ICERM Lecture 2

(geodesic planes in ∞-vol hyp mflds)

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$$G = PSL_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3)$$

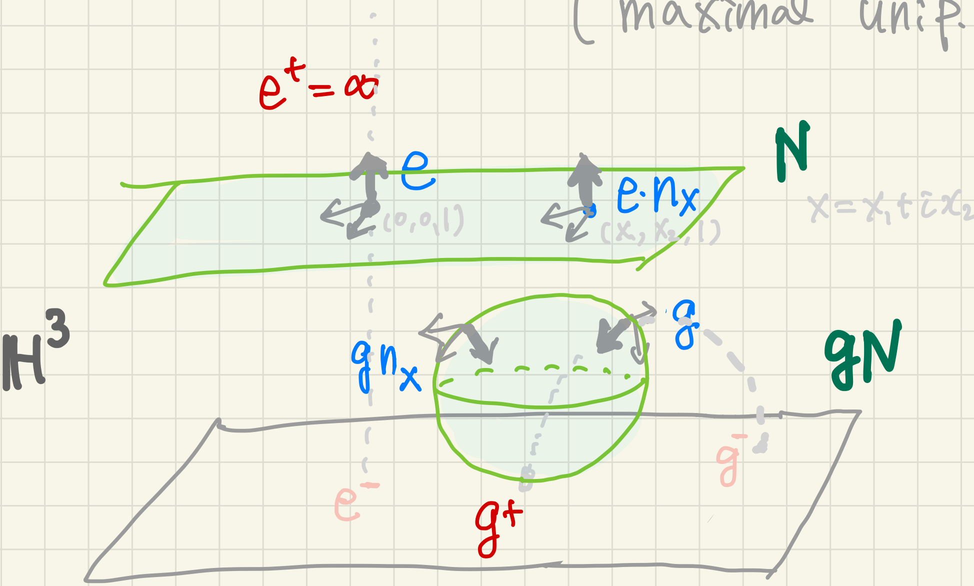
$$\boxed{\begin{array}{ccc} F(\mathbb{H}^3) & \Leftrightarrow & PSL_2\mathbb{C} \\ \downarrow & & \\ T^1(\mathbb{H}^3) & \Leftrightarrow & PSL_2\mathbb{C}/SO(2) \\ \downarrow & & \\ \mathbb{H}^3 & \Leftrightarrow & PSL_2\mathbb{C}/PSU(2) \end{array}}$$

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{C} \right\}$$
$$= \{ g \in G \mid a_{-t}\, g\, a_t \to e \text{ as } t \to +\infty \}$$

contracting horospherical subgp

(maximal unip. subgp)



$$G = \mathrm{Isom}^+(\mathbb{H}^n) = SO^\circ(Q)$$

$$Q(x_1, \ldots, x_{n+1}) = 2x_1 x_{n+1} + \sum_{i=2}^{n} x_i^2$$

$$A = \left\{ a_t = \begin{pmatrix} e^t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

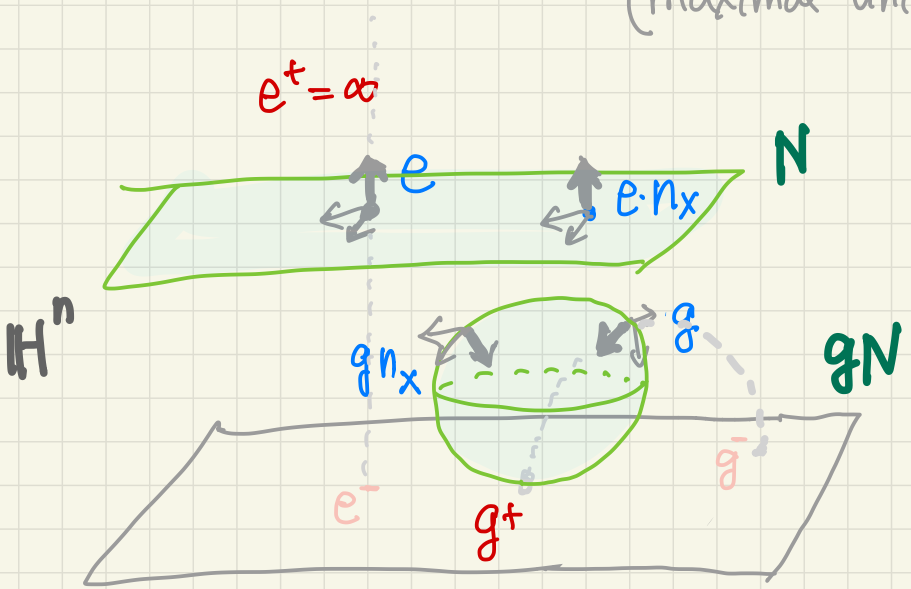
$$N = \left\{ n_x = \begin{pmatrix} 1 & x & \frac{1}{2} x \cdot x^t \\ & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & x^t \\ & & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R}^{n-1} \right\} \cong \mathbb{R}^{n-1}$$

$n_x \Leftrightarrow x$

$$= \{ g \in G \mid a_{-t}\, g\, a_t \to e \quad \text{as } t \to +\infty \}$$

contracting horospherical subgp.

(maximal unipotent subgp)



For $2 \le k \le n$, $\quad U_k \simeq \mathbb{R}^{k-1} < N \simeq \mathbb{R}^{n-1}$

$$H(U_k) = \langle U_k, U_k^t \rangle \simeq SO^\circ(k,1)$$

Any conn. subgp W of G generated by unipotents is conjugate to $\begin{cases} U_k \\ H(U_k) \end{cases}$ $2 \le k \le n$

<u>Thm</u> (McMullen-Mohammadi-O. $n=3$, Lee-O. $n \ge 4$)

Let $\Gamma \backslash \mathbb{H}^n$ have Fuchsian ends

$$\Omega = RFM = \{[g] \in \Gamma \backslash G \mid g^{\pm} \in \Lambda\}$$

$$\forall x \in \Omega, \quad \overline{xW} \cap \Omega = xL \cap \Omega$$

for some $W < L < G$

Moreover, $\overline{xH(U_k)} = xH(U_m)C \cap RF_+M \cdot H(U_k)$

where $RF_+M = \{[g] \in \Gamma \backslash G \mid g^+ \in \Lambda\}$

Induction:

either $\overline{xH(U_k)} = xH(U_k)C$ $\quad C \subset C_G(H(U_k)) \simeq SO(n-k)$

or $\overline{xH(U_k)} \supset \overline{yU_m} \supset yH(U_m)$

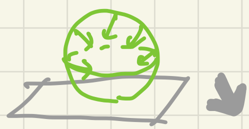
$m > k$

$\rightsquigarrow$ Need to understand

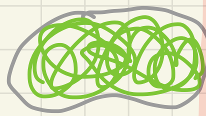
N-orbit closures.

Thm (Furstenberg, Hedlund, Veech)

$\Gamma < G$ cocpt lattice



- N-action on $\Gamma \backslash G$ is minimal



- N-action on $\Gamma \backslash G$ is uniquely ergodic

the G-inv measure on $\Gamma \backslash G$ is the only N-inv measure.

This can be deduced from mixing of a_t-action, i.e, the frame flow on $\mathcal{F}(M) = \Gamma\backslash G$.

Thm (Howe-Moore) $\mathrm{vol}(\Gamma\backslash G) = 1$

$\forall\, f_1, f_2 \in C_c(\Gamma\backslash G)$, as $|t| \to \infty$

$$\int_{\Gamma\backslash G} f_1(x a_t) f_2(x)\, dx \longrightarrow \int f_1\, dx \cdot \int f_2\, dx$$

To show $xN \cap \mathcal{O} \neq \phi$ $\forall$ open $\mathcal{O} \subset \Gamma\backslash G$,

ETS $xNG_\varepsilon \cap \mathcal{O} \neq \phi$

$P^+ := \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ $\quad a_{-t} P_\varepsilon^+ a_t \supset P_\varepsilon^+ \quad t \geq 0$

$$xNG_\varepsilon \supset x(a_t N_1 a_{-t}) P_\varepsilon^+ \supset (\underbrace{x a_t}_{\approx y}) N_1 P_\varepsilon^+ a_{-t} \supset y G_\varepsilon a_{-t}$$

$G_\varepsilon = N_\varepsilon P_\varepsilon^+$ $\qquad$ for $t \gg 1$

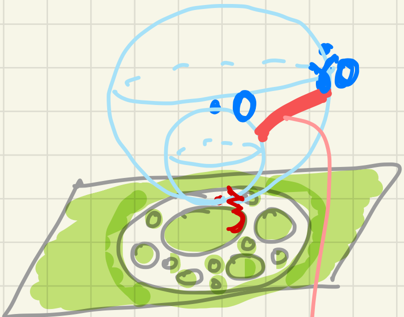
$$xNG_\varepsilon \cap \mathcal{O} \supset yG_\varepsilon a_{-t} \cap \mathcal{O} \neq \phi$$

by Mixing

Γ convex cocpt (or geom. finite)

Two important geometric measures on $\Gamma \backslash G$

$\boxed{\text{Sullivan}}$ $o \in \mathbb{H}^n$



$\exists!$ Γ-conformal measure ν_o on Λ of dim $\delta = \delta_\Gamma$

$$\frac{d\gamma_* \nu_o}{d\nu_o}(\xi) = e^{-\delta \beta_\xi(\gamma o, o)}$$

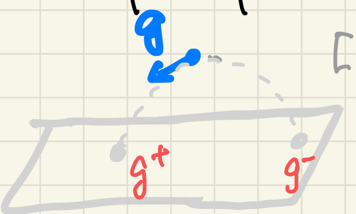
Patterson-Sullivan measure

$$\beta_\xi(\gamma o, o) = \lim_{t \to \infty} d(\gamma o, \xi_t) - d(o, \xi_t)$$

<u>Sullivan</u> ν_o = δ-dim'l H'ff measure on Λ.

$\rightsquigarrow \left\{ \begin{array}{l} \text{BMS} \\ \text{BR} \end{array} \right.$ **measures on $\Gamma \backslash G$**

Hopf parametrization



$$[g] \longmapsto (g^+,\ g^-,\ \underset{s}{\underbrace{\beta_{\xi}(o, go)}})$$

$$T^1(\mathbb{H}^n) = G/SO(n-1) \simeq \partial\mathbb{H}^n \times \partial\mathbb{H}^n \times \mathbb{R}$$

$$dm^{BMS}[g] = e^{\delta\beta_{g^+}(o,go) + \delta\beta_{g^-}(o,go)}\, d\nu_o(g^+)\, d\nu_o(g^-)\, ds$$

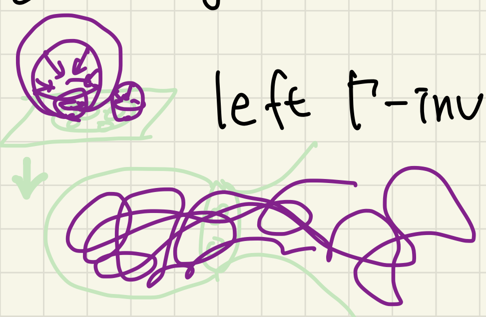
left Γ-inv & right A-inv measure

$\rightsquigarrow$ m^{BMS} on $\Gamma \backslash G$ $\qquad$ $\mathrm{supp}\, m^{BMS} = \{ g^{\pm} \in \Lambda \}$

$= RFM = \Omega$

Leb measure

$$dm^{BR}[g] = e^{\delta\beta_{g^+}(o,go) + (n-1)\beta_{g^-}(o,go)}\, d\nu_o(g^+)\, dm_o(g^-)\, ds$$



left Γ-inv & right N-inv measure

$\rightsquigarrow$ m^{BR} on $\Gamma \backslash G$

$\mathrm{supp}\, m^{BR} = \{ g^+ \in \Lambda \}$

$= RF_+M = \mathcal{E}$

Thm Γ geom. finite & $\Gamma < SO^{\circ}(n,1)$ Z. dense

- m^{BMS} is A-ergodic (Sullivan, Winter) & measure of max. entropy
- m^{BMS} is mixing (Babillot, Winter)

Using the BMS-mixing, Winter proved:

Thm (Burger, Roblin, Winter)

Γ convex cocpt (geom. finite)

- N-action on $RF_+M = \mathcal{E}$ is minimal
- N-action on RF_+M is uniquely-ergodic,

m^{BR} is the uniq N-inv measure on RF_+M

In particular, m^{BR} is N-ergodic.

$\mathbb{R}^{k-1} \simeq U \underset{\neq}{\leq} N \simeq \mathbb{R}^{n-1}$ conn. unipotent subgp

m^{BR} is not U-ergodic in general

Thm (Mohammadi-O. Maucourant-Schapira)

- If $\delta > \text{co-dim}_N U = (n-k)$, m^{BR} is U-ergodic

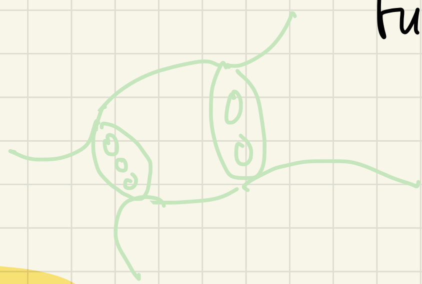
  a.e U-orbits are dense in RF_+M

- If $\delta < \text{co-dim}_N(U) = (n-k)$,

  m^{BR} is totally dissipative.

  a.e U-orbits are proper immersion of $U \simeq \mathbb{R}^{k-1}$.

$M = \Gamma \backslash \mathbb{H}^n$ Convex Cocpt with Fuchsian ends



$\delta > n-2$

Corollary For any conn unip subgp $U < N$ (even $\dim U = 1$), a.e U-orbits are dense in RF_+M.

xA
$\cap$

For m^{BMS} a.e $x \in \Gamma \backslash G$, $x SO^\circ(k,1)$ is dense in $RF_+M \cdot SO(k,1)$

$\cap\ xU$

For m^{BR} a.e $x \in \Gamma \backslash G$, $x SO^\circ(k,1)$ is dense in $RF_+M \cdot SO(k,1)$

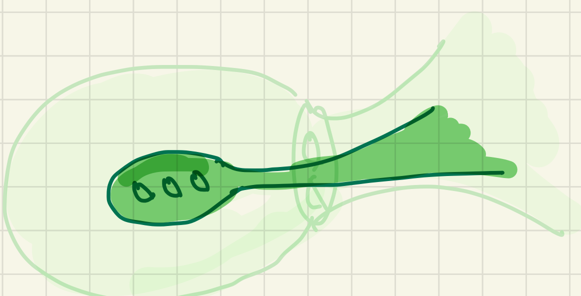
Measure-theoretic analogue of MMO & LO thm ?

Conj $U < N$

Any U-inv erg measure on RF_+M is of the form

$$m^{BR}_{xH(\hat{U})C}$$

for some closed $xH(\hat{U})C$ with $U \subset \hat{U}$



Thank You !