Exponential Mixing of Frame Flows for Geometrically Finite Hyperbolic Manifolds

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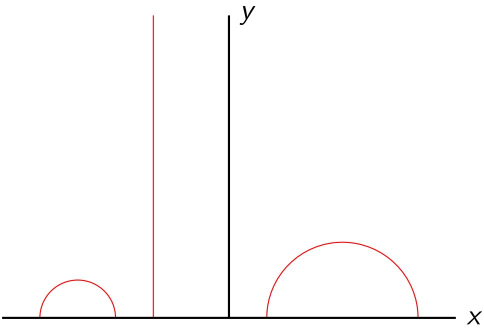
Overview

Outline

Hyperbolic plane

- $\mathbb{H}^2 = \{x + iy : y > 0\}$
- Constant sectional curvature $K = -1$
- $ds_{\mathbb{H}^2} = \dfrac{ds_{\mathbb{C}}}{y}$



Group of isometries

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) : ad - bc = 1 \right\}$$

Example: $A = \{a_t = \mathrm{diag}(e^{t/2}, e^{-t/2})\}$

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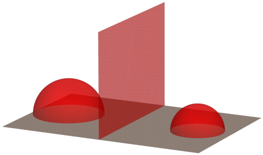
$$G = \mathrm{PSL}_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Hyperbolic space

- $\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$
- Constant sectional curvature $K = -1$
- $ds_{\mathbb{H}^n} = \frac{ds_{\mathbb{R}^n}}{x_n}$



Group of isometries

$$\mathrm{SO}(n,1) = \{X \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{R}) : {}^{\mathrm{t}}XJX = J, \det(X) = 1\}$$

$$J = \mathrm{diag}(1, 1, \ldots, 1, -1)$$

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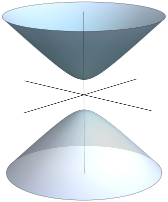
$$A = \left\{ a_t = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix} : t \in \mathbb{R} \right\}$$

Group of isometries

$$\mathrm{SO}(n,1) \curvearrowright \mathbb{R}^{n,1}$$
$$G = \mathrm{SO}(n,1)^\circ \curvearrowright \mathbb{H}^n$$



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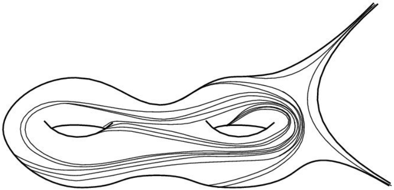
- Example: $\mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_2(\mathbb{R})$
- $X = \Gamma \backslash \mathbb{H}^n$. If Γ is a lattice, X has finite volume.

Picture



Unit tangent and frame bundles for $\mathbb{H}^n$

$$\mathrm{F}(\mathbb{H}^n)$$

project to the first basis vector $\downarrow$

$$\mathrm{T}^1(\mathbb{H}^n)$$

project to the basepoint $\downarrow$

$$\mathbb{H}^n$$

Unit tangent and frame bundles for $\mathbb{H}^n$

$$\begin{array}{ccc} \mathrm{F}(\mathbb{H}^n) & = & G \\ \downarrow \text{ project to the first basis vector} & & \downarrow \\ \mathrm{T}^1(\mathbb{H}^n) & = & G/M \\ \downarrow \text{ project to the basepoint} & & \downarrow \\ \mathbb{H}^n & = & G/K \end{array}$$

Unit tangent and frame bundles for $X = \Gamma\backslash\mathbb{H}^n$

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X

Unit tangent and frame bundles for $X = \Gamma\backslash\mathbb{H}^n$

$$\begin{array}{rccc} & \mathrm{F}(X) & = & \Gamma\backslash G \\ \text{project to the first basis vector} & \downarrow & & \downarrow \\ & \mathrm{T}^1(X) & = & \Gamma\backslash G/M \\ \text{project to the basepoint} & \downarrow & & \downarrow \\ & X & = & \Gamma\backslash G/K \end{array}$$

Geodesic flow

- Geodesic flow on $\mathrm{T}^1(X)$: moves a unit tangent vector along a geodesic through it.

- Geodesic flow: $\mathrm{T}^1(X) = \Gamma \backslash G / M \curvearrowleft A$ (by matrix multiplication).

Frame flow

- Frame flow on $\mathrm{F}(X)$: moves a frame (positively oriented orthonormal basis) by parallel transport along a geodesic through, say, the first basis vector.

- Frame flow: $\mathrm{F}(X) = \Gamma\backslash G \curvearrowleft A$ (by matrix multiplication).

Outline

Theorem (Howe–Moore '79)

Let $X = \Gamma \backslash \mathbb{H}^n$ *be of finite volume. The frame flow on* $\mathrm{F}(X) = \Gamma \backslash G$ *is mixing:*

$\forall \phi, \psi \in L^2(\Gamma \backslash G)$, *we have*

$$\lim_{t \to +\infty} \int_{\Gamma \backslash G} \phi(x a_t) \psi(x) \, d\mu(x) = \mu(\phi) \cdot \mu(\psi).$$

Question

Can we say something stronger? More precisely, what is the **rate** of mixing?

Theorem (Ratner, Moore '87)

Let $X = \Gamma\backslash\mathbb{H}^n$ be of finite volume. The frame flow on $\mathrm{F}(X) = \Gamma\backslash G$ is exponentially mixing:

$\exists C > 0, \eta > 0$ such that $\forall \phi, \psi \in C^1(\Gamma\backslash G)$, we have

$$\left| \int_{\Gamma\backslash G} \phi(xa_t)\psi(x)\, d\mu(x) - \mu(\phi)\cdot\mu(\psi) \right| \leq Ce^{-\eta t}\|\phi\|_{C^1}\cdot\|\psi\|_{C^1}.$$

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Can we prove similar theorems for **infinite volume** hyperbolic manifolds?

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- The limit set Λ is the set of limit points of any orbit $\Gamma \cdot o$ in $\partial\mathbb{H}^n$.

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- If $\#\Lambda > 2$, then Γ is said to be non-elementary.
- The critical exponent δ_Γ is the abscissa of convergence of $\mathscr{P}(s) = \sum_{\gamma\in\Gamma} e^{-s\cdot d(o,\gamma\cdot o)}$.

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- If $\text{Core}(X)$ is compact, then Γ and X are said to be convex cocompact.
- If $\text{Core}(X)_\epsilon$ for any $\epsilon > 0$ has finite volume, then Γ and X are said to be geometrically finite.

BMS measure

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- On $\mathrm{F}(\mathbb{H}^n)$, the BMS measure is $d\nu(f) = e^{\delta_\Gamma \beta_{f^+}(o,f)} e^{\delta_\Gamma \beta_{f^-}(o,f)} \, d\nu^{\mathrm{PS}}(f^+) \, d\nu^{\mathrm{PS}}(f^-) \, dt \, dm$

Outline

Theorem (Babillot '02, Winter '15)

Let $\Gamma < G = \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{K}})$ *be geometrically finite and Zariski dense. The frame flow on* $\mathrm{H}(X) = \Gamma \backslash G$ *is mixing with respect to the BMS measure:*

$\forall \phi, \psi \in C_c(\Gamma \backslash G)$, *we have*

$$\lim_{t \to +\infty} \int_{\Gamma \backslash G} \phi(x a_t) \psi(x) \, d\nu(x) = \nu(\phi) \cdot \nu(\psi).$$

Theorem (Mohammadi–Oh '15)

Let $\Gamma < G$ be geometrically finite and Zariski dense with $\delta_\Gamma > \max\left\{\frac{n-1}{2}, n-2\right\}$. The frame flow on $\mathrm{F}(X) = \Gamma\backslash G$ is exponentially mixing with respect to the BMS measure:

$\exists C > 0, \eta > 0$, and $\ell \in \mathbb{N}$ such that $\forall \phi, \psi \in C_c^\infty(\Gamma\backslash G)$, we have

$$\left|\int_{\Gamma\backslash G} \phi(xa_t)\psi(x)\, d\nu(x) - \nu(\phi)\cdot\nu(\psi)\right| \leq Ce^{-\eta t}\|\phi\|_{\mathcal{S}^\ell}\cdot\|\psi\|_{\mathcal{S}^\ell}.$$

Theorem (S.–Winter '21)

Let $\Gamma < G$ be convex cocompact and Zariski dense. The frame flow on $\mathrm{F}(X) = \Gamma \backslash G$ is exponentially mixing with respect to the BMS measure:

$\exists C > 0, \eta > 0$ *such that* $\forall \phi, \psi \in C_c^1(\Gamma \backslash G)$, *we have*

$$\left| \int_{\Gamma \backslash G} \phi(xa_t)\psi(x)\, d\nu(x) - \nu(\phi) \cdot \nu(\psi) \right| \leq Ce^{-\eta t} \|\phi\|_{C^1} \cdot \|\psi\|_{C^1}.$$

Theorem (Chow–S. '22)

Let $\Gamma < G = \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{K}})$ *be convex cocompact and Zariski dense. The frame flow on* $\mathrm{H}(X) = \Gamma\backslash G$ *is exponentially mixing with respect to the BMS measure:*

$\exists C > 0, \eta > 0$ *such that* $\forall \phi, \psi \in C^1_{\mathrm{c}}(\Gamma\backslash G)$, *we have*

$$\left| \int_{\Gamma\backslash G} \phi(xa_t)\psi(x)\, d\nu(x) - \nu(\phi)\cdot\nu(\psi) \right| \leq Ce^{-\eta t}\|\phi\|_{C^1}\cdot\|\psi\|_{C^1}.$$

Theorem (Li–Pan–S. '23)

Let $\Gamma < G$ be geometrically finite and Zariski dense. The frame flow on $\mathrm{F}(X) = \Gamma \backslash G$ is exponentially mixing with respect to the BMS measure:

$\exists C > 0, \eta > 0$ *such that* $\forall \phi, \psi \in C^1(\Gamma \backslash G)$, *we have*

$$\left| \int_{\Gamma \backslash G} \phi(xa_t)\psi(x)\, d\nu(x) - \nu(\phi) \cdot \nu(\psi) \right| \leq Ce^{-\eta t} \|\phi\|_{C^1} \cdot \|\psi\|_{C^1}.$$

Applications

- Counting orbit points, counting geodesics

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- Spectral gaps

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- Non-concentration property (NCP)
- Large deviation property (LDP)

LNIC

We need a strong form of non-integrability when dealing with the frame flow:

$$[\mathfrak{n}^+, \mathfrak{n}^-] = \mathfrak{a} \oplus \mathfrak{m}.$$

Integrability would be $[\mathfrak{n}^+, \mathfrak{n}^-] = 0$.

NCP

Not all frames accessible due to fractal nature of supp(ν). To deal with this, we need the non-concentration property:

$\exists \delta > 0$ such that $\forall x \in \Lambda$, $\epsilon > 0$, and direction ω, $\exists y \in \Lambda \cap B_\epsilon(x)$ such that $|\langle y - x, \omega \rangle| \geq \epsilon\delta$.

True when Γ is convex cocompact. **Not true** when Γ is geometricaly finite **with cusps**! Replace Λ with a certain large subset $\Lambda_\epsilon \subset \Lambda$.

LDP

When Γ is geometricaly finite **with cusps**, we need a large deviation property which ensures that under a certain random walk, we are mostly in Λ_ϵ:

$\exists\kappa \in (0,1)$ such that $\forall\epsilon > 0$ and $n \in \mathbb{N}$, we have
$\nu^{\mathrm{PS}}\{x \in \Lambda_0 : \#\{j \in \mathbb{N} : j \leq n, T^j(x) \in \Lambda_\epsilon\} < \kappa n\} \leq e^{-\kappa n}$.