# ICERM Lecture 1

(Geodesic planes in $\infty$-vol hyp mflds)

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# Geodesic planes in hyperbolic mflds of ∞ vol.

Ref • Geodesic planes in hyp 3-mflds
( McMullen-Mohammadi-O. – Inventiones 2017)
36p.

• Horocycles in hyp 3-mflds (MMO,
– GAFA 2016
12p

• Geodesic planes in the convex core of an acylindrical 3-mfld (MMO
– Duke 2022) 31p.

• Geodesic planes in geom. finite acylindrical 3-mflds (Benoist-O.
– ETDS 2022)
39p.

• Orbit closures of unip flows for hyp mflds with Fuchsian ends
(Minju Lee-O. To appear G&T )
101 p.

* Dynamics for discrete subgps of $SL_2\mathbb{C}$

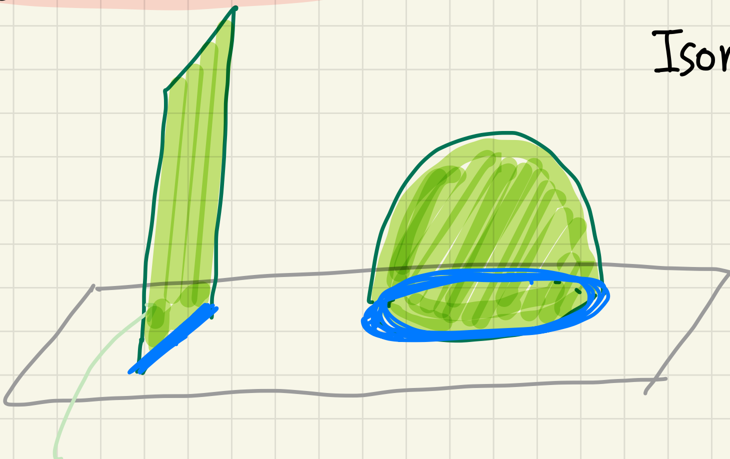
in Dynamics, Geometry, Number theory (Margulis vol)

$ds = \frac{d_{Euc}}{y}$

geodesic planes in $\mathbb{H}^3 = \{(x_1, x_2, y) \mid y > 0\}$

$\mathrm{Isom}^+(\mathbb{H}^3) = PSL_2\mathbb{C}$

$\simeq SO^\circ(3,1)$



$\partial \mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\}$

$= \mathbb{S}^2$

$\mathbb{H}^2 = \{(x_1, 0, y) \mid y > 0\}$

$\tilde{P} = g(\mathbb{H}^2) \quad g \in PSL_2\mathbb{C}$

$\Updownarrow$

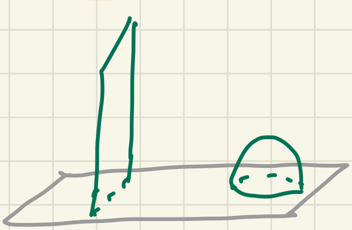
$\partial \tilde{P} = g(\hat{\mathbb{R}})$

geodesic planes in $\mathbb{H}^3$

$\Updownarrow$

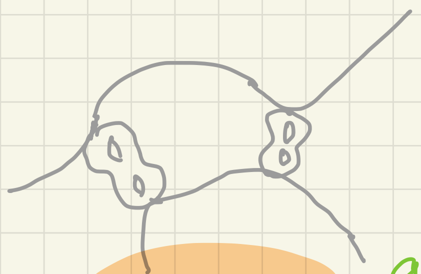
circles in $\mathbb{S}^2$

$M = \Gamma \backslash \mathbb{H}^3 \quad \Gamma < PSL_2\mathbb{C}$ t.f discrete



$\tilde{P}$

$\xrightarrow{\pi}$

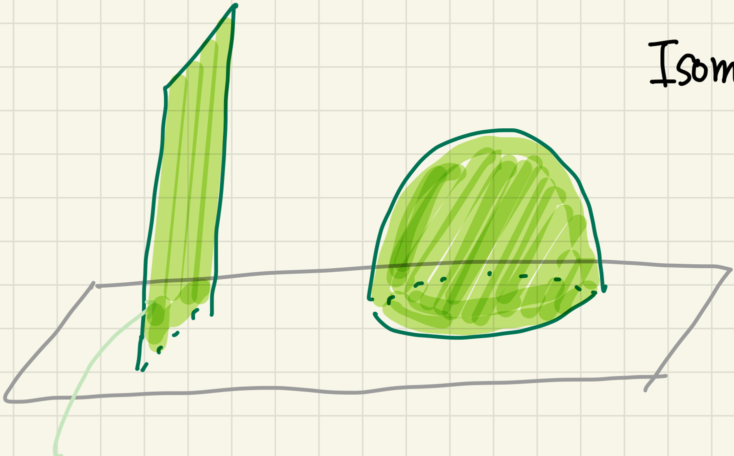


$P = \pi(\tilde{P})$

geod. planes in $M$

geodesic $k$ planes in $\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, y) \mid y > 0\}$

$$\mathrm{Isom}^+(\mathbb{H}^n) \simeq \underset{\displaystyle G}{\underset{\shortparallel}{SO^\circ(n,1)}}$$



$\partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$
$= \mathbb{S}^{n-1}$

$\mathbb{H}^k = \{(x_1, \dots, x_k, 0, \dots, 0, y) \mid y > 0\}$

$\tilde{P} = g(\mathbb{H}^k) \quad g \in G$

$\Updownarrow$

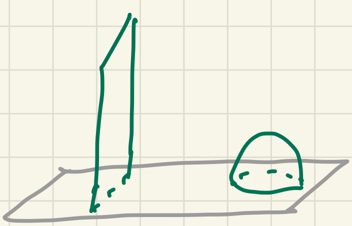
$\partial \tilde{P} = g(\hat{\mathbb{R}}^{k-1})$

geodesic $k$ planes in $\mathbb{H}^n$

$\Updownarrow$

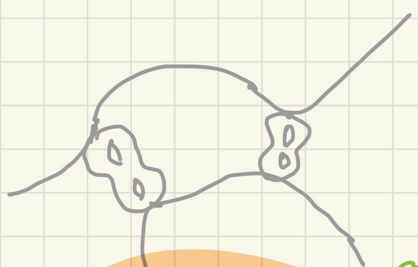
$(k-1)$ spheres in $\mathbb{S}^{n-1}$

$M = \Gamma \backslash \mathbb{H}^n \qquad \Gamma < G \qquad$ t.f discrete



$\tilde{P}$

$\xrightarrow{\pi}$



$P = \pi(\tilde{P})$

geod. $k$ planes in $M$

## Question

$P$ geod $k$-plane in $M = \Gamma\backslash\mathbb{H}^n$

$2 \le k \le n-1$

Describe $\overline{P}$ !

$\boxed{A}$ **Thm** (Ratner, Shah ~90)

$M = \Gamma\backslash\mathbb{H}^n$ finite vol

$P \subset M$ geod $k$-plane $2 \le k \le n-1$

$\Rightarrow$ $\overline{P}$ is a geod $m$-plane $k \le m \le n$

(e.g If $n=3$, $\overline{P} = P$ or $\overline{P} = M$)

What about $\mathrm{vol}(M) = \infty$ ?

$M = {}_{\Gamma}\backslash \mathbb{H}^n$ convex cocpt

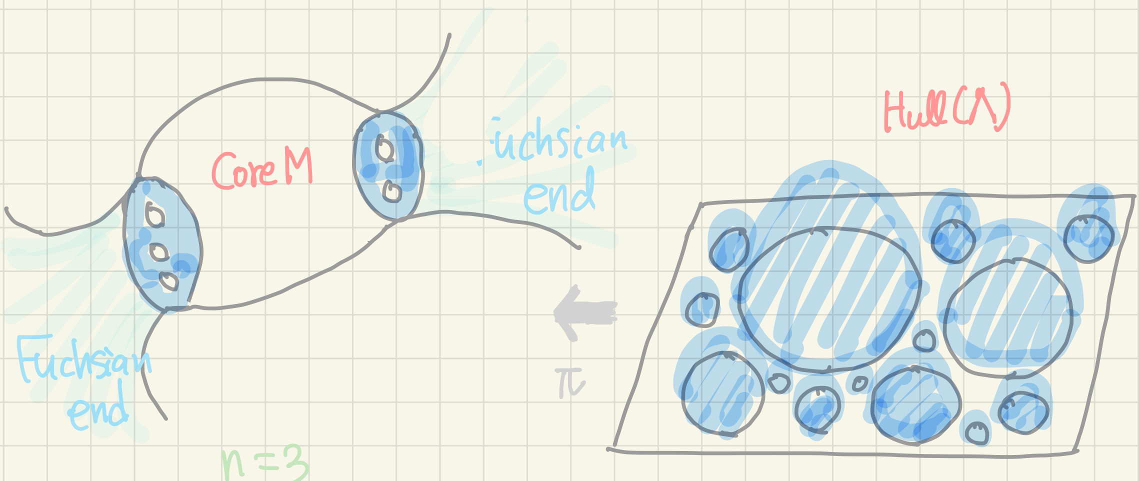
$\mathrm{Int}(\mathrm{Core}\, M) \neq \phi$ $\left(\Rightarrow \Gamma < SO^\circ(n,1) = G\right.$ Zariski dense$\left.\right)$

$\boxed{\mathbf{B}}$ Def M has Fuchsian ends

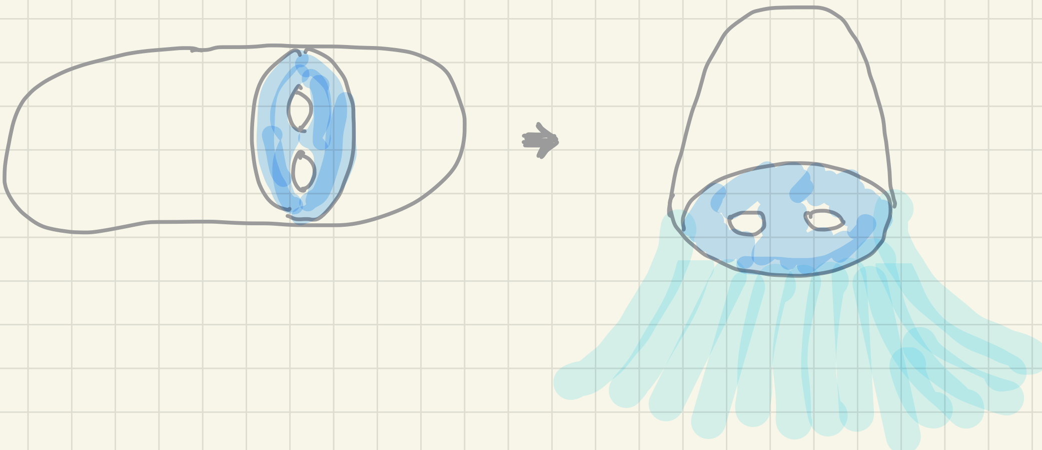
$\Leftrightarrow$ $\partial(\mathrm{Core}\, M)$ is totally geodesic

$\Leftrightarrow$ $S^{n-1} - \Lambda = \bigcup_i B_i$ $\quad \overline{B_i} \cap \overline{B_j} = \phi$

round balls



$$\left\{\begin{array}{c}\text{closed hyp}\\ n\text{-mfld}\end{array}\right\} \supset \left\{\begin{array}{c}\text{closed hyp } n\text{-mfld}\\ \text{with properly embedded}\\ \text{codim 1 geod. planes}\end{array}\right\}$$



Two types of planes:

$$\begin{cases} P \cap \mathrm{Int}(\mathrm{Core}\, M) \neq \phi \\ P \cap \mathrm{Int}(\mathrm{Core}\, M) = \phi \end{cases}$$

$$\Rightarrow \quad \overline{P} \subset M - \mathrm{Int}(\mathrm{Core}\, M)$$ "Fuchsian ends"

Can use Ratner-Shah to describe $\overline{P}$.

**Thm** ( McMullen – Mohammadi – O. $n=3$
Minju Lee – O. $n \geq 4$ )

$M = \Gamma \backslash \mathbb{H}^n$ : hyp mfld with Fuchsian ends

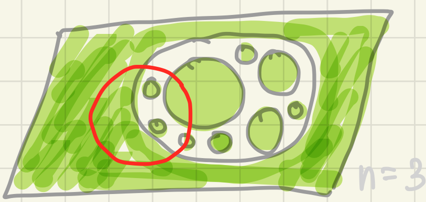
$P \subset M$ geod $k$-plane $2 \leq k \leq n-1$

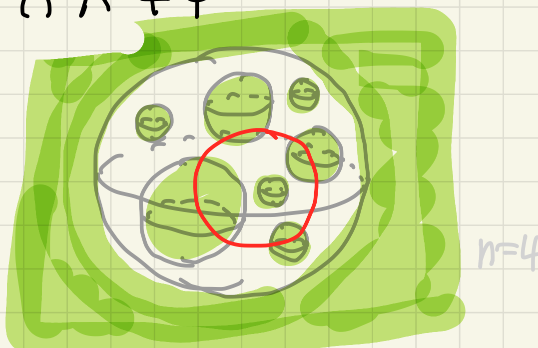
$P \cap \mathrm{Int}(\mathrm{Core}\, M) \neq \phi$

$\Rightarrow$ $\overline{P}$ = geod $m$-plane $k \leq m \leq n$

For $n=3$, $\overline{P} = P$ or $\overline{P} = M$.

$$\mathcal{C}_\Lambda = \left\{ C \subset \mathbb{S}^{n-1} \;\middle|\; \begin{array}{l} (k-1) \text{ sphere} \\ C \cap \Lambda \neq \phi \end{array} \right\}$$




$C \in \mathcal{C}_\Lambda$ $|C \cap \Lambda| \geq 2$

$\Rightarrow$ $\overline{\Gamma C} = \{ D \in \mathcal{C}_\Lambda \mid D \subset \Gamma S \}$ for some $(m-1)$ sphere $S \subset \mathbb{S}^{n-1}$

For $n=3$, $\overline{\Gamma C} = \Gamma C$ or $\overline{\Gamma C} = \mathcal{C}_\Lambda$

$\boxed{C}$ $M = \Gamma\backslash\mathbb{H}^3$ Convex Cocpt

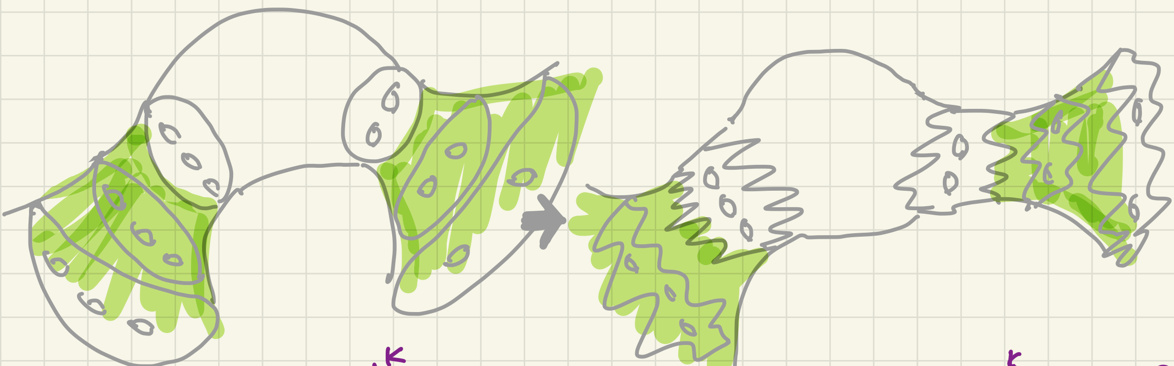
<u>Def</u> $M$ has quasi-Fuchsian ends

$=$ $M \underset{Q.I}{\sim} M_0 = \Gamma_0\backslash\mathbb{H}^3$ CC with Fuchsian ends

$=$ $\Gamma$ is a Quasi-conf. def of $\Gamma_0$

$=$ $S^2 - \Lambda = \bigcup B_i$ $\qquad \overline{B_i} \cap \overline{B_j} = \phi$

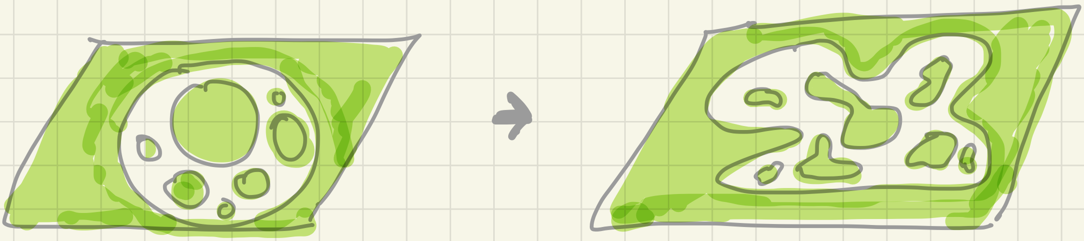
Jordan disks



$\partial \, \mathrm{Core}\, M_0 = \bigcup_{i=1}^{k} S_i$

$Q.I(M_0) = Q.C(\Gamma_0) = \prod_{i=1}^{k} \mathrm{Teich}(S_i)$

$\simeq \prod \mathbb{R}^{6g_i - 6}$



**Thm** (McMullen-Mohammadi-O.)

$M = \Gamma \backslash \mathbb{H}^3$ quasi-Fuchsian ends

$P \subset M$ geod plane with $P \cap \mathrm{Int}(\mathrm{Core}\, M) \neq \phi$

$\Rightarrow$ $P$ is either closed or dense in $\mathrm{Int}(\mathrm{Core}\, M)$

i.e.,
$$\overline{P} \cap \mathrm{Int}(\mathrm{Core}\, M) = \begin{cases} P \cap \mathrm{Int}(\mathrm{Core}\, M) \\ \mathrm{Int}(\mathrm{Core}\, M) \end{cases}$$

**Thm** (Yongquan Zhang)

$\exists$ $P \subset M$ s.t

$P$ is closed in $\mathrm{Int}(\mathrm{Core}\, M)$

but not closed in $M$.

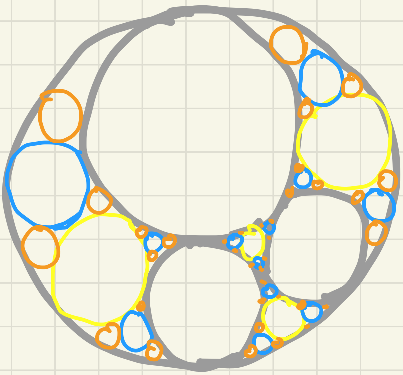
<u>Rmk</u> $\overline{P}$ for $P \subset$ Ends of $M$

is not completely understood.

<u>Rmk</u> Analogous results for some geom. finite hyp 3-mflds were obtained by Benoist-O.

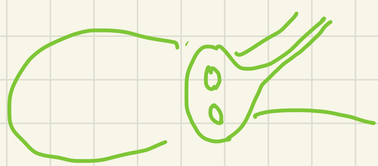
## Open Problem

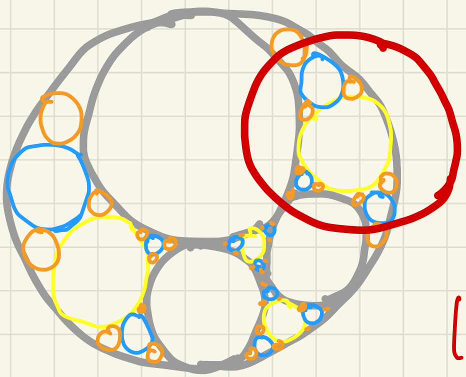
$\mathcal{P}$:



Apollonian circle packing

$$\Gamma = \{ g \in PSL_2\mathbb{C} \mid g\mathcal{P} = \mathcal{P} \}$$





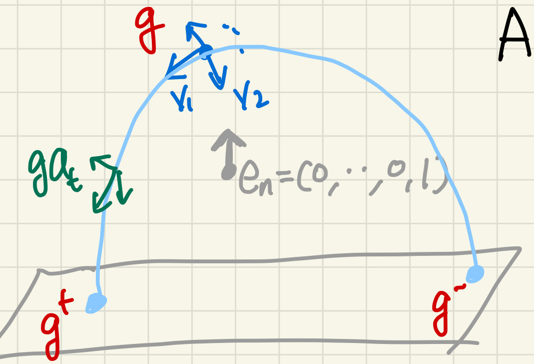
$$\overline{\Gamma C} \overset{?}{=} \begin{cases} \Gamma C \\ C_\Lambda \end{cases}$$

$|C \cap \Lambda| \geq 2$

# Homogeneous dynamics.

$$G = \mathrm{Isom}^+(\mathbb{H}^n) = SO^\circ(Q) \qquad Q = 2x_1x_{n+1} + \sum_{i=2}^{n} x_i^2$$

$$A = \left\{ a_t = \begin{pmatrix} e^t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & e^{-t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$




$$F(\mathbb{H}^n) = \overset{SO^\circ(n,1)}{\overset{\shortparallel}{G}} \curvearrowleft a_t \quad \text{Frame flow}$$

$$\downarrow$$

$$T^1(\mathbb{H}^n) = G/SO(n-1) \curvearrowleft a_t \quad \text{geodesic flow}$$

$$\downarrow$$

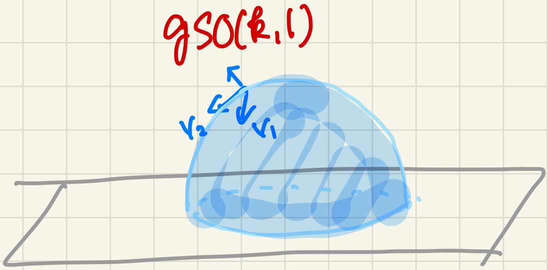
$$\mathbb{H}^n = G/SO(n)$$

$gSO(k,1)$ : $SO(k,1)$ orbits

$$\downarrow$$

(oriented)

$g\mathbb{H}^k$ : geodesic $k$-planes




$$F(M) = \Gamma\backslash G \curvearrowleft a_t$$ Frame flow

$$\big\downarrow$$

$$T^1(M) = \Gamma\backslash G/SO(n-1) \curvearrowleft a_t$$ geodesic flow

$$\big\downarrow$$

$$M = \Gamma\backslash G/SO(n)$$

$[g]SO(k,1)$ : $SO(k,1)$ orbits

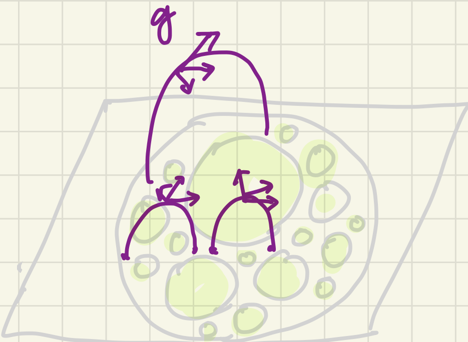
$$\big\downarrow$$

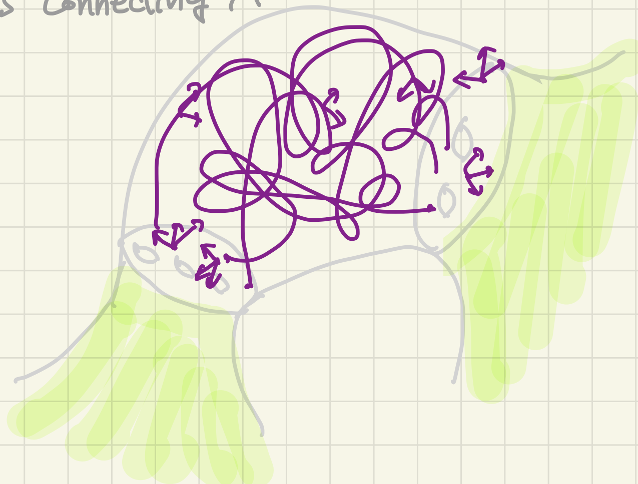
$\pi(g\mathbb{H}^k)$: geodesic $k$-planes

Describe $\overline{[g]SO(k,1)}$ in $\Gamma\backslash G$

$\Omega = \mathrm{RFM} = \{ [g] \in \Gamma \backslash G \mid g^{\pm} \in \Lambda \}$ cpt A-inv subset

union of all geodesics connecting $\Lambda$



Let $M = \Gamma \backslash \mathbb{H}^n$ have Fuchsian ends

**Thm** (McMullen-Mohammadi-O. $n=3$ Lee-O. $n \geq 4$)

$W < G$ conn subgp gen. by unipotent elts

(& normalized by $A$)

Any $W$-orbit closure is relative homogeneous in $\Omega$

$\forall x \in \Omega$, $\overline{xW} \cap \Omega = xL \cap \Omega$

where $W < L < G$ & $xL$ closed.

Moreover, $\quad 2 \le k \le n-1$

$$\overline{x\,SO(k,1)} \cap \Omega = x\,SO(m,1)C \cap \Omega$$

where $C$ closed subgp $\subset$ Centralizer $SO(m,1)$ $\simeq SO(n-m)$

$$\overline{x\,SO(k,1)} = x\,SO(m,1)C \cap \Omega_+ SO(k,1)$$

where $\Omega_+ = \{[g] \in \Gamma \backslash G \mid g^+ \in \Lambda\}$

If $M$ has empty Fuchsian ends (i.e., $M$ cpt),

$\Omega = \Gamma \backslash G$ & this is Ratner & Shah.

Thank you !