(X,\mathbb{Z})$

Polarization: $$\langle\omega_1,\omega_2\rangle = \frac{i}{2}\int_X \omega_1\wedge\overline{\omega}_2.$$

[C,D] = usual intersection form on $H_1(X,\mathbb{Z})$

Hodge norm: $$\|C\|_X = \sup\left\{\left|\int_C \omega\right| \;:\; \langle\omega,\omega\rangle = 1\right\}.$$

Real multiplication

End(A) = ring of endomorphisms of A
as a complex Lie group

K totally real field of degree g = dim(A).

A has *real multiplication* by K if we are given a map

$$K \to \mathrm{End}(A)\otimes\mathbb{Q}$$

such that T_k is self-adjoint for all k in K.

Eg. f in Aut(X) has order n ⇒ T = f+f^{-1} generates real mult. by $\mathbb{Q}$(cos(2π/n)).

The projective line $\mathbb{P}^1_A(K)$

$$K \subset \mathrm{End}(A)\otimes\mathbb{Q}$$

$$H_1(A,\mathbb{Q}) \cong K^2$$

$\mathbb{P}^1_A(K)$ = space of K-lines in $H_1(A,\mathbb{Q}) \cong \mathbb{Q}^{2g}$

Hodge norm at a place v

Diagonalize K on Ω(A) and H_1(A)

$T_k\,\omega_v = \rho_v(k)\,\omega_v$ orthonormal eigenforms

$$H_1(A,\mathbb{R}) = \oplus_v S_v \xrightarrow{\pi_v} S_v$$

$$\|C\|_v = \|\pi_v(C)\|_A = \left|\int_C \omega_v\right|$$

$$|C|_v = \|C\|_v^{1/g}$$ `Hodge valuation'

Height $H_A(x)$ on $\mathbb{P}^1_A(K)$

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$x \in \mathbb{P}^1_A(K)$

$C \in H_1(A, \mathbb{Z})$

$[x] = [C]$ (same K line)

Why a height?

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v$$

Theorem. Given a linear isomorphism

$\iota : \mathbb{P}^1_A(K) \to \mathbb{P}^1(K)$

we have $H(\iota(x)) \asymp H_A(x).$

How to make A with RM?

K = totally real field degree g over $\mathbb{Q}$

$\mathcal{O}$ = ring of integers in K

$\tau = (\tau_1, \ldots, \tau_g) \in \mathbb{H}^g$

$A = \mathbb{C}^g / \mathcal{O} \oplus \mathcal{O}^\vee \tau$

$(a, b) \mapsto (a_i + b_i \tau_i)$

Polarization = usual inner product on $\mathbb{C}^g$

$$T_k(z) = (k_i z_i)$$

$\mathcal{O} \subset \mathrm{End}(A) \implies A$ has real multiplication by K

Height on $\mathbb{P}^1$(K) from τ

$$H_\tau(x) = \inf_{\substack{x=b/a \\ a,b \in \mathcal{O}}} \left(\prod_i \frac{|a_i + b\tau_i|^2}{\mathrm{Im}\, \tau_i} \right)^{1/2d}$$

$$\delta(\tau) = \inf_x H_\tau(x) > 0$$

descends to a proper function on X_K

Case of a torus

$$A = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \qquad H_1(A,\mathbb{Z}) \cong \mathbb{Z}^2$$

$$K = \mathbb{Q}$$

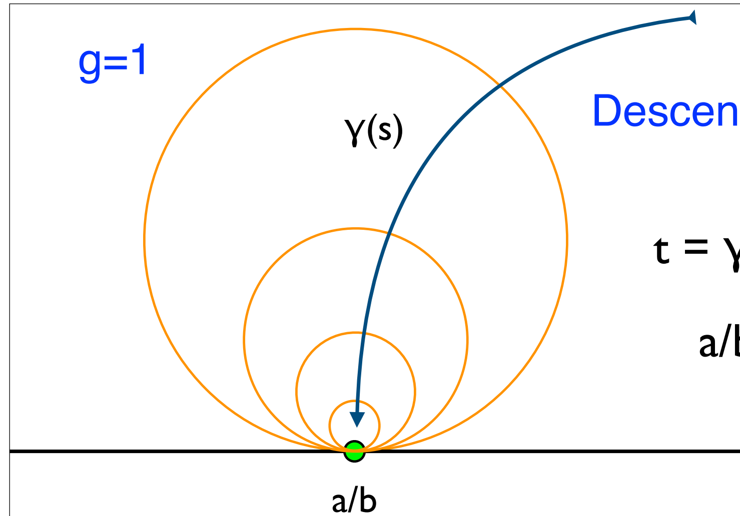
$$\|C\|_A^2 = \left| \int_C \omega \right|^2 = \frac{|a+b\tau|^2}{\operatorname{Im}\tau}$$

Hodge norm

$H_\tau(x)$ = length of geodesic with slope $x = a/b$

---

g=1



Descent for $SL_2\mathbb{Z}$

$t = \gamma(s) \in \mathbb{H}$

$a/b \in \mathbb{Q}$

$H_t(a/b)$

$A_t = \mathbb{C} / \mathbb{Z} \oplus t\mathbb{Z}$

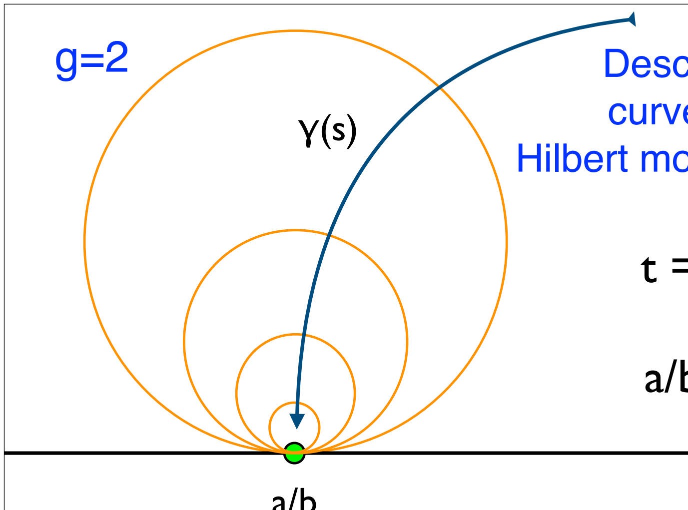
decreases like exp(-s)

no lower bound on H_t

⇒ a/b is a cusp

---

g=2



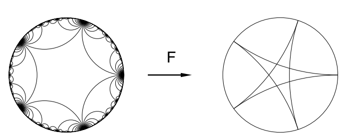
Descent on a curve V on a Hilbert modular surface

$t = \gamma(s) \in \mathbb{H}$

$a/b \in \mathbb{Q}(\sqrt{D})$

$H_\tau(a/b)$ $\qquad$ $\tau = (t, F(t))$

$A_\tau = \mathbb{C}^2 / \mathcal{O} \oplus \tau\mathcal{O}^\vee$



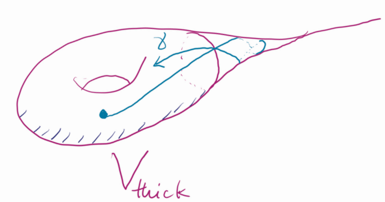
To show a/b in K is a cusp:

$H_\tau(a/b) \sim$ (t term) x (F(t) term)

$\leq \exp(-s) \exp(|F'|\, s)$

When t lies over V_{thick} :

$H_\tau(a/b) \geq 1$
$|F'(t)| < \delta < 1$



So γ spends only a finite amount of time over V_{thick}

⇒

a/b is a cusp

QED Theorem Q

IV. Hidden arithmetic and modular symbols

What about matrix entries in $\Delta(2,5,\infty)$?

M = all nonzero matrix entries

$\delta M = \{m'/m : m \text{ is in } M\}$

$R = -\gamma^{-2} \cdot \delta M.$

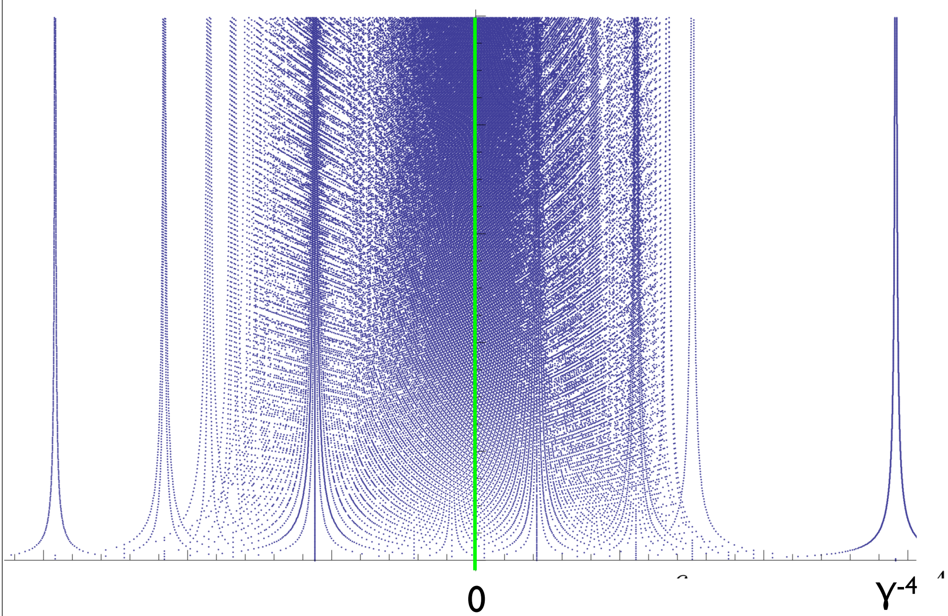
Theorem

The closure of R is a countable semigroup in [-1,1], homeomorphic to $\omega^\omega + 1$.

(Whereas $\delta\, \mathbb{Z}[\gamma]$ is dense in $\mathbb{R}$.)

cf. Hilbert theorem 90.

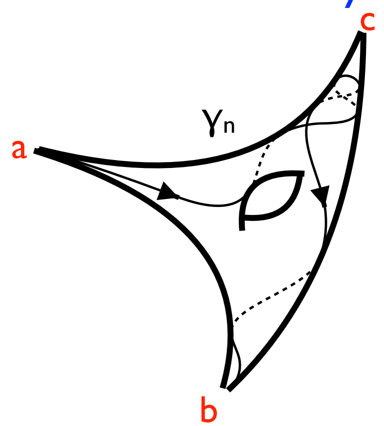
Image of M under $(m'/m, H(m))$



Compare to ω^ω in

Pisot numbers,
Weyl spectrum,
3D hyperbolic volumes, …

Proof uses modular symbols



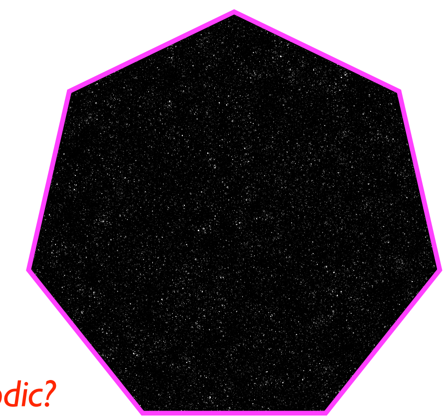
nonabelian

= *formal products of geodesics between cusps*

The space of modular symbols is itself homeomorphic to ω^{ω}.

V. The heptagon

Open problem



Regular 7-gon

K = $\mathbb{Q}(\cos(2\pi/7))$
(cubic)

(i) Which slopes are periodic?

Shown: $L(s)=7$, $L(2\,s) = 2190$.

(ii) How long do we have to wait to test periodicity?!

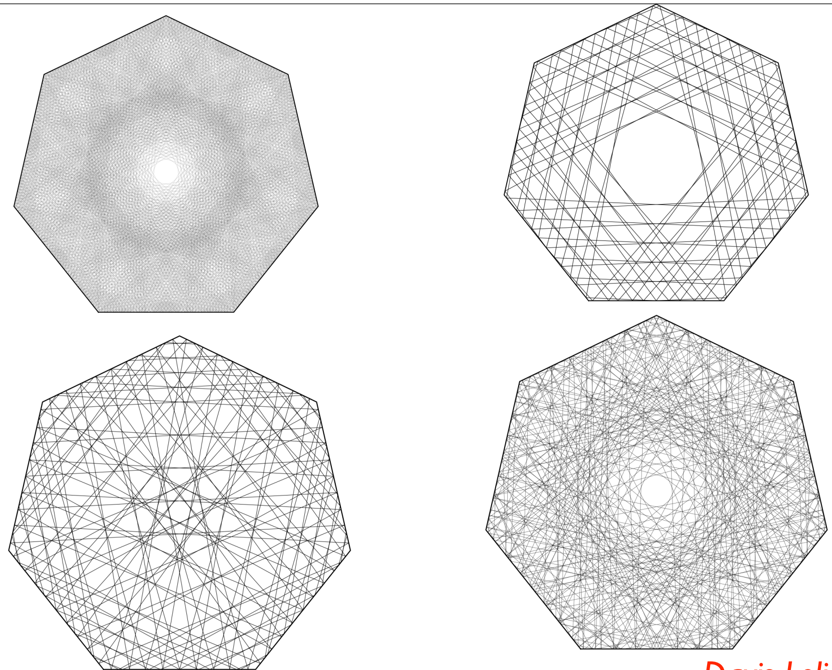
Is there any algorithm at all?!!

Bold Conjecture

K = $\mathbb{Q}(\cos(2\pi/7))$

Every x in K is the fixed point of a parabolic or hyperbolic element g in $\Delta(2,7,\infty)$.

Due independently to Hanson-Merberg-Towse-Yudovina, and Boulanger; further investigations by K. Winsor.



Davis-Lelievre

VII. A spectral gap for triangles

Cusps

Theorem

Every Teichmüller curve $V \to \mathcal{M}_g$ has a cusp.

↑ most cases

Theorem

\`Every' geodesic curve $V \to X_K$ has a cusp, provided $\dim(X_K)=2$

What happens when $\dim(X_K) > 2$?

What happens if $\dim X_K > 2$?

Theorem

There exists a compact geodesic curve V on a 6D Hilbert modular variety,

$$V = \mathbb{H}/\Delta' \to X_K$$

such that there is no compact Shimura variety with $V \subset S \subset X_K$.

Spectral Gap

Theorem

For all but finitely many Δ(p,q,r),
 # spherical and # hyperbolic places are about the same.

about 1/3 spherical

Cor (Takeuchi)

There are only finitely many arithmetic triangle groups.

Cor (Waterman-Maclachlan)

There are only finitely many purely hyperbolic triangle groups.

Galois conjugate triangles

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{7}\right) \sim \left(\frac{1}{2},\frac{1}{3},\frac{2}{7}\right) \sim \left(\frac{1}{2},\frac{1}{3},\frac{3}{7}\right)$$

hyperbolic spherical spherical

Δ(2,3,7) is arithmetic

38 arithmetic triangle groups

[commensurability classes]

	(e_1,e_2,e_3)	Field	Ram
1	$(2,3,\infty),(2,4,\infty),(2,6,\infty),(2,\infty,\infty),(3,3,\infty),(3,\infty,\infty),(4,4,\infty),(6,6,\infty),(\infty,\infty,\infty)$	$\mathbb{Q}$	$\emptyset$
2	$(2,4,6),(2,6,6),(3,4,4),(3,6,6)$	$\mathbb{Q}$	2, 3
3	$(2,3,8),(2,4,8),(2,6,8),(2,8,8),(3,3,4),(3,8,8),(4,4,4),(4,6,6),(4,8,8)$	$\mathbb{Q}(\sqrt{2})$	$\mathcal{P}_2$
4	$(2,3,12),(2,6,12),(3,3,6),(3,4,12),(3,12,12),(6,6,6)$	$\mathbb{Q}(\sqrt{3})$	$\mathcal{P}_2$
5	$(2,4,12),(2,12,12),(4,4,6),(6,12,12)$	$\mathbb{Q}(\sqrt{3})$	$\mathcal{P}_3$
6	$(2,4,5),(2,4,10),(2,5,5),(2,10,10),(4,4,5),(5,10,10)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_2$
7	$(2,5,6),(3,5,5)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_3$
8	$(2,3,10),(2,5,10),(3,3,5),(5,5,5)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_5$
9	$(3,4,6)$	$\mathbb{Q}(\sqrt{6})$	$\mathcal{P}_2$
10	$(2,3,7),(2,3,14),(2,4,7),(2,7,7),(2,7,14),(3,3,7),(7,7,7)$	$\mathbb{Q}(\cos\pi/7)$	$\emptyset$
11	$(2,3,9),(2,3,18),(2,9,18),(3,3,9),(3,6,18),(9,9,9)$	$\mathbb{Q}(\cos\pi/9)$	$\emptyset$
12	$(2,4,18),(2,18,18),(4,4,9),(9,18,18)$	$\mathbb{Q}(\cos\pi/9)$	$\mathcal{P}_2,\mathcal{P}_3$
13	$(2,3,16),(2,8,16),(3,3,8),(4,16,16),(8,8,8)$	$\mathbb{Q}(\cos\pi/8)$	$\mathcal{P}_2$
14	$(2,5,20),(5,5,10)$	$\mathbb{Q}(\cos\pi/10)$	$\mathcal{P}_2$
15	$(2,3,24),(2,12,24),(3,3,12),(3,8,24),(6,24,24),(12,12,12)$	$\mathbb{Q}(\cos\pi/12)$	$\mathcal{P}_2$
16	$(2,5,30),(5,5,15)$	$\mathbb{Q}(\cos\pi/15)$	$\mathcal{P}_3$
17	$(2,3,30),(2,15,30),(3,3,15),(3,10,30),(15,15,15)$	$\mathbb{Q}(\cos\pi/15)$	$\mathcal{P}_5$
18	$(2,5,8),(4,5,5)$	$\mathbb{Q}(\sqrt{2},\sqrt{5})$	$\mathcal{P}_2$
19	$(2,3,11)$	$\mathbb{Q}(\cos\pi/11)$	$\emptyset$

Takeuchi

Maclachlan-Reid

Purely hyperbolic

$$\left(\frac{1}{3},\frac{1}{10},\frac{1}{10}\right) \sim \left(\frac{1}{3},\frac{3}{10},\frac{3}{10}\right)$$

hyperbolic hyperbolic

$$\left(\frac{1}{14},\frac{1}{21},\frac{1}{42}\right) \sim \left(\frac{1}{14},\frac{8}{21},\frac{13}{42}\right) \sim \left(\frac{3}{14},\frac{4}{21},\frac{17}{42}\right) \sim$$

$$\left(\frac{3}{14},\frac{10}{21},\frac{11}{42}\right) \sim \left(\frac{5}{14},\frac{2}{21},\frac{19}{42}\right) \sim \left(\frac{5}{14},\frac{5}{21},\frac{5}{42}\right)$$

all hyperbolic

11 purely hyperbolic triangle groups

M, Maclachlan-Waterman

Conj. These are all!

```
----------------Delta{2, 4, 6}--------------
 DegK, DegK0: {4, 1} {hyp,sph}: {1, 0}
Ramification0: [ 2, 3][]

----------------Delta{2, 6, 6}--------------
 DegK, DegK0: {2, 1} {hyp,sph}: {1, 0}
Ramification0: [ 2, 3][]

----------------Delta{2, 6, 10}--------------
 DegK, DegK0: {8, 2} {hyp,sph}: {2, 0}
Ramification0: [ Prime [5, 0] [-1, 2], Prime [3, 0]][]

----------------Delta{3, 4, 4}--------------
 DegK, DegK0: {2, 1} {hyp,sph}: {1, 0}
Ramification0: [ 2, 3][]

----------------Delta{3, 6, 6}--------------
 DegK, DegK0: {2, 1} {hyp,sph}: {1, 0}
Ramification0: [ 2, 3][]

----------------Delta{3, 10, 10}--------------
 DegK, DegK0: {4, 2} {hyp,sph}: {2, 0}
Ramification0: [ Prime [5, 0] [-1, 2], Prime [3, 0]][]

----------------Delta{4, 6, 12}--------------
 DegK, DegK0: {4, 2} {hyp,sph}: {2, 0}
Ramification0: [ Prime [3, 0] [0, 1], Prime [2, 0] [1, 1]][]

----------------Delta{5, 6, 6}--------------
 DegK, DegK0: {4, 2} {hyp,sph}: {2, 0}
Ramification0: [ Prime [5, 0] [-1, 2], Prime [3, 0]][]

----------------Delta{6, 9, 18}--------------
 DegK, DegK0: {6, 3} {hyp,sph}: {3, 0}
Ramification0: [ Prime [3, 0, 0] [2, 1, 0], Prime [2, 0, 0]][]

----------------Delta{6, 10, 15}--------------
 DegK, DegK0: {8, 4} {hyp,sph}: {4, 0}
Ramification0: [ Prime [3, 0, 0, 0] [0, 1, 0, 1], Prime [5, 0, 0, 0]
[2, 3, 0, 0]][]

----------------Delta{14, 21, 42}--------------
 DegK, DegK0: {12, 6} {hyp,sph}: {6, 0}
Ramification0: [][]
```

VII. The (14,21,42) triangle group

Fallacy

$\Delta = \Delta(p,q,r) \subset SL_2(\mathbb{R})$

$K = \mathbb{Q}(\text{traces of elements in } \Delta)$

Δ can be realized as a subgroup of $SL_2(K)$

correction

$\Leftrightarrow$ quaternion algebra $B = \mathbb{Q}(\Delta)$ splits over K

$\Rightarrow$ Δ is purely hyperbolic

Theorem

Among the 11 known purely hyperbolic cocompact triangle groups, only $\Delta(14,21,42)$ is also split at all finite places.

Cor

$\Delta(14,21,42)$ embeds in SL_2(its inv. trace field K).

up to finite index

From (14,21,42) to an exotic curve

$K = \mathbb{Q}(\cos \pi/21)$ *degree 6*

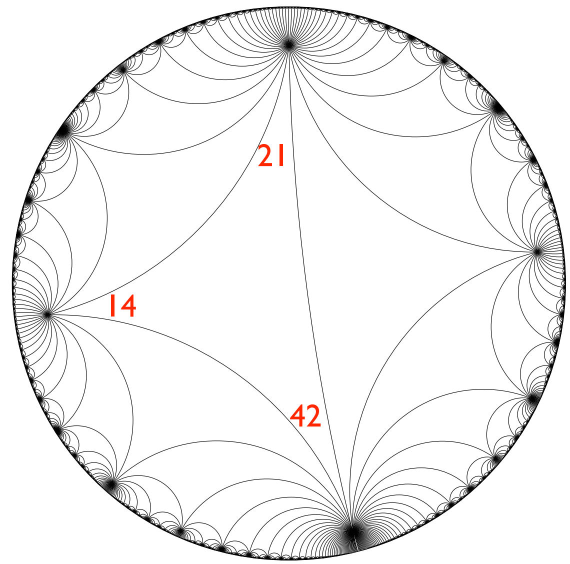
$V = \mathbb{H}/\Delta' \to X_K$ geodesic curve on a 6D Hilbert modular variety

Theorem

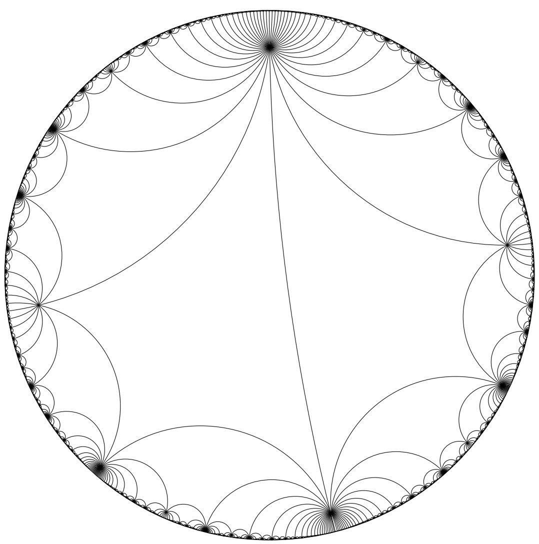
V is a compact geodesic curve, but there is no compact Shimura variety with $V \subset S \subset X_K$.

Proof: Δ is Zariski dense.

Start with Δ(14,21,42)



Pass to index 2



$V = \mathbb{H}/\Delta(14,21,42)'$

Construct 5 maps $\mathbb{H}$ to $\mathbb{H}$

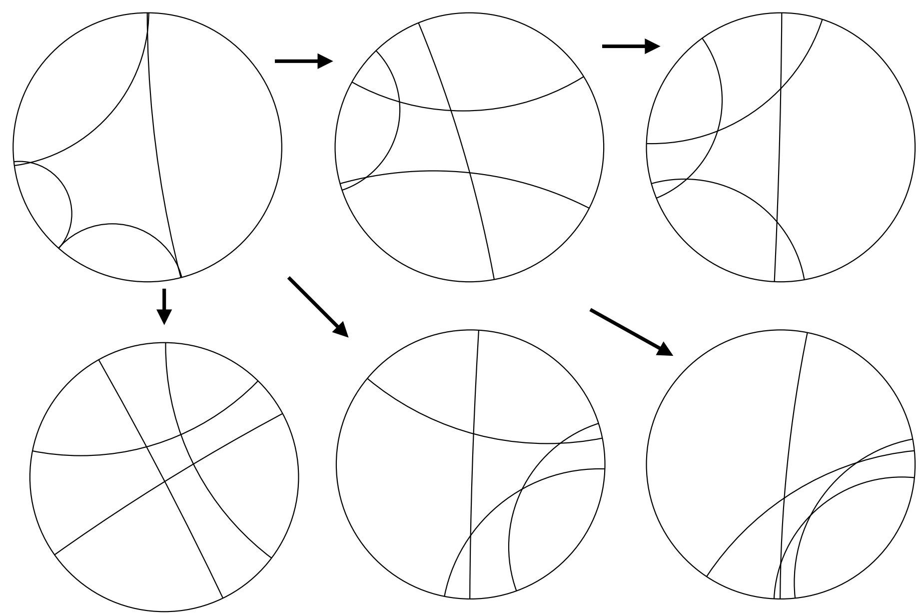
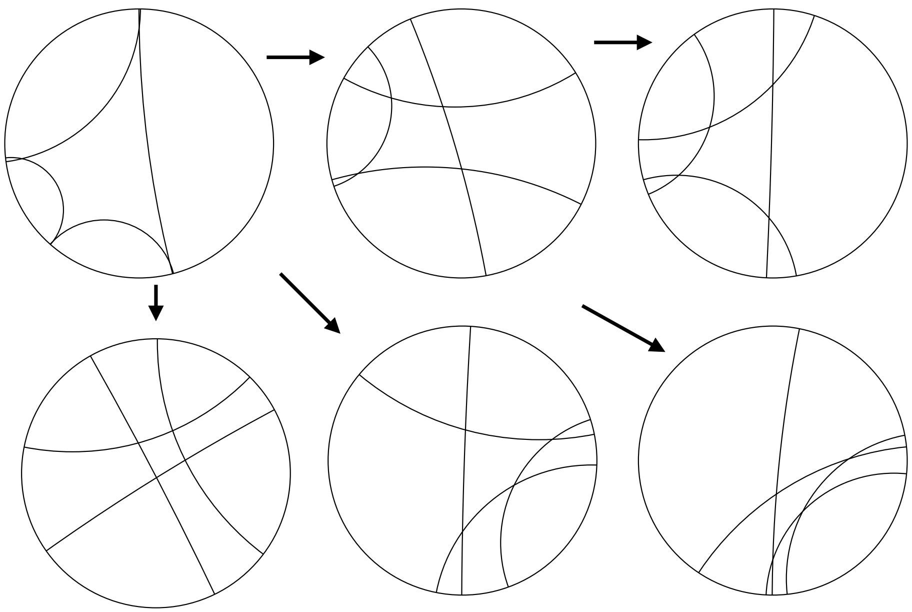


Image covers exotic V in X_K



Conjecture

$\Delta(14,21,42)$ is the only compact triangle group which virtually embeds in SL_2(its inv. trace field K).

Problem

Are there more examples of exotic curves?

For example, with dim $X_K = 3$?

Conclusion

The moving tablecloth game

Key step in proofs